Technical Appendix (Separate Document, Not for Publication)

Infrequent Portfolio Decisions: A Solution to the Forward Discount Puzzle

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This Technical Appendix describes all technical details related to the paper. We first describe the two-country model, which is divided into the following sections:

1. Full general equilibrium
2. Solution to the portfolio maximization problems
3. Linearization of the market equilibrium condition
4. Solution for the equilibrium exchange rate
5. Exchange rate determinacy
6. Computation of the threshold cost
7. Computation of excess return predictability
8. Expectations conditioned on limited information
9. An additional asset

After that we describe an extension of the model to more than two countries, which is again broken into the same subsections described above (except for sections 5 and 9). We conclude by discussing alternative modeling strategies. We consider three topics: (1) equilibrium in a more general setup that allows for nominal rigidities, (2) equilibrium in a real model with equity and (3) equilibrium in a model where the government budget is balanced through taxation rather than government spending.

1 Two-Country Model

1.1 General Equilibrium

As discussed in the text, the model is ultimately driven by the Foreign bond market equilibrium condition, which determines the equilibrium exchange rate. We show that this is indeed the case by describing the full general equilibrium of the model in this subsection.

First, consider the bond market equilibrium conditions. We describe the general setup with both agents making frequent and infrequent portfolio decisions. As described in the text, the Foreign bond market equilibrium condition is

$$n_F \sum_{k=1}^{T} b^F_{t-k+1,t} W^F_{t-k+1,t} + n_I \sum_{k=1}^{T} b^I_{t-k+1} W^I_{t-k+1,t} + X_t = BS_t \tag{1}$$

Investors are indifferent between Home bonds and the riskfree technology. The aggregate demand for Home bonds plus the riskfree technology is

$$n_F \sum_{k=1}^{T} (1 - b^F_{t-k+1,t}) W^F_{t-k+1,t} + n_I \sum_{k=1}^{T} (1 - b^I_{t-k+1}) W^I_{t-k+1,t} - X_t + K^g \tag{2}$$

where K^g is the quantity of riskfree technology held by the Home and Foreign governments, which is constant. The capital stock K_t, which is the aggregate claim on the riskfree technology, is then equal to (2) minus the Home bond supply B_H:

$$K_t = n_F \sum_{k=1}^{T} (1 - b^F_{t-k+1,t}) W^F_{t-k+1,t} + n_I \sum_{k=1}^{T} (1 - b^I_{t-k+1}) W^I_{t-k+1,t} - X_t + K^g - B_H \tag{3}$$

Since K_t does not enter the Foreign bond market equilibrium condition, which determines the equilibrium exchange rate, it plays no role in the analysis of the paper.

We then turn to the goods market equilibrium condition, which is

$$n + K_{t-1} \left(e^{\bar{r}} - 1\right) = n_F W^F_{t-T,t} + n_I W^I_{t-T,t} + G_t + C^L_t + K_t - K_{t-1} \tag{4}$$

The supply is on the left hand side while demand for goods is on the right hand side. The supply is the endowment of the newborn plus the production through the riskfree technology. There are four sources of demand: by the agents who are dying and consuming their wealth, by the governments (G_t), by the liquidity traders (C_t^L) and investment ($K_t - K_{t-1}$).

Liquidity traders born at $t-1$ invest X_{t-1} in the Foreign bond and $-X_{t-1}$ in the Home bond and riskfree technology (both measured in terms of the Home currency). At t they consume the net return on these claims:

$$C_t^L = X_{t-1} e^{i^*_{t-1}} \frac{S_t}{S_{t-1}} - X_{t-1} e^{\bar{r}} \tag{5}$$

The sum of Home and Foreign government consumption is equal to earnings on the constant holding of K^g of the riskfree technology minus interest payments. Total government consumption (aggregating over the two governments) is then

$$G_t = (e^{\bar{r}} - 1)K^g - (e^{\bar{r}} - 1)B_H - (e^{i^*_{t-1}} - 1)BS_t \tag{6}$$

Since the other markets are in equilibrium, goods market equilibrium follows automatically from Walras' Law. We can verify this as follows. First define the wealth at time t of all investors alive as

$$W_t = n_F \sum_{k=1}^{T} W^F_{t-k+1,t} + n_I \sum_{k=1}^{T} W^I_{t-k+1,t} \tag{7}$$

Wealth at t is equal to the aggregate return on wealth from $t-1$ to t, minus consumption by the generation that exits at time t, plus the endowment of the newborn at t. Therefore

$$W_t = W_{t-1} R^p_t - n_F W^F_{t-T,t} - n_I W^I_{t-T,t} + n \tag{8}$$

where $W_{t-1}R^p_t$ is the gross asset return from $t-1$ to t aggregated across all investors (other than liquidity traders):

$$\begin{aligned} W_{t-1}R^p_t \;=\; & n_F \sum_{k=1}^{T} W^F_{t-k,t-1} \left((1 - b^F_{t-k,t-1}) e^{\bar{r}} + b^F_{t-k,t-1} \frac{S_t}{S_{t-1}} e^{i^*_{t-1}} \right) + \\ & n_I \sum_{k=1}^{T} W^I_{t-k,t-1} \left((1 - b^I_{t-k}) e^{\bar{r}} + b^I_{t-k} \frac{S_t}{S_{t-1}} e^{i^*_{t-1}} \right) \end{aligned} \tag{9}$$

Using (1) and (3) at $t-1$ we can write this as

$$W_{t-1}R^p_t = e^{\bar{r}} \left(K_{t-1} + X_{t-1} - K^g + B_H \right) + \frac{S_t}{S_{t-1}} e^{i^*_{t-1}} \left(BS_{t-1} - X_{t-1} \right) \tag{10}$$

We now substitute the various equations above into the goods market clearing condition. First, using (8) we can rewrite (4) as:

$$W_t - W_{t-1}R_t^p + K_{t-1}e^{\bar{r}} - K_t = G_t + C_t^L \tag{11}$$

Substituting (5), (6) and (10) into (11) the goods market clearing condition implies:

$$W_t = K_t + B_H - K^g + BS_t \tag{12}$$

It is easily verified that this equation can be derived by taking the sum of (1) and (3) and using the definition (7) of W_t.

1.2 Portfolio Choice

In this section we will derive the optimal portfolios and value functions for both investors that make frequent and infrequent portfolio decisions.

1.2.1 Investors with Actively Managed Portfolios

Consider an agent born at time t. We will compute the optimal portfolio and value function at $t+k$ for $k = 0, .., T-1$. The state of the world consists of all shocks to interest rates and liquidity trade that have happened up to that point in time. However, we will truncate the state space to $\bar{T}$ periods before the present date. This only applies to interest rate shocks. For liquidity demand we can summarize the state by $\hat{x}$, which is a variable that captures the impact of liquidity trade on the exchange rate and follows an AR process with AR coefficient ρ_x and variance of innovations of $b_1^2\sigma_x^2$. We can then write the state space at $t+k$ as

$$Y_{t+k} = \begin{pmatrix} \epsilon_{t+k}^u \\ \cdots \\ \epsilon_{t+k-\bar{T}+1}^u \\ \hat{x}_{t+k} \\ 1 \end{pmatrix} \tag{13}$$

We know from the Bellman equation that the value function at $t+k$ is equal to the expected value function at $t+k+1$:

$$V_{t+k} = E_{t+k}V_{t+k+1} \tag{14}$$

and that the terminal value function at $t+T$ is

$$V_{t+T} = W_{t+T}^{1-\gamma}/(1-\gamma) \tag{15}$$

We can solve the value function with backward induction. Make the following guess:

$$V_{t+k} = e^{Y'_{t+k}H_kY_{t+k}}(1-\tau)^{(1-\gamma)(T-k)}W_{t+k}^{1-\gamma}/(1-\gamma) \tag{16}$$

where H_k is a square matrix of size $\bar{T}+2$. We show that this is correct by assuming that the value function takes this form for $t+k+1$ and then deriving the time $t+k$ value function from the Bellman equation. This also gives us an updating formula relating H_k to H_{k+1}. Clearly H_T is a matrix with only zeros.

We know that

$$W_{t+k+1} = (1-\tau)W_{t+k}e^{r^p_{t+k+1}} \tag{17}$$

We adopt the following second order approximation for the log return:

$$r^p_{t+k+1} = \bar{r} + b^F_{t,t+k}q_{t+k+1} + 0.5b^F_{t,t+k}(1-b^F_{t,t+k})\sigma_F^2 \tag{18}$$

where σ_F^2 is the conditional variance of next period's excess return. After substituting (17) and (18) into the Bellman equation we have

$$E_{t+k}e^{v_{t+k+1}} = e^{Y'_{t+k}H_kY_{t+k}} \tag{19}$$

where

$$v_{t+k+1} = (1-\gamma)\bar{r}+(1-\gamma)b^F_{t,t+k}q_{t+k+1}+(1-\gamma)0.5b^F_{t,t+k}(1-b^F_{t,t+k})\sigma_F^2+Y'_{t+k+1}H_{k+1}Y_{t+k+1} \tag{20}$$

We will now show that there is a matrix G_k such that

$$E_{t+k}e^{v_{t+k+1}} = e^{Y'_{t+k}G_kY_{t+k}} \tag{21}$$

The Bellman equation then implies $H_k = G_k$.

We can write

$$q_{t+k+1} = M_1^kY_{t+k} + M_2^k\epsilon_{t+k+1} \tag{22}$$

where

$$\epsilon_{t+k+1} = \begin{pmatrix} \epsilon^u_{t+k+1} \\ \epsilon^x_{t+k+1} \end{pmatrix} \tag{23}$$

We will derive below that

$$q_{t+k+1} = \lambda_u(1,1)\epsilon^u_{t+k+1} + b_1\epsilon^x_{t+k+1} + \sum_{v=0}^{\infty}\delta_u(v,1)\epsilon^u_{t+k-v} + (\rho_x - 1)\hat{x}_{t+k} - \bar{r} \tag{24}$$

Therefore $M_1^k(v+1) = \delta_u(v,1)$ for $v = 0,..,\bar{T}-1$, $M_1^k(\bar{T}+1) = \rho_x - 1$, $M_1^k(\bar{T}+2) = -\bar{r}$, $M_2^k(1) = \lambda_u(1,1)$ and $M_2^k(2) = b_1$.

We can also write

$$Y_{t+k+1} = N_1^k Y_{t+k} + N_2^k \epsilon_{t+k+1} \tag{25}$$

The matrix N_1^k is $\bar{T}+2$ by $\bar{T}+2$. It has zeros in the first row. We have $N_1^k(v+1,v) = 1$ for $v = 1,..,\bar{T}-1$, $N_1^k(\bar{T}+1,\bar{T}+1) = \rho_x$ and $N_1^k(\bar{T}+2,\bar{T}+2) = 1$. The matrix N_2^k is $\bar{T}+2$ by 2. We have $N_2^k(1,1) = 1$, $N_2^k(\bar{T}+1,2) = b_1$ and zeros elsewhere.

It follows that

$$v_{t+k+1} = (1-\gamma)\bar{r} + (1-\gamma)b^F_{t,t+k}M_1^k Y_{t+k} + (1-\gamma)0.5b^F_{t,t+k}(1-b^F_{t,t+k})\sigma_F^2 + Y'_{t+k}(N_1^k)'H_{k+1}N_1^k Y_{t+k} + C_1^k\epsilon_{t+k+1} + \epsilon'_{t+k+1}C_2^k\epsilon_{t+k+1} \tag{26}$$

where

$$C_1^k = (1-\gamma)b^F_{t,t+k}M_2^k + 2Y'_{t+k}(N_1^k)'H_{k+1}N_2^k \tag{27}$$

$$C_2^k = (N_2^k)'H_{k+1}N_2^k \tag{28}$$

We know that $\epsilon_{t+k+1} \sim N(0,\Sigma)$ where $\Sigma[1,1] = \sigma_u^2$, $\Sigma[2,2] = \sigma_x^2$ and the off-diagonal elements are zero. Therefore

$$E_{t+k}e^{C_1^k\epsilon_{t+k+1}+\epsilon'_{t+k+1}C_2^k\epsilon_{t+k+1}} = \frac{1}{|\Sigma|^{0.5}2\pi}\int e^{C_1^k\epsilon_{t+k+1}+\epsilon'_{t+k+1}C_2^k\epsilon_{t+k+1}-0.5\epsilon'_{t+k+1}\Sigma^{-1}\epsilon_{t+k+1}}d\epsilon_{t+k+1} = \frac{1}{|\Sigma|^{0.5}2\pi}\int e^{C_1^k\epsilon_{t+k+1}+\epsilon'_{t+k+1}(C_2^k-0.5\Sigma^{-1})\epsilon_{t+k+1}}d\epsilon_{t+k+1} = \frac{|\Omega^k|^{0.5}}{|\Sigma|^{0.5}}e^{0.5C_1^k\Omega^k(C_1^k)'} \tag{29}$$

Here the last equality uses the moment generating function of the normal distribution and

$$\Omega^k = (\Sigma^{-1} - 2C_2^k)^{-1} \tag{30}$$

It follows that

$$E_{t+k}e^{v_{t+k+1}} = e^{\hat{v}_{t+k}} \tag{31}$$

where

$$\hat{v}_{t+k} = (1-\gamma)\bar{r} + (1-\gamma)b^F_{t,t+k}M^k_1 Y_{t+k} + 0.5(1-\gamma)b^F_{t,t+k}(1-b^F_{t,t+k})\sigma^2_F + Y'_{t+k}(N^k_1)'H_{k+1}N^k_1 Y_{t+k} + 0.5ln(|\Omega^k|/|\Sigma|) + 0.5C^k_1\Omega^k(C^k_1)' \quad (32)$$

¿From the definition of C^k_1 we have

$$C^k_1\Omega^k(C^k_1)' = (1-\gamma)^2\hat{\sigma}^2_F(k)(b^F_{t,t+k})^2 + 2(1-\gamma)D^k_1 Y_{t+k}b^F_{t,t+k} + 2Y'_{t+k}D^k_2 Y_{t+k} \quad (33)$$

where

$$\hat{\sigma}^2_F(k) = M^k_2\Omega^k(M^k_2)' \quad (34)$$

$$D^k_1 = 2M^k_2\Omega^k(N^k_2)'H_{k+1}N^k_1 \quad (35)$$

$$D^k_2 = 2(N^k_1)'H_{k+1}N^k_2\Omega^k(N^k_2)'H_{k+1}N^k_1 \quad (36)$$

Therefore

$$\hat{v}_{t+k} = A^k + B^k b^F_{t,t+k} + C^k(b^F_{t,t+k})^2 \quad (37)$$

where

$$A^k = (1-\gamma)\bar{r} + Y'_{t+k}((N^k_1)'H_{k+1}N^k_1 + D^k_2)Y_{t+k} + 0.5ln(|\Omega^k|/|\Sigma|) \quad (38)$$

$$B^k = (1-\gamma)(M^k_1 + D^k_1)Y_{t+k} + (1-\gamma)0.5\sigma^2_F \quad (39)$$

$$C^k = -0.5(1-\gamma)\sigma^2_F + 0.5(1-\gamma)^2\hat{\sigma}^2_F(k) \quad (40)$$

At $t+k$ the investor chooses the portfolio to maximize the expected $t+k+1$ value function, which is equivalent to maximizing $\hat{v}_{t+k}$. This yields

$$b^F_{t,t+k} = \frac{-B^k}{2C^k} = \frac{(M^k_1 + D^k_1)Y_{t+k} + 0.5\sigma^2_F}{(\gamma-1)\hat{\sigma}^2_F(k) + \sigma^2_F} \quad (41)$$

Note that $M_1 Y_{t+k} = E_{t+k}q_{t+k+1}$. The hedge term is proportional to $D^k_1 Y_{t+k}$. With $\tilde{q} = q + \bar{r}$ the excess return in deviation from steady state, the optimal portfolio can then be written as

$$b^F_{t,t+k} = \bar{b}^F(k) + \frac{E_{t+k}(\tilde{q}_{t+k+1})}{(\gamma-1)\hat{\sigma}^2_F(k) + \sigma^2_F} + \frac{1}{(\gamma-1)\hat{\sigma}^2_F(k) + \sigma^2_F}D^k_1 Y_{t+k} \quad (42)$$

where

$$\bar{b}^F(k) = \frac{0.5\sigma^2_F - \bar{r}}{(\gamma-1)\hat{\sigma}^2_F(k) + \sigma^2_F} \quad (43)$$

Substituting the optimal portfolio in $\hat{v}_{t+k}$ we have

$$\hat{v}_{t+k} = A^k - \frac{(B^k)^2}{4C^k} = G_k^1 + G_k^2 Y_{t+k} + Y_{t+k}' G_k^3 Y_{t+k} \tag{44}$$

where

$$G_k^1 = (1-\gamma)\bar{r} + 0.5ln(|\Omega^k|/|\Sigma|) + \frac{1}{8}\frac{(1-\gamma)\sigma_F^4}{\sigma_F^2 + (\gamma-1)\hat{\sigma}_F^2(k)} \tag{45}$$

$$G_k^2 = \frac{0.5(1-\gamma)\sigma_F^2}{\sigma_F^2 + (\gamma-1)\hat{\sigma}_F^2(k)}(M_1^k + D_1^k) \tag{46}$$

$$G_k^3 = (N_1^k)' H_{k+1} N_1^k + D_2^k + \frac{0.5(1-\gamma)}{\sigma_F^2 + (\gamma-1)\hat{\sigma}_F^2(k)}(M_1^k + D_1^k)'(M_1^k + D_1^k) \tag{47}$$

We can write

$$G_k^1 + G_k^2 Y_{t+k} + Y_{t+k}' G_k^3 Y_{t+k} = Y_{t+k}' G_k Y_{t+k} \tag{48}$$

where $G_k = G_k^3 + G_k^4 + G_k^5$, G_k^4 is a square $\bar{T}+2$ matrix with the vector G_k^2 in the last row and otherwise zeros and G_k^5 is a square $\bar{T}+2$ matrix with the element $(\bar{T}+2, \bar{T}+2)$ equal to G_k^1 and all other elements zero. It follows that $H_k = G_k$.

1.2.2 Investors Making Infrequent Portfolio Decisions

Now consider an investor born at time t making one portfolio decision for the next T periods. The time t value function is

$$V_t = E_t e^{(1-\gamma)(r_{t+1}^p + .. + r_{t+T}^p)}/(1-\gamma) \tag{49}$$

We again adopt the second order approximation for the log return:

$$r_{t+k}^p = \bar{r} + b_t^I q_{t+k} + 0.5 b_t^I (1 - b_t^I) var_t(q_{t+k}) \tag{50}$$

We then have

$$V_t = E_t e^{(1-\gamma)\bar{r}T + (1-\gamma)b_t^I q_{t,t+T} + 0.5(1-\gamma)b_t^I(1-b_t^I)\sum_{k=1}^T var_t(q_{t+k})}/(1-\gamma) =$$
$$e^{(1-\gamma)\bar{r}T + (1-\gamma)b_t^I E_t q_{t,t+T} + 0.5(1-\gamma)^2 (b_t^I)^2 var_t(q_{t,t+T}) + 0.5(1-\gamma)b_t^I(1-b_t^I)\sum_{k=1}^T var_t(q_{t+k})}/(1-\gamma)$$

We can write the term in the exponential as

$$A + Bb_t^I + C(b_t^I)^2 \tag{51}$$

where

$$\begin{aligned}
A &= (1-\gamma)\bar{r}T && (52)\\
B &= (1-\gamma)E_t q_{t,t+T} + 0.5(1-\gamma)\sum_{k=1}^{T} var_t(q_{t+k}) && (53)\\
C &= 0.5(1-\gamma)^2 var_t(q_{t,t+T}) - 0.5(1-\gamma)\sum_{k=1}^{T} var_t(q_{t+k})\\
&= -0.5\gamma(1-\gamma)\sigma_I^2 && (54)
\end{aligned}$$

where

$$\sigma_I^2 = \left(1-\frac{1}{\gamma}\right) var_t(q_{t,t+T}) + \frac{1}{\gamma}\sum_{k=1}^{T} var_t(q_{t+k}) \tag{55}$$

The optimal portfolio is

$$b_t^I = \frac{-B}{2C} = b^I + \frac{E_t q_{t,t+T}}{\gamma\sigma_I^2} \tag{56}$$

where

$$b^I = \frac{0.5\sum_{k=1}^{T} var_t(q_{t+k})}{\gamma\sigma_I^2} \tag{57}$$

Writing $\tilde{q}_{t,t+T}$ as the excess return in deviation from steady state, we have

$$b_t^I = \bar{b}^I + \frac{E_t\tilde{q}_{t,t+T}}{\gamma\sigma_I^2} \tag{58}$$

where

$$\bar{b}^I = \frac{0.5\sum_{k=1}^{T} var_t(q_{t+k}) - \bar{r}T}{\gamma\sigma_I^2} \tag{59}$$

Substituting the optimal portfolio, the value function becomes

$$V_t = e^{A-\frac{B^2}{4C}}/(1-\gamma) \tag{60}$$

We will derive below that $E_t q_{t,t+T} = M_1 Y_t$, where M_1 is a $\bar{T}+2$ vector with $M_1(v+1) = \delta_u(v,T)$ for $v = 0,..,\bar{T}-1$, $M_1(\bar{T}+1) = \rho_x^T - 1$ and $M_1(\bar{T}+2) = -\bar{r}T$. We have

$$A - \frac{B^2}{4C} = G_1 + G_2 Y_t + Y_t' G_3 Y_t \tag{61}$$

where

$$G_1 = (1-\gamma)\bar{r}T + \frac{1-\gamma}{\gamma}\frac{(\sum_{k=1}^{T} var_t(q_{t+k}))^2}{8\sigma_I^2} \tag{62}$$

$$G_2 = \frac{1-\gamma}{\gamma}\frac{\sum_{k=1}^{T} var_t(q_{t+k})}{2\sigma_I^2} M_1 \tag{63}$$

$$G_3 = \frac{1-\gamma}{\gamma}\frac{1}{2\sigma_I^2}(M_1)'(M_1) \tag{64}$$

We can write

$$G_1 + G_2 Y_t + Y_t' G_3 Y_t = Y_t' G Y_t \tag{65}$$

where $G = G_3 + G_4 + G_5$. G_4 is a square $\bar{T}+2$ matrix with the vector G_2 in the last row and otherwise zeros and G_5 is a square $\bar{T}+2$ matrix with the element $(\bar{T}+2, \bar{T}+2)$ equal to G_1 and all other elements zero. The time t value function is then

$$V_t = e^{Y_t' G Y_t}/(1-\gamma) \tag{66}$$

1.3 Linearization of Market Equilibrium Condition

The market equilibrium condition (see equation 13 of text) is

$$n_F \sum_{k=1}^{T} b_{t-k+1,t}^F W_{t-k+1,t}^F + n_I \sum_{k=1}^{T} b_{t-k+1}^I W_{t-k+1,t}^I + (\bar{x}+x_t)\bar{W} = Be^{s_t} \tag{67}$$

where

$$W_{t-k+1,t}^F = \prod_{i=1}^{k-1} R_{t-k+i+1}^p (1-\tau)^{k-1} \tag{68}$$

$$R_{t-k+i+1}^p = (1-b_{t-k+1,t-k+i}^F)e^{i_{t-k+i}} + b_{t-k+1,t-k+i}^F e^{s_{t-k+i+1}-s_{t-k+i}+i_{t-k+i}^*} \tag{69}$$

and

$$W_{t-k+1,t}^I = \prod_{i=1}^{k-1} R_{t-k+i+1}^p \tag{70}$$

$$R_{t-k+i+1}^p = (1-b_{t-k+1}^I)e^{i_{t-k+i}} + b_{t-k+1}^I e^{s_{t-k+i+1}-s_{t-k+i}+i_{t-k+i}^*} \tag{71}$$

We differentiate this budget constraint around the point where the exchange rate and asset returns are zero, $\tau = 0$, and portfolio shares are equal to their steady state values. The first order Taylor approximation of the term $b^F_{t-k+1,t} W^F_{t-k+1,t}$ is:

$$\begin{aligned} & b^F_{t-k+1,t} W^F_{t-k+1,t} = \\ & b^F_{t-k+1,t} + \bar{b}^F(k-1) \sum_{i=1}^{k-1} \left[\bar{r} - \tau + \bar{b}^F(i-1) q_{t-k+i+1} \right] \end{aligned} \tag{72}$$

therefore a

$$\sum_{k=1}^{T} b^F_{t-k+1,t} W^F_{t-k+1,t} = \sum_{k=1}^{T} b^F_{t-k+1,t} + \bar{k}^F + \sum_{k=1}^{T-1} k^F(k) q_{t-k+1} \tag{73}$$

where

$$\begin{aligned} \bar{k}^F &= \sum_{k=1}^{T-1} \bar{b}^F(k) k (\bar{r} - \tau) \\ k^F(k) &= \sum_{j=1}^{T-k} \bar{b}^F(j-1) \bar{b}^F(j+k-1) \end{aligned}$$

Similarly

$$\sum_{k=1}^{T} b^I_{t-k+1} W^I_{t-k+1,t} = \sum_{k=1}^{T} b^I_{t-k+1} + \bar{k}^I + \sum_{k=1}^{T-1} k^I(k) q_{t-k+1} \tag{74}$$

where

$$\begin{aligned} \bar{k}^I &= \sum_{k=1}^{T-1} \bar{b}^I k \bar{r} \\ k^I(k) &= (T-k)(\bar{b}^I)^2 \end{aligned}$$

Finally, a first order Taylor approximation of the right hand side of the budget constraint is $Be^{s_t} = Bs_t$. The budget constraint can then be rewritten as

$$\begin{aligned} & n_F \sum_{k=1}^{T} b^F_{t-k+1,t} + n_I \sum_{k=1}^{T} b^I_{t-k+1} + n_F \bar{k}^F + n_I \bar{k}^I + \\ & \sum_{k=1}^{T-1} (n_F k^F(k) + n^I k^I(k)) q_{t-k+1} + (\bar{x} + x_t) \bar{W} = B + Bs_t \end{aligned} \tag{75}$$

We define $\bar{W}$ as equal to total steady state wealth when evaluated at returns on Home and Foreign bonds equal to their steady state levels ($\bar{r}$ for Home bonds and 0 for Foreign bonds), the fraction invested in Foreign bonds is b for all investors (the steady state relative supply of Foreign bonds) and a zero portfolio decision cost for investors making frequent portfolio decisions.

Based on that definition we have

$$\bar{W} = wnT \tag{76}$$

where

$$w = \sum_{k=1}^{T} \left(\bar{R}^p\right)^{k-1} /T \tag{77}$$

$$\bar{R}^p = (1-b)e^{\bar{r}} + b \tag{78}$$

Dividing (75) by nT yields

$$f\frac{1}{T}\sum_{k=1}^{T} b^F_{t-k+1,t} + (1-f)\frac{1}{T}\sum_{k=1}^{T} b^I_{t-k+1} + \frac{1}{T}(f\bar{k}^F + (1-f)\bar{k}^I) + \sum_{k=1}^{T-1}\frac{1}{T}(fk^F(k) + (1-f)k^I(k))q_{t-k+1} + w\bar{x} + wx_t = wb + wbs_t \tag{79}$$

where $b = B/\bar{W}$ is the steady state supply of Foreign bonds divided by steady state wealth, which is the steady state fraction invested in Foreign bonds.

Let $\tilde{q}$ again denote the excess return in deviation from its steady state. Then subtracting the steady state from both the right and left hand sides of (79), and substituting the expressions for optimal portfolios, gives

$$f\frac{E_t\tilde{q}_{t+1}}{\gamma\sigma^2} + fDY_t + (1-f)\frac{1}{T}\sum_{k=1}^{T}\frac{E_{t-k+1}\tilde{q}_{t-k+1,t-k+1+T}}{\gamma\sigma_I^2} + \sum_{k=1}^{T-1}\frac{1}{T}(fk^F(k) + (1-f)k^I(k))\tilde{q}_{t-k+1} + wx_t = wbs_t \tag{80}$$

where

$$D = \frac{1}{T}\sum_{k=1}^{T}\frac{D_1^{k-1}}{(\gamma-1)\hat{\sigma}_F^2(k-1) + \sigma_F^2}$$

$$\frac{1}{\sigma^2} = \frac{1}{T}\sum_{k=1}^{T}\frac{\gamma}{(\gamma-1)\hat{\sigma}_F^2(k-1) + \sigma_F^2}$$

The constant term in the portfolio of liquidity traders is set such that the market clearing condition holds in steady state for a given real interest rate $\bar{r}$. From (79) this is the case when is

$$f\bar{b}^F + (1-f)\bar{b}^I + \frac{1}{T}(f\bar{k}^F + (1-f)\bar{k}^I) - \sum_{k=1}^{T-1}\frac{1}{T}(fk^F(k) + (1-f)k^I(k))\bar{r} + w\bar{x} = wb \tag{81}$$

where

$$\bar{b}^F = \frac{1}{T}\sum_{k=1}^{T} \bar{b}^F(k-1) \tag{82}$$

The steady state $\bar{x}$ plays no role in the analysis below.

1.4 Solution Equilibrium Exchange Rate

In this section we will discuss how the equilibrium exchange rate is solved for a given fraction f of investors making frequent portfolio decisions. As explained in section 2.5 of the paper, we will assume that $\psi = 0$, so that $i_t - i_t^* = u_t + \bar{r}$ and $\tilde{q}_{t+1} = s_{t+1} - s_t - u_t$. There are two stochastic innovations in the model, associated with the forward discount (foreign interest rate) and liquidity trade. We conjecture that the equilibrium exchange rate at time t depends linearly on present and past innovations:

$$s_t = A(L)\epsilon_t^u + B(L)\epsilon_t^x \tag{83}$$

where

$$A(L) = a_1 + a_2 L + a_3 L^2 + ... \tag{84}$$

$$B(L) = b_1 + b_2 L + b_3 L^2 + ... \tag{85}$$

The number of lags is infinite in both lag operators. Substituting (83) into the market equilibrium condition (80), we obtain an equilibrium exchange rate equation. We then need to equate the conjectured to the equilibrium exchange rate equation. We choose the process

$$x_t = C(L)\epsilon_t^x = (c_1 + c_2 L + c_3 L^2 + ...)\epsilon_t^x \tag{86}$$

such that $b_{k+1} = \rho_x b_k$ for $k \geq 1$. Normalize such that $c_1 = 1$. Below we will show how to solve $A(L)$, $C(L)$ and b_1 by imposing foreign bond market equilibrium. We will write $\hat{x}_t = B(L)\epsilon_t^x$. Therefore

$$s_t = A(L)\epsilon_t^u + \hat{x}_t \tag{87}$$

$$\hat{x}_t = \rho_x \hat{x}_{t-1} + b_1 \epsilon_t^x \tag{88}$$

In order to solve for the equilibrium exchange rate equation we need to write both sides of the market equilibrium equation as a function of the underlying innovations and then equate the coefficients multiplying these innovations on the right and left side of the equation. The overall approach is rather straightforward, but the algebra is a bit lengthy.

We first need to compute first and second order moments in the market equilibrium condition. Starting with the second moments, we need to compute (i) $var_t(q_{t+k}),\ \ k = 1, ..T$; and (ii) $var_t(q_{t,t+T})$. Starting with the former,

$$\begin{aligned} &\tilde{q}_{t+k} = s_{t+k} - s_{t+k-1} - u_{t+k-1} \equiv \\ &\sum_{v=1}^{k} \left(\bar{\lambda}_u(v,k)\epsilon^u_{t+v} + \bar{\lambda}_x(v,k)\epsilon^x_{t+v} \right) + \\ &\sum_{v=0}^{\infty} \bar{\delta}_u(v,k)\epsilon^u_{t-v} + \rho_x^{k-1}(\rho_x - 1)\hat{x}_t \end{aligned} \tag{89}$$

where

$$\begin{aligned} \bar{\lambda}_u(k,k) &= a_1 \\ \bar{\lambda}_u(v,k) &= a_{k+1-v} - a_{k-v} - \rho^{k-v-1} \quad v = 1, .., k-1 \\ \bar{\delta}_u(v,k) &= a_{k+1+v} - a_{k+v} - \rho^{k-1+v} \quad v \geq 0 \end{aligned}$$

and

$$\begin{aligned} \bar{\lambda}_x(k,k) &= b_1 \\ \bar{\lambda}_x(v,k) &= b_{k+1-v} - b_{k-v} \quad v = 1, .., k-1 \end{aligned}$$

We will also write

$$\rho_x^{k-1}(\rho_x - 1)\hat{x}_t = \sum_{v=0}^{\infty} \bar{\delta}_x(v,k)\epsilon^x_{t-v}$$

where $\bar{\delta}_x(v,k) = b_{k+1+v} - b_{k+v}$ for $v \geq 0$.

Therefore

$$var_t(q_{t+k}) = \sum_{v=1}^{k} \left(\bar{\lambda}_u(v,k)^2\sigma_u^2 + \bar{\lambda}_x(v,k)^2\sigma_x^2 \right) \tag{90}$$

Now consider the variance of $q_{t,t+T}$. We have

$$\begin{aligned} \tilde{q}_{t,t+T} = \ &\sum_{k=1}^{T} \tilde{q}_{t+k} = s_{t+T} - s_t - i^*_t - .. - i^*_{t+T-1} = \\ &s_{t+T} - s_t - \sum_{v=1}^{T} u_{t+v-1} \end{aligned} \tag{91}$$

We can write

$$\textstyle\sum_{v=1}^{T} u_{t+v-1} = \left(\sum_{v=1}^{T}\rho^{v-1}\right)\sum_{v=0}^{\infty}\rho^{v}\epsilon_{t-v}^{u} + \sum_{v=1}^{T-1}\left(\sum_{k=1}^{T-v}\rho^{k-1}\right)\epsilon_{t+v}^{u} = \tag{92}$$
$$\frac{1-\rho^{T}}{1-\rho}\sum_{v=0}^{\infty}\rho^{v}\epsilon_{t-v}^{u} + \sum_{v=1}^{T-1}\frac{1-\rho^{T-v}}{1-\rho}\epsilon_{t+v}^{u} \equiv$$
$$\rho(0,T)\sum_{v=0}^{\infty}\rho^{v}\epsilon_{t-v}^{u} + \sum_{v=1}^{T-1}\rho(v,T)\epsilon_{t+v}^{u}$$

where $\rho(v,T) = (1-\rho^{T-v})/(1-\rho)$.

Substituting (83) and (92) into (91) we get

$$\tilde{q}_{t,t+T} = a_1\epsilon_{t+T}^{u} + \sum_{v=1}^{T-1}\left(-\rho(v,T) + a_{T-v+1}\right)\epsilon_{t+v}^{u} + \tag{93}$$
$$b_1\epsilon_{t+T}^{x} + \sum_{v=1}^{T-1} b_{T-v+1}\epsilon_{t+v}^{x} +$$
$$\sum_{v=0}^{\infty}\left(-\rho(0,T)\rho^{v} + a_{T+v+1} - a_{v+1}\right)\epsilon_{t-v}^{u} + (\rho_x^T - 1)\hat{x}_t \equiv$$
$$\sum_{v=1}^{T}\left(\lambda_u(v,T)\epsilon_{t+v}^{u} + \lambda_x(v,T)\epsilon_{t+v}^{x}\right) + \sum_{v=0}^{\infty}\delta_u(v,T)\epsilon_{t-v}^{u} + (\rho_x^T - 1)\hat{x}_t$$

At times we will also write

$$(\rho_x^T - 1)\hat{x}_t = \sum_{v=0}^{\infty}\delta_x(v,T)\epsilon_{t-v}^{x}$$

It follows that

$$var_t(q_{t,t+T}) = \sum_{v=1}^{T}\left(\lambda_u(v,T)^2\sigma_u^2 + \lambda_x(v,T)^2\sigma_x^2\right) \tag{94}$$

Next consider the first moments. Using (93) for $T = 1$ we have

$$E_t\tilde{q}_{t+1} = E_t\tilde{q}_{t,t+1} = \sum_{v=0}^{\infty}\delta_u(v,1)\epsilon_{t-v}^{u} + (\rho_x - 1)\hat{x}_t =$$
$$\sum_{v=0}^{\infty}\delta_u(v,1)\epsilon_{t-v}^{u} + \sum_{v=0}^{\infty}\delta_x(v,1)\epsilon_{t-v}^{x} \tag{95}$$

and

$$\sum_{k=1}^{T} E_{t-k+1}q_{t-k+1,t-k+1+T} = \left(1 + L + ... + L^{T-1}\right)E_t q_{t,t+T} =$$

$$\left(1 + L + ... + L^{T-1}\right) \sum_{v=0}^{\infty} \left(\delta_u(v,T)\epsilon^u_{t-v} + \delta_x(v,T)\epsilon^x_{t-v}\right) =$$

$$\sum_{v=0}^{\infty} \sum_{k=0}^{min(v,T-1)} \delta_u(v-k,T)\epsilon^u_{t-v} + \sum_{v=0}^{\infty} \sum_{k=0}^{min(v,T-1)} \delta_x(v-k,T)\epsilon^x_{t-v} \equiv$$

$$\sum_{v=0}^{\infty} \eta^T_u(v)\epsilon^u_{t-v} + \sum_{v=0}^{\infty} \eta^T_x(v)\epsilon^x_{t-v} \tag{96}$$

Therefore

$$f\frac{E_t\tilde{q}_{t+1}}{\gamma\sigma^2} + (1-f)\frac{1}{T}\sum_{k=1}^{T} \frac{E_{t-k+1}\tilde{q}_{t-k+1,t-k+1+T}}{\gamma\sigma^2_I} \equiv$$

$$\sum_{v=0}^{\infty} \eta_u(v)\epsilon^u_{t-v} + \sum_{v=0}^{\infty} \eta_x(v)\epsilon^x_{t-v} \tag{97}$$

where

$$\eta_u(v) = f\eta^1_u(v)\frac{1}{\gamma\sigma^2} + (1-f)\frac{1}{T}\eta^T_u\frac{1}{\gamma\sigma^2_I} \tag{98}$$

$$\eta_x(v) = f\eta^1_x(v)\frac{1}{\gamma\sigma^2} + (1-f)\frac{1}{T}\eta^T_x\frac{1}{\gamma\sigma^2_I} \tag{99}$$

We finally need to compute, as a function of model innovations, the expression

$$\frac{1}{T}\sum_{k=1}^{T-1}(fk^F(k) + (1-f)k^I(k))\tilde{q}_{t-k+1} \tag{100}$$

in the linearized market clearing condition. Using (93) with t replaced by $t-k$ and T by 1, we have for $k > 1$

$$\begin{aligned} \tilde{q}_{t+1-k} = \; & \lambda_u(1,1)\epsilon^u_{t+1-k} + \lambda_x(1,1)\epsilon^x_{t+1-k} + \\ & \sum_{v=0}^{\infty}\left(\delta_u(v,1)\epsilon^u_{t-k-v} + \delta_x(v,1)\epsilon^x_{t-k-v}\right) \equiv \\ & \sum_{v=0}^{\infty}\left(\omega_u(v,k)\epsilon^u_{t-v} + \omega_x(v,k)\epsilon^x_{t-v}\right) \end{aligned} \tag{101}$$

where

$$\begin{aligned} \omega_u(v,k) &= 0 \quad v = 0,..,k-2 \\ \omega_u(v,k) &= \lambda_u(1,1) \quad v = k-1 \\ \omega_u(v,k) &= \delta_u(v-k,1) \quad v \geq k \\ \omega_x(v,k) &= 0 \quad v = 0,..,k-2 \\ \omega_x(v,k) &= \lambda_x(1,1) \quad v = k-1 \\ \omega_x(v,k) &= \delta_x(v-k,1) \quad v \geq k \end{aligned}$$

It follows that

$$\frac{1}{T}\sum_{k=1}^{T-1}(fk^F(k)+(1-f)k^I(k))\tilde{q}_{t-k+1}\equiv$$
$$\sum_{v=0}^{\infty}\left(\mu_u(v)\epsilon^u_{t-v}+\mu_x(v)\epsilon^x_{t-v}\right) \tag{102}$$

where

$$\mu_u(v)=\frac{1}{T}\sum_{k=1}^{T-1}(fk^F(k)+(1-f)k^I(k))\omega_u(v,k)$$
$$\mu_x(v)=\frac{1}{T}\sum_{k=1}^{T-1}(fk^F(k)+(1-f)k^I(k))\omega_x(v,k)$$

Substituting (97) and (102) into (80) we get

$$\begin{aligned} wbs_t = \ & \sum_{v=0}^{\infty}\left(\eta_u(v)+\mu_u(v)\right)\epsilon^u_{t-v}+\sum_{v=0}^{\bar{T}-1}fD(v+1)\epsilon^u_{t-v}+fD(\bar{T}+1)\hat{x}_t+ \\ & \sum_{v=0}^{\infty}\left(\eta_x(v)+\mu_x(v)+wc_{v+1}\right)\epsilon^x_{t-x} \end{aligned}$$

Equating the conjectured to the equilibrium exchange rate equation then gives

$$wba_{v+1}=\eta_u(v)+\mu_u(v)+fD(v+1)\quad\forall v=0,..,\bar{T}-1 \tag{103}$$
$$wba_{v+1}=\eta_u(v)+\mu_u(v)\quad\forall v\geq\bar{T} \tag{104}$$
$$wbb_{v+1}=\eta_x(v)+\mu_x(v)+wc_{v+1}+fD(\bar{T}+1)b_1\rho_x^v\quad\forall v\geq 0 \tag{105}$$

We can solve for $A(L)$ and b_1 using (103) for $v=0,..,\bar{T}-1$ and (105) for $v=0$ (using $c_1=1$). We therefore omit equation (104). This is equivalent to truncating the state space after $\bar{T}$ periods. We only solve for $a_1,..,a_{\bar{T}}$ and set $a_{\bar{T}+k}=0$ for $k>0$. This is a very close approximation for large $\bar{T}$ as the impact of interest rate shocks on the exchange rate dies out over time. Once $A(L)$ and b_1 have been solved, $c(v+1)$ for $\forall v>0$ can be solved from (105). But these parameters are inconsequential for the analysis.

In practice we set $\bar{T}=60$, so that there are 61 equations to be solved. Further increasing $\bar{T}$ has a negligible impact on the results.

1.5 Exchange Rate Determinacy

Some comments are in order regarding exchange rate determinacy in the context of the monetary policy rule. Abstracting from infrequent portfolio decisions, it is well-known that a Wicksellian monetary policy rule is sufficient to give a unique price level solution. In the context of the Foreign monetary policy rule (1) in the paper, this corresponds to $\psi > 0$. However, we have set $\psi = 0$ in the parameterization and in the solution to the equilibrium above.

The assumption $\psi = 0$ raises the issue of price level (and therefore exchange rate) determinacy. It is well-known that an exogenous monetary policy rule can lead to price level indeterminacy. This is not always the case though. The price level is well-determined even under an exogenous (or passive) monetary policy rule when combined with an active fiscal policy rule (see contributions by Leeper, Journal of Monetary Economics, 1991, Sims, Economic Theory, 1994 or Woodford, Economic Theory 1994). In that case price level determinacy is generated by the government budget constraint (fiscal theory of the price level).

In our paper fiscal policy is passive (adjusts to satisfy the budget constraint), so the fiscal theory of the price level does not hold. But the model presents another case where passive monetary policy does not lead to price level indeterminacy. This mechanism is associated with valuation effects and their effect on risk-premia. Intuitively, a depreciation of the Home currency leads to an increase in the relative supply of Foreign bonds due to a standard valuation effect. This raises the risk premium on Foreign bonds, which in turn leads to an appreciation of the Home currency. The exchange rate is solved from the resulting fixed point problem. In order to show this algebraically, we will focus on the case where $T = 1$ and all investors make a portfolio decision each period. We abstract both from gradual portfolio adjustments and from liquidity traders as these are not central to the argument. We go back to the general monetary policy rule, where ψ can be either zero or positive.

The linearized version of the Foreign bond market clearing condition in this case becomes simply

$$b_t = b + bs_t \tag{106}$$

The left hand side is the fraction invested in Foreign bonds. The right hand side is the linearized supply of Foreign bonds. Here b is the steady state relative supply of

Foreign bonds and bs_t represents changes in the relative supply of Foreign bonds due to exchange rate changes (valuation effects).

The optimal portfolio share depends on the expected excess return, which we can write as

$$b_t = \bar{b} + \lambda(E_t s_{t+1} - s_t - fd_t) \tag{107}$$

where $\bar{b}$ is a constant, λ depends on the rate of risk-aversion and the variance of the excess return and fd_t is the forward discount, given by (setting $\bar{r} = 0$):

$$fd_t = i_t - i_t^* = u_t + \psi s_t \tag{108}$$

Substituting (107) into (75), we have a standard difference equation:

$$s_t = \frac{\lambda}{\lambda(1+\psi)+b} E_t s_{t+1} - \frac{\lambda}{\lambda(1+\psi)+b} u_t - \tilde{b} \tag{109}$$

with $\tilde{b}$ a constant that depends on b and $\bar{b}$. Ruling out bubbles, the above difference equation has a well defined, and unique, solution when either $b > 0$ (positive bond supply) or $\psi > 0$. The solution is

$$s_t = -\frac{\lambda}{\lambda(1+\psi)+b} \sum_{k=0}^{\infty} \left(\frac{\lambda}{\lambda(1+\psi)+b} \right)^k E_t u_{t+k} - \frac{\lambda(1+\psi)+b}{b+\psi\lambda} \tilde{b} \tag{110}$$

Since u_t follows an AR process, the solution can be written as:

$$s_t = -\frac{\lambda}{\lambda(1+\psi-\rho)+b} fd_t - \frac{\lambda}{\lambda(1+\psi)+b} \tag{111}$$

The main point is that $\psi > 0$ is not a necessary condition for determinacy. Even when ψ is zero, there is still a unique equilibrium as long as $b > 0$. A positive level of bonds is necessary for valuation effects to be active, which is key in the intuition described above.

The solution above implicitly assumes a stochastic exchange rate equilibrium, so that there is uncertainty about the future exchange rate. Otherwise λ is not well-defined (it depends on the inverse of the variance of the excess return). To complete the analysis, we need to consider a possible "no-risk" solution. In the absence of exchange rate risk we have

$$s_{t+1} - s_t - fd_t = 0 \tag{112}$$

Under the Wicksellian policy rule, where $\psi > 0$, this implies

$$s_t = \frac{1}{1+\psi} s_{t+1} - \frac{1}{1+\psi} u_t \tag{113}$$

Adding expectations on both sides and ruling out bubbles, the solution is

$$s_t = -\frac{1}{1+\psi} \sum_{k=0}^{\infty} \left(\frac{1}{1+\psi} \right)^k E_t u_{t+k} = -\frac{1}{1+\psi-\rho} u_t \tag{114}$$

Clearly, there is uncertainty about s_{t+1} in this case, which is inconsistent with a "no-risk" equilibrium. So the only equilibrium is (111).

This is the case as long as $\psi > 0$, even when ψ is infinitesimal. When ψ is exactly equal to zero, the discount rate in (113) is exactly 1, and the bizarre solution

$$s_t = a - u_{t-1} - u_{t-2} - ... \tag{115}$$

applies for any constant a. While one may rule out this "no-risk" solution on a-priori grounds (e.g. hard to implement such a solution as one needs to pre-commit to the exchange rate next period, without regard to monetary shocks at that time), the advantage of $\psi > 0$ (even when infinitesimal) is that this type of "no-risk" equilibria is ruled out entirely. In practice though, it makes little difference to set $\psi = 0$ as we focus on the unique stochastic equilibrium in that case. The only difference when setting ψ positive but infinitesimally small is that the strange no-risk equilibrium described above is ruled out.

1.6 Computation of the Threshold Cost

In this section we compute the threshold cost such that investors break even between making frequent or infrequent portfolio decisions. Consider an investor born at time t. If the investor makes frequent portfolio decisions the time t value function is

$$V_t = e^{Y_t' H_0 Y_t} (1-\tau)^{(1-\gamma)T} / (1-\gamma) \tag{116}$$

The decision whether to make frequent portfolio decisions or not is made when born before the state is observed. Therefore we need to compare the unconditional value functions at time t, before observing the state vector Y_t. We therefore need to compute the unconditional expectation of

$$e^{Y_t' H_0 Y_t} \tag{117}$$

Define

$$H_0 = \begin{pmatrix} \bar{H} & \bar{h}_1 \\ \bar{h}_2' & h \end{pmatrix} \tag{118}$$

where $\bar{H}$ is a $\bar{T}+1$ by $\bar{T}+1$ matrix, $\bar{h}_1$ and $\bar{h}_2$ are $\bar{T}+1$ by 1 vectors and h is a scalar. Also write

$$Y_t = \begin{pmatrix} \widehat{Y}_t \\ 1 \end{pmatrix} \tag{119}$$

Then

$$V_t = e^{h+2\widehat{Y}_t'\bar{h}+\widehat{Y}_t'\bar{H}\widehat{Y}_t}/(1-\gamma) \tag{120}$$

where

$$\bar{h} = \frac{\bar{h}_1 + \bar{h}_2}{2} \tag{121}$$

We have $\widehat{Y}_t \sim N(0, \Sigma_y)$, where $\Sigma_y(k,k) = \sigma_u^2$ for $k = 1,..,\bar{T}$, $\Sigma_y(\bar{T}+1, \bar{T}+1) = b_1^2\sigma_x^2/(1-\rho_x^2)$. The other elements of Σ_y are zero.

Therefore

$$\begin{aligned} &E_t e^{h+2\widehat{Y}_t'\bar{h}+\widehat{Y}_t'\bar{H}\widehat{Y}_t} = \\ &\frac{1}{|\Sigma_y|^{0.5}(2\pi)^{0.5\bar{T}+0.5}} \int e^{h+2\bar{h}'\widehat{Y}_t+\widehat{Y}_t'\bar{H}\widehat{Y}_t-0.5\widehat{Y}_t'\Sigma_y^{-1}\widehat{Y}_t} d\widehat{Y}_t = \\ &\frac{1}{|\Sigma_y|^{0.5}(2\pi)^{0.5\bar{T}+0.5}} \int e^{h+2\bar{h}'\widehat{Y}_t+\widehat{Y}_t'(\bar{H}-0.5\Sigma_y^{-1})\widehat{Y}_t} d\widehat{Y}_t = \frac{|\Omega_y|^{0.5}}{|\Sigma_y|^{0.5}} e^{h+2\bar{h}'\Omega_y\bar{h}} \end{aligned} \tag{122}$$

where

$$\Omega_y = (\Sigma_y^{-1} - 2\bar{H})^{-1} \tag{123}$$

The unconditional expectation of the value function for investors making infrequent portfolio decisions is computed in exactly the same way. If we write the expected value for investors making frequent portfolio decisions as

$$v^F(1-\tau)^{(1-\gamma)T} \tag{124}$$

and the value function for investors making infrequent portfolio decisions as v^I, then the threshold cost

$$\tau = 1 - (v_I/v_F)^{\frac{1}{(1-\gamma)T}} \tag{125}$$

1.7 Computation of Excess Return Predictability

In this section we discuss how to compute in the context of the model the coefficients β_k in the regression of the excess return q_{t+k} on the forward discount at time t. It is equal to

$$\frac{cov(q_{t+k}, fd_t)}{var(fd_t)} \tag{126}$$

We need to compute both the numerator and denominator. The forward discount is equal to

$$fd_t = u_t + \bar{r} \tag{127}$$

The numerator of (126) then becomes

$$cov(q_{t+k}, fd_t) = \sum_{v=0}^{\infty} \delta_u(v+k-1, 1)\rho^v \sigma_u^2 \tag{128}$$

The denominator of (126) is

$$var(fd_t) = \sum_{v=0}^{\infty} (\rho^v)^2 \sigma_u^2 \tag{129}$$

1.8 Expectations Conditioned on Limited Information

We now consider the case where agents only use the past M interest rates to form expectations about future exchange rates. Most of the solution procedure is the same as above. We therefore only note the changes.

First consider investors making frequent portfolio decisions. The relevant state space at $t+k$ is now

$$Y_{t+k} = \begin{pmatrix} u_{t+k} \\ \cdots \\ u_{t+k-M+1} \\ 1 \end{pmatrix} \tag{130}$$

We make the same guess for the value function as before, although now the matrix H_k in the exponential of the value function is a square matrix of size $M+1$.

Define the vector

$$\epsilon_{t+s+1} = \begin{pmatrix} \epsilon^u_{t+k+1} \\ q_{t+k+1} - E_{t+k} q_{t+k+1} \end{pmatrix} \tag{131}$$

Below we will derive

$$E_{t+k}q_{t+k+1} = \beta^1(1)u_{t+k} + .. + \beta^1(M)u_{t+k-M+1} \tag{132}$$

Below we will also derive the conditional variance $var_{t+k}(\epsilon_{t+k+1}(2))$. Clearly $var_{t+k}(\epsilon_{t+k+1}(1)) = \sigma_u^2$ and $cov_{t+k}(\epsilon_{t+k+1}(2), \epsilon_{t+k+1}(1)) = \lambda_u(1,1)\sigma_u^2$. These define the elements of the matrix $\Sigma = var_{t+k}(\epsilon_{t+k+1})$. The vector M_1^k now has $M+1$ elements with $M_1^k(v) = \beta_v$ for $v = 1, .., M$ and $M_1^k(M+1) = -\bar{r}$. The vector M_2^k has two elements with $M_2^k(1) = 0$ and $M_2^k(2) = 1$.

The matrix N_1^k is now $M+1$ by $M+1$. We have $N_1^k(1,1) = \rho$, $N_1^k(v, v-1) = 1$ for $v = 2, .., M$, $N_1^k(M+1, M+1) = 1$. All other elements are zero. N_2^k is a $M+1$ by 2 matrix. It has $N_2^k(1,1) = 1$ and all other elements zero.

The computation of the unconditional value function, used to compute the threshold cost, is also very similar when expectations are conditioned on limited information. In this case $\bar{H}$ is a M by M matrix, $\bar{h}_1$ and $\bar{h}_2$ are M by 1 vectors and h is a scalar. We have $\widehat{Y}_t = (u_t, .., u_{t-M+1})'$, so that

$$var(\widehat{Y}_t) = \begin{pmatrix} 1 & \rho & \rho^2 & ... & \rho^{M-1} \\ \rho & 1 & \rho & ... & \rho^{M-2} \\ ... & & & ... & ... \\ \rho^{M-1} & ... & & & 1 \end{pmatrix} \frac{\sigma_u^2}{1-\rho^2} \tag{133}$$

The rest is the same as before.

The expressions for the optimal portfolio and value function for investors making infrequent portfolio decisions are still the same, except that the first and second moments of the excess return are now different. In particular, we now have $E_t q_{t,t+T} = M_1 Y_t$, where M_1 is now a vector of length $M+1$ with $M_1(v) = \beta^T(v)$ for $v = 1, .., M$ and $M_1(M+1) = -\bar{r}T$. We can still write the value function for investors making infrequent portfolio decisions as

$$V_t = e^{A - \frac{B^2}{4C}}/(1-\gamma) \tag{134}$$

with

$$A - \frac{B^2}{4C} = G_1 + G_2 Y_t + Y_t' G_3 Y_t \tag{135}$$

The matrices G_1, G_2 and G_3 are the same as before. We again have

$$G_1 + G_2 Y_{t+k} + Y_{t+k}' G_3 Y_{t+k} = Y_{t+k}' G Y_{t+k} \tag{136}$$

where $G = G_3 + G_4 + G_5$. G_4 is now a square $M+1$ matrix with the vector G_2 in the last row and otherwise zeros and G_5 is a square $M+1$ matrix with the element $(M+1, M+1)$ equal to G_1 and all other elements zero. The computation of the unconditional value function proceeds as before.

The first and second moments of excess returns, needed to compute the equilibrium exchange rate, optimal portfolios and value function, are now different. First consider the expectation $E_t\widetilde{q}_{t,t+T}$. The expression (93) of $\widetilde{q}_{t,t+T}$ remains correct. If $M > 1$, the expectation of $\widetilde{q}_{t,t+T}$ can be computed by regressing $\widetilde{q}_{t,t+T}$ on $u_t, .., u_{t-M+1}$. We can write this regression as

$$\widetilde{q}_{t,t+T} = \widehat{Y}_t'\beta^T + \epsilon \tag{137}$$

The regression yields

$$\beta^T = (var(\widehat{Y}_t))^{-1}cov(\widehat{Y}_t, \widetilde{q}_{t,t+T}) \tag{138}$$

where $var(\widehat{Y}_t)$ is given by (133) and

$$cov(\widehat{Y}_t, \widetilde{q}_{t,t+T}) = \begin{pmatrix} \sum_{v=0}^{\infty} \delta_u(v,T)\rho^v \\ ... \\ \sum_{v=M-1}^{\infty} \delta_u(v,T)\rho^{v-M+1} \end{pmatrix} \sigma_u^2 \tag{139}$$

We then have

$$E_t\widetilde{q}_{t,t+T} = \sum_{v=1}^{\infty} z_u(v,T)\epsilon_{t-v}^u \tag{140}$$

where

$$z_u(v,T) = \sum_{k=1}^{v+1} \beta^T(k)\rho^{v+1-k} \quad v = 0, .., M-1$$
$$z_u(v,T) = \rho z_u(v-1,T) \quad \forall v > M-1$$

We now have

$$\sum_{k=1}^{T} E_{t-k+1}\widetilde{q}_{t-k+1,t-k+1+T} = \zeta_u^T(v)\epsilon_{t-v}^u \tag{141}$$

where

$$\zeta_u^T(v) = \sum_{k=0}^{min(v,T-1)} z_u(v-k,T) \tag{142}$$

The following change now needs to be made to (98)-(99). If investors making frequent portfolio decisions condition expectations on limited information we need to replace $\eta_u^1(v)$ with $\zeta_u^1(v)$ and $\eta_x^1(v)$ with 0. If investors making infrequent portfolio decisions condition expectations on limited information we need to replace $\eta_u^T(v)$ with $\zeta_u^T(v)$ and $\eta_x^T(v)$ with 0.

If investors making infrequent portfolio decisions condition expectations on limited information we also need to compute $var_t(\widetilde{q}_{t,t+T})$ and $var_t(\widetilde{q}_{t+k})$ $(k = 1,..,T)$. The former is

$$var_t(\widetilde{q}_{t,t+T}) = \sum_{v=1}^{T} \left(\lambda_u(v,T)^2\sigma_u^2 + \lambda_x(v,T)^2\sigma_x^2\right) + \sum_{v=0}^{\infty} \left((\delta_u(v,T) - z_u(v,T))^2\sigma_u^2\right) + \frac{(\rho_x^T - 1)^2 b_1^2 \sigma_x^2}{1-\rho_x^2} \quad (143)$$

$var_t(\widetilde{q}_{t+k})$ can be computed similarly as the conditional variance of $\widetilde{q}_{t,t+T}$. Following the same procedure as above with $\widetilde{q}_{t+k}$ on the left-hand side we get

$$E_t\widetilde{q}_{t+k} = \sum_{v=1}^{\infty} \bar{z}_u(v,k)\epsilon_{t-v}^u \quad (144)$$

Using (89) it follows that analogous to $\widetilde{q}_{t,t+T}$ we have

$$\widetilde{q}_{t+k} = \sum_{v=1}^{k} \left(\bar{\lambda}_u(v,k)\epsilon_{t+v}^u + \bar{\lambda}_x(v,k)\epsilon_{t+v}^x\right) + \sum_{v=0}^{\infty} \left(\bar{\delta}_u(v,k)\epsilon_{t-v}^u + \bar{\delta}_x(v,k)\epsilon_{t-v}^x\right) \quad (145)$$

where

$$\begin{aligned} \bar{\lambda}_u(k,k) &= a_1 \\ \bar{\lambda}_u(v,k) &= a_{k+1-v} - a_{k-v} - \rho^{k-v-1} \quad v = 1,..,k-1 \\ \bar{\delta}_u(v,k) &= a_{k+1+v} - a_{k+v} - \rho^{k-1+v} \quad v \geq 0 \end{aligned}$$

and

$$\begin{aligned} \bar{\lambda}_x(k,k) &= b_1 \\ \bar{\lambda}_x(v,k) &= b_{k+1-v} - b_{k-v} \quad v = 1,..,k-1 \\ \bar{\delta}_x(v,k) &= b_{k+1+v} - b_{k+v} \end{aligned}$$

It therefore follows that

$$var_t(\widetilde{q}_{t+k}) = \sum_{v=1}^{k} \left(\bar{\lambda}_u(v,k)^2\sigma_u^2 + \bar{\lambda}_x(v,k)^2\sigma_x^2\right) + \sum_{v=0}^{\infty} \left((\bar{\delta}_u(v,k) - \bar{z}_u(v,k))^2\sigma_u^2\right) + \frac{(\rho_x^k - \rho_x^{k-1})^2 b_1^2 \sigma_x^2}{1-\rho_x^2} \quad (146)$$

Finally, it should be pointed out that when investors making frequent portfolio decisions condition expectations on limited information the matrix D in the hedge term is now of length $M+1$ and $D(M+1)=0$. Therefore in equations (103)-(105) we now have that the elements $D(k)$ for k larger than M are zeros.

We finally need to discuss the case where $M=0$, so that no information is processed to compute the expected change of the exchange rate. The exchange rate is assumed to be a random walk. We only consider this when all agents make infrequent portfolio decisions. We assume that there is knowledge of the current interest rate and projected future interest rates. In that case

$$E_t\tilde{q}_{t,t+T} = -\sum_{v=0}^{\infty}\rho(0,T)\rho^v\epsilon^u_{t-v} \tag{147}$$

Therefore $z_u(v,T) = -\rho(0,T)\rho^v$. Everything else follows as before. In this case we also have

$$E_t\tilde{q}_{t+k} = -\sum_{v=0}^{\infty}\rho^{k-1+v}\epsilon^u_{t-v} \tag{148}$$

so that $\bar{z}_u(v,k) = -\rho^{k-1+v}$.

1.9 An Additional Asset

Now consider adding to the model an additional asset whose return is uncorrelated with the excess return on foreign investments. In fact, it is uncorrelated with both of the other shocks in the model. First consider an investor making frequent portfolio decisions born at t. The portfolio return from $t+k-1$ to $t+k$ now becomes

$$R^p_{t+k} = e^{i_{t+k-1}} + b^F_{t,t+k}(e^{s_{t+1}-s_{t+k-1}+i^*_{t+k-1}} - e^{i_{t+k-1}}) + d^F_{t,t+k}(e^{\Delta_{t+k}} - e^{i_{t+k-1}}) \tag{149}$$

where $d^F_{t,t+k}$ is the fraction invested in the third asset and Δ_{t+k} is its return. It is uncorrelated over time and distributed $N(\bar{\Delta},\sigma^2_\Delta)$. The second order approximation of the log return then becomes

$$r^p_{t+k+1} = \bar{r} + b^F_{t,t+k}q_{t+k+1} + 0.5b^F_{t,t+k}(1-b^F_{t,t+k})\sigma^2_F + d^F_{t,t+k}(\Delta_{t+k}-\bar{r}) + 0.5d^F_{t,t+k}(1-d^F_{t,t+k})\sigma^2_\Delta \tag{150}$$

We now have

$$v_{t+k+1} = Y'_{t+k+1}H_{k+1}Y_{t+k+1} + (1-\gamma)r^p_{t+k+1} = \tag{151}$$

$$(1-\gamma)\bar{r} + (1-\gamma)b^F_{t,t+k}q_{t+k+1} + (1-\gamma)0.5b^F_{t,t+k}(1-b^F_{t,t+k})\sigma^2_F +$$
$$(1-\gamma)d^F_{t,t+k}(\Delta_{t+k} - \bar{r}) + (1-\gamma)0.5d^F_{t,t+k}(1-d^F_{t,t+k})\sigma^2_\Delta +$$
$$Y'_{t+k+1}H_{k+1}Y_{t+k+1} \tag{152}$$

Substituting $q_{t+k+1} = M^k_1 Y_{t+k} + M^k_2 \epsilon_{t+k+1}$ and $Y_{t+k+1} = N^k_1 Y_{t+k} + N^k_2 \epsilon_{t+k+1}$, this becomes

$$v_{t+k+1} = (1-\gamma)\bar{r} + (1-\gamma)b^F_{t,t+k}M^k_1 Y_{t+k} + (1-\gamma)0.5b^F_{t,t+k}(1-b^F_{t,t+k})\sigma^2_F +$$
$$Y'_{t+k}(N^k_1)'H_{k+1}N^k_1 Y_{t+k} + C^k_1\epsilon_{t+k+1} + \epsilon'_{t+k+1}C^k_2\epsilon_{t+k+1}$$
$$+(1-\gamma)d^F_{t,t+k}(\Delta_{t+k} - \bar{r}) + (1-\gamma)0.5d^F_{t,t+k}(1-d^F_{t,t+k})\sigma^2_\Delta \tag{153}$$

It follows that

$$E_{t+k}e^{v_{t+k+1}} = e^{\hat{v}_{t+k}} \tag{154}$$

where

$$\hat{v}_{t+k} = (1-\gamma)\bar{r} + (1-\gamma)b^F_{t,t+k}M^k_1 Y_{t+k} + 0.5(1-\gamma)b^F_{t,t+k}(1-b^F_{t,t+k})\sigma^2_F +$$
$$Y'_{t+k}(N^k_1)'H_{k+1}N^k_1 Y_{t+k} + 0.5ln(|\Omega^k|/|\Sigma|) + 0.5C^k_1\Omega^k(C^k_1)' +$$
$$(1-\gamma)d^F_{t,t+k}(\bar{\Delta} - \bar{r} + 0.5\sigma^2_\Delta) - 0.5\gamma(1-\gamma)(d^F_{t,t+k})^2\sigma^2_\Delta \tag{155}$$

This can be written as

$$\hat{v}_{t+k} = A^k + B^k b^F_{t,t+k} + C^k(b^F_{t,t+k})^2 + B^k_\Delta d^F_{t,t+k} + C^k_\Delta(d^F_{t,t+k})^2 \tag{156}$$

where

$$B^k_\Delta = (1-\gamma)(\bar{\Delta} - \bar{r} + 0.5\sigma^2_\Delta) \tag{157}$$
$$C^k_\Delta = -0.5\gamma(1-\gamma)\sigma^2_\Delta \tag{158}$$

At $t+k$ the investor chooses the portfolio to maximize the expected $t+k+1$ value function, which is equivalent to maximizing $\hat{v}_{t+k}$. The portfolio for $b^F_{t,t+k}$ is the same as before. For the third asset we get

$$d^F_{t,t+k} = \frac{-B^k_\Delta}{2C^k_\Delta} = \frac{1}{2\gamma} + \frac{\bar{\Delta} - \bar{r}}{\gamma\sigma^2_\Delta} \tag{159}$$

Substituting the optimal portfolios in $\hat{v}_{t+k}$ we have

$$\hat{v}_{t+k} = A^k - \frac{(B^k)^2}{4C^k} - \frac{(B^k_\Delta)^2}{4C^k_\Delta} = G^1_k + G^2_k Y_{t+k} + Y'_{t+k}G^3_k Y_{t+k} \tag{160}$$

with the only change from before that to the previous expression for G_k^1 we need to add

$$0.5\frac{1-\gamma}{\gamma}\frac{(\bar{\Delta}-\bar{r}+0.5\sigma_{\Delta}^2)^2}{\sigma_{\Delta}^2} \tag{161}$$

Next consider investors making infrequent portfolio decisions. For investors born at time t the portfolio return from $t+k-1$ to $t+k$ is

$$R_{t+k}^p = e^{i_{t+k-1}} + b_t^I(e^{s_{t+1}-s_{t+k-1}+i_{t+k-1}^*} - e^{i_{t+k-1}}) + d_t^I(e^{\Delta_{t+k}} - e^{i_{t+k-1}}) \tag{162}$$

Log-linearization gives

$$r_{t+k+1}^p = \bar{r} + b_t^I q_{t+k+1} + 0.5b_t^I(1-b_t^I)\sigma_F^2 + d_t^I(\Delta_{t+k}-\bar{r}) + 0.5d_t^I(1-d_t^I)\sigma_{\Delta}^2 \tag{163}$$

The value function when born is

$$V_t = E_t e^{(1-\gamma)(r_{t+1}^p+..+r_{t+T}^p)}/(1-\gamma) = e^{\hat{v}}/(1-\gamma) \tag{164}$$

where

$$\hat{v} = A + Bb_t^I + C(b_t^I)^2 + B_{\Delta}d_t^I + C_{\Delta}(d_t^I)^2 \tag{165}$$

Here A, B and C are as before and

$$B_{\Delta} = (1-\gamma)T(\bar{\Delta}-\bar{r}+0.5\sigma_{\Delta}^2) \tag{166}$$

$$C_{\Delta} = -0.5\gamma(1-\gamma)T\sigma_{\Delta}^2 \tag{167}$$

The optimal portfolio b_t^I is as before. For the third asset the optimal portfolio is

$$d_t^I = \frac{-B_{\Delta}}{2C_{\Delta}} = \frac{1}{2\gamma} + \frac{\bar{\Delta}-\bar{r}}{\gamma\sigma_{\Delta}^2} \tag{168}$$

Substituting back in the value function we have

$$\hat{v} = G_1 + G_2Y_t + Y_t'G_3Y_t \tag{169}$$

with the only change from before that to the previous expression for G^1 we need to add

$$0.5T\frac{1-\gamma}{\gamma}\frac{(\bar{\Delta}-\bar{r}+0.5\sigma_{\Delta}^2)^2}{\sigma_{\Delta}^2} \tag{170}$$

Next consider the market equilibrium conditions. To simplify matters assume that investors making infrequent portfolio decisions, and that have already previously made a portfolio decision, reinvest an increase in wealth due to bond returns in the bond market and reinvest returns in the third market into that market. This simplifying assumption implies that the return on Foreign bonds does not affect the foreign exchange return through a portfolio rebalancing effect. This way the FX return remains uncorrelated with the return on the third asset. Portfolio rebalancing only applies within the bond market, between Home and Foreign bonds. In that case nothing changes for the Foreign bonds market. We need to keep in mind though that $b = B/\bar{W}$ is now smaller. Previously is was set at 0.5. If Home and Foreign bonds are in equal supply and the steady state fraction of the third asset in the total supply is d, then $b = 0.5(1-d)$.

Regarding the third asset, note that the optimal portfolio is not time-varying and is the same for investors making frequent and infrequent portfolio decisions. Equating steady state demand and supply for the third asset gives

$$\frac{1}{2\gamma} + \frac{\bar{\Delta} - \bar{r}}{\gamma \sigma_{\Delta}^2} = d \tag{171}$$

If we pick d, then this implies a value for $\bar{\Delta}$. Since we assumed that returns from this third market are reinvested in that market, in deviation from the steady state demand will change with returns in this market. Simply assume that supply changes accordingly, so that there is no need for time variation in the expected return in this third market, which unnecessarily complicates matters.

We can summarize that there are two changes. First, $b = 0.5(1-d)$. Second, to the terms G_1^k for frequent traders and G_1 for infrequent trades we need to add a constant. This constant term can be simplified as follows. For the frequent traders it is

$$-\frac{(B_{\Delta}^k)^2}{4C_{\Delta}^k} = -\left(\frac{-B_{\Delta}^k}{2C_{\Delta}^k}\right)^2 C_{\Delta}^k = d^2\gamma(1-\gamma)\sigma_{\Delta}^2 \tag{172}$$

For the infrequent traders it is

$$-\frac{(B_{\Delta})^2}{4C_{\Delta}} = -\left(\frac{-B_{\Delta}}{2C_{\Delta}}\right)^2 C_{\Delta} = d^2 T\gamma(1-\gamma)\sigma_{\Delta}^2 \tag{173}$$

This implies that both for frequent and infrequent traders the value function when born is multiplied by

$$e^{d^2 T\gamma(1-\gamma)\sigma_{\Delta}^2}$$

This therefore has no effect on the threshold cost τ. In order words, the only effect on predictability and the threshold cost comes through a drop in the steady state relative supply of the Foreign bond to $b = 0.5(1-d)$.

2 Multiple Currencies

2.1 The setup

There are now $N > 2$ currencies. For the Home country the nominal interest rate still equal to $\bar{r}$ and its price level is constant. For the other $N-1$ countries the interest rate follows an AR process. For country n the nominal interest rate is $i_t^n = -u_t^n$ where

$$u_t^n = \rho u_{t-1}^n + \epsilon_t^n \tag{174}$$

The innovations have a $N(0, \sigma_u^2)$ distribution, are uncorrelated over time and have a joint contemporaneous correlation matrix of Σ_u. The latter is a $N-1$ by $N-1$ matrix. We will assume perfect symmetry for the $N-1$ currencies (other than the Home currency, which is currency N). This has the advantage that we can simply talk about *the* predictability coefficient rather than $N-1$ predictability coefficients. It also has the advantage that there are far fewer parameters to solve for in the equilibrium exchange rate. Adopting this symmetry implies assuming that all the off-diagonal elements of Σ_u are the same. We will adopt similar symmetry of the liquidity shocks, which we will get to later. We also assume an equal steady state bond supply of the $N-1$ Foreign bonds.

The portfolio return from $t+k$ to $t+k+1$ of investors born at time t making infrequent portfolio decisions is

$$R_{t+k+1}^p = \left(1 - \sum_{n=1}^{N-1} b_t^{n,I}\right) e^{\bar{r}} + \sum_{n=1}^{N-1} b_t^{n,I} e^{s_{t+k+1}^n - s_{t+k}^n + i_{t+k}^n} \tag{175}$$

Here $b_t^{n,I}$ is the fraction invested in Foreign bond n and s_t^n is the exchange rate of currency n relative to the Home currency (units Home currency per unit of currency n).

The portfolio return of investors born at time t making frequent portfolio decisions is

$$R_{t+k+1}^p = \left(1 - \sum_{n=1}^{N-1} b_{t,t+k}^{n,F}\right) e^{\bar{r}} + \sum_{n=1}^{N-1} b_{t,t+k}^{n,F} e^{s_{t+k+1}^n - s_{t+k}^n + i_{t+k}^n} \tag{176}$$

where $b_{t,t+k}^{n,F}$ is the fraction investors born at time t invest in Foreign bond n at time $t+k$.

2.2 Portfolio Choice

In this section we will derive the optimal portfolios and value functions for both investors that make frequent and infrequent portfolio decisions.

2.2.1 Investors with Actively Managed Portfolios

Consider an agent born at time t. We will compute the optimal portfolio and value function at $t+k$ for $k=0,..,T-1$. The state of the world consists of all shocks to interest rates and liquidity trade that have happened up to that point in time. However, we will truncate the state space to $\bar{T}$ periods before the present date. This only applies to interest rate shocks. For liquidity demand we can summarize the state by $\hat{x}=\left(\hat{x}^1,..,\hat{x}^{N-1}\right)'$, where $\hat{x}^n$ captures the impact of liquidity trade on the exchange rate of currency n and follows an AR process with AR coefficient ρ_x. Also write $\epsilon^u_{t+k}=\left(\epsilon^1_{t+k},..,\epsilon^{N-1}_{t+k}\right)'$. We can then write the state space at $t+k$ as

$$Y_{t+k}=\begin{pmatrix}\epsilon^u_{t+k}\\ \cdots\\ \epsilon^u_{t+k-\bar{T}+1}\\ \hat{x}_{t+k}\\ 1\end{pmatrix} \tag{177}$$

We know from the Bellman equation that the value function at $t+k$ is equal to the expected value function at $t+k+1$:

$$V_{t+k}=E_{t+k}V_{t+k+1} \tag{178}$$

and that the terminal value function at $t+T$ is

$$V_{t+T}=W^{1-\gamma}_{t+T}/(1-\gamma) \tag{179}$$

We can solve the value function with backward induction. Make the following guess:

$$V_{t+k}=e^{Y'_{t+k}H_kY_{t+k}}(1-\tau)^{(1-\gamma)(T-k)}W^{1-\gamma}_{t+k}/(1-\gamma) \tag{180}$$

where H_k is a square matrix of size $(\bar{T}+1)(N-1)+1$. We show that this is correct by assuming that the value function takes this form for $t+k+1$ and then deriving the time $t+k$ value function from the Bellman equation. This also gives us an updating formula relating H_k to H_{k+1}. Clearly H_T is a matrix with only zeros.

We know that

$$W_{t+k+1} = (1-\tau)W_{t+k}e^{r^p_{t+k+1}} \tag{181}$$

Write the vector of portfolio choices as $b^F_{t,t+k} = \left(b^{1,F}_{t,t+k}, .., b^{N-1,F}_{t,t+k}\right)'$. Also write the conditional variance-covariance matrix of excess returns $q_{t+k+1} = \left(q^1_{t+k+1}, .., q^{N-1}_{t+k+1}\right)'$ as Σ_q and the vector of conditional variances as $v\bar{a}r(q)$, which contains the diagonal elements of Σ_q. We again adopt a continuous time approximation of the log portfolio return (based on Ito's Lemma):

$$r^p_{t+k+1} = \bar{r} + (b^F_{t,t+k})'(q_{t+k+1} + 0.5v\bar{a}r(q)) - 0.5(b^F_{t,t+k})'\Sigma_q b^F_{t,t+k} \tag{182}$$

After substituting (181) and (182) into the Bellman equation we have

$$E_{t+k}e^{v_{t+k+1}} = e^{Y'_{t+k}H_k Y_{t+k}} \tag{183}$$

where

$$\begin{aligned} v_{t+k+1} &= (1-\gamma)\bar{r} + (1-\gamma)(b^F_{t,t+k})'(q_{t+k+1} + 0.5v\bar{a}r(q)) \\ &-0.5(1-\gamma)(b^F_{t,t+k})'\Sigma_q b^F_{t,t+k} + Y'_{t+k+1}H_{k+1}Y_{t+k+1} \end{aligned} \tag{184}$$

We will now show that there is a matrix G_k such that

$$E_{t+k}e^{v_{t+k+1}} = e^{Y'_{t+k}G_k Y_{t+k}} \tag{185}$$

The Bellman equation then implies $H_k = G_k$.

We can write

$$q_{t+k+1} = M^k_1 Y_{t+k} + M^k_2 \epsilon_{t+k+1} \tag{186}$$

where

$$\epsilon_{t+k+1} = \begin{pmatrix} \epsilon^u_{t+k+1} \\ \epsilon^x_{t+k+1} \end{pmatrix} \tag{187}$$

We will derive below that

$$q_{t+k+1} = \lambda_u(1,1)\epsilon^u_{t+k+1} + b_1\epsilon^x_{t+k+1} + \sum_{v=0}^{\infty}\delta_u(v,1)\epsilon^u_{t+k-v} + (\rho_x - 1)\hat{x}_{t+k} - \bar{r} \tag{188}$$

Therefore $M^k_1(1:N-1, 1+v(N-1):(v+1)(N-1)) = \delta_u(v,1)$ for $v = 0, .., \bar{T}-1$, $M^k_1(1:N-1,(N-1)\bar{T}+1:(N-1)(\bar{T}+1)) = (\rho_x - 1)I_{N-1,N-1}$, $M^k_1(1:N-1,(N-1)(\bar{T}+1)+1) = -\bar{r}\iota$, $M^k_2(1:N-1, 1:N-1) = \lambda_u(1,1)$ and $M^k_2(1:N-1, N:2N-2) = b_1$. ι is a vector of ones of length $N-1$.

We can also write

$$Y_{t+k+1} = N_1^k Y_{t+k} + N_2^k \epsilon_{t+k+1} \tag{189}$$

The matrix N_1^k is $(\bar{T}+1)(N-1)+1$ by $(\bar{T}+1)(N-1)+1$. It has zeros in the first $N-1$ rows. We have $N_1^k(v(N-1)+1:(v+1)(N-1),(v-1)(N-1)+1:v(N-1)) = I_{N-1,N-1}$ for $v=1,..,\bar{T}-1$, $N_1^k(\bar{T}(N-1)+1:(\bar{T}+1)(N-1),\bar{T}(N-1)+1:(\bar{T}+1)(N-1)) = \rho_x I_{N-1,N-1}$ and $N_1^k((\bar{T}+1)(N-1)+1,(\bar{T}+1)(N-1)+1) = 1$. The matrix N_2^k is $(\bar{T}+1)(N-1)+1$ by $2(N-1)$. We have $N_2^k(1:N-1,1:N-1) = I_{N-1,N-1}$, $N_2^k(\bar{T}(N-1)+1:(\bar{T}+1)(N-1),N:2N-2) = b_1$ and zeros elsewhere.

It follows that

$$\begin{aligned}
&v_{t+k+1} = (1-\gamma)\bar{r} + (1-\gamma)(b^F_{t,t+k})' M_1^k Y_{t+k} + 0.5(1-\gamma)(b^F_{t,t+k})' v\bar{a}r(q) - \\
&0.5(1-\gamma)(b^F_{t,t+k})'\Sigma_q b^F_{t,t+k} + Y'_{t+k}(N_1^k)' H_{k+1} N_1^k Y_{t+k} + \\
&C_1^k \epsilon_{t+k+1} + \epsilon'_{t+k+1} C_2^k \epsilon_{t+k+1}
\end{aligned} \tag{190}$$

where

$$C_1^k = (1-\gamma)(b^F_{t,t+k})' M_2^k + 2Y'_{t+k}(N_1^k)' H_{k+1} N_2^k \tag{191}$$

$$C_2^k = (N_2^k)' H_{k+1} N_2^k \tag{192}$$

We know that $\epsilon_{t+k+1} \sim N(0,\Sigma)$ where $\Sigma[1:N-1,1:N-1] = \Sigma_u$, $\Sigma[N:2N-2,N:2N-2] = \Sigma_x$ and other elements zero. Here Σ_x is the variance-covariance matrix of the $N-1$ liquidity shock innovations. Therefore

$$\begin{aligned}
&E_{t+k} e^{C_1^k \epsilon_{t+k+1} + \epsilon'_{t+k+1} C_2^k \epsilon_{t+k+1}} = \\
&\frac{1}{|\Sigma|^{0.5} 2\pi} \int e^{C_1^k \epsilon_{t+k+1} + \epsilon'_{t+k+1} C_2^k \epsilon_{t+k+1} - 0.5\epsilon'_{t+k+1}\Sigma^{-1}\epsilon_{t+k+1}} d\epsilon_{t+k+1} = \\
&\frac{1}{|\Sigma|^{0.5} 2\pi} \int e^{C_1^k \epsilon_{t+k+1} + \epsilon'_{t+k+1}(C_2^k - 0.5\Sigma^{-1})\epsilon_{t+k+1}} d\epsilon_{t+k+1} = \frac{|\Omega^k|^{0.5}}{|\Sigma|^{0.5}} e^{0.5 C_1^k \Omega^k (C_1^k)'}
\end{aligned} \tag{193}$$

Here the last equality uses the moment generating function of the normal distribution and

$$\Omega^k = (\Sigma^{-1} - 2C_2^k)^{-1} \tag{194}$$

It follows that

$$E_{t+k} e^{v_{t+k+1}} = e^{\hat{v}_{t+k}} \tag{195}$$

where

$$\hat{v}_{t+k} = (1-\gamma)\bar{r} + (1-\gamma)(b^F_{t,t+k})' M_1^k Y_{t+k} + 0.5(1-\gamma)(b^F_{t,t+k})' v\bar{a}r(q)$$
$$-0.5(1-\gamma)(b^F_{t,t+k})'\Sigma_q b^F_{t,t+k} + Y'_{t+k}(N_1^k)' H_{k+1} N_1^k Y_{t+k} +$$
$$0.5ln(|\Omega^k|/|\Sigma|) + 0.5 C_1^k \Omega^k (C_1^k)' \tag{196}$$

¿From the definition of C_1^k we have

$$C_1^k\Omega^k(C_1^k)' = (1-\gamma)^2(b^F_{t,t+k})'\hat{\sigma}^2_F(k) b^F_{t,t+k} + 2(1-\gamma)(b^F_{t,t+k})' D_1^k Y_{t+k} + 2Y'_{t+k} D_2^k Y_{t+k} \tag{197}$$

where

$$\hat{\sigma}^2_F(k) = M_2^k \Omega^k (M_2^k)' \tag{198}$$
$$D_1^k = 2M_2^k\Omega^k(N_2^k)' H_{k+1} N_1^k \tag{199}$$
$$D_2^k = 2(N_1^k)' H_{k+1} N_2^k \Omega^k (N_2^k)' H_{k+1} N_1^k \tag{200}$$

Therefore

$$\hat{v}_{t+k} = A^k + (b^F_{t,t+k})' B^k + (b^F_{t,t+k})' C^k b^F_{t,t+k} \tag{201}$$

where

$$A^k = (1-\gamma)\bar{r} + Y'_{t+k}((N_1^k)' H_{k+1} N_1^k + D_2^k) Y_{t+k} + 0.5ln(|\Omega^k|/|\Sigma|) \tag{202}$$
$$B^k = (1-\gamma)(M_1^k + D_1^k) Y_{t+k} + (1-\gamma)0.5 v\bar{a}r(q) \tag{203}$$
$$C^k = -0.5(1-\gamma)\Sigma_q + 0.5(1-\gamma)^2 \hat{\sigma}^2_F(k) \tag{204}$$

At $t+k$ the investor chooses the portfolio to maximize the expected $t+k+1$ value function, which is equivalent to maximizing $\hat{v}_{t+k}$. This yields

$$b^F_{t,t+k} = -0.5(C^k)^{-1} B^k = (\bar{C}^k)^{-1}(M_1^k + D_1^k) Y_{t+k} + 0.5(\bar{C}^k)^{-1} v\bar{a}r(q) \tag{205}$$

where $\bar{C}^k = \Sigma_q + (\gamma-1)\hat{\sigma}^2_F(k)$. Note that $M_1 Y_{t+k} = E_{t+k} q_{t+k+1}$. The hedge term is proportional to $D_1^k Y_{t+k}$. With $\tilde{q} = q + \bar{r}\iota$ the excess returns in deviation from steady state, the optimal portfolio can then be written as

$$b^F_{t,t+k} = \bar{b}^F(k) + (\bar{C}^k)^{-1} E_{t+k}(\tilde{q}_{t+k+1}) + (\bar{C}^k)^{-1} D_1^k Y_{t+k} \tag{206}$$

where

$$\bar{b}^F(k) = (\bar{C}^k)^{-1}(0.5 v\bar{a}r(q) - \bar{r}\iota) \tag{207}$$

Substituting the optimal portfolio in $\hat{v}_{t+k}$ we have

$$\hat{v}_{t+k} = A^k - (1/4)(B^k)' \left((C^k)^{-1}\right) B_k = G_k^1 + G_k^2 Y_{t+k} + Y_{t+k}' G_k^3 Y_{t+k} \tag{208}$$

where

$$G_k^1 = (1-\gamma)\bar{r} + 0.5ln(|\Omega^k|/|\Sigma|) + \frac{1}{8}(1-\gamma)v\bar{a}r(q)'(\bar{C}^k)^{-1}v\bar{a}r(q) \tag{209}$$

$$G_k^2 = 0.5(1-\gamma)v\bar{a}r(q)'(\bar{C}^k)^{-1}(M_1^k + D_1^k) \tag{210}$$

$$G_k^3 = (N_1^k)'H_{k+1}N_1^k + D_2^k + 0.5(1-\gamma)(M_1^k + D_1^k)'(\bar{C}^k)^{-1}(M_1^k + D_1^k) \tag{211}$$

We can write

$$G_k^1 + G_k^2 Y_{t+k} + Y_{t+k}' G_k^3 Y_{t+k} = Y_{t+k}' G_k Y_{t+k} \tag{212}$$

where $G_k = G_k^3 + G_k^4 + G_k^5$, G_k^4 is a square $(\bar{T}+1)(N-1)+1$ matrix with the vector G_k^2 in the last row and otherwise zeros and G_k^5 is a square $(\bar{T}+1)(N-1)+1$ matrix with the element $((\bar{T}+1)(N-1)+1, (\bar{T}+1)(N-1)+1)$ equal to G_k^1 and all other elements zero. It follows that $H_k = G_k$.

2.2.2 Investors Making Infrequent Portfolio Decisions

Now consider an investor born at time t making one portfolio decision for the next T periods. The time t value function is

$$V_t = E_t e^{(1-\gamma)(r_{t+1}^p + .. + r_{t+T}^p)}/(1-\gamma) \tag{213}$$

We again adopt the continuous time approximation for the log return:

$$r_{t+k}^p = \bar{r} + (b_t^I)'(q_{t+k+1} + 0.5v\bar{a}r_k(q)) - 0.5(b_t^I)'var_t(q_{t+k})b_t^I \tag{214}$$

where $v\bar{a}r_k(q)$ is a vector that contains conditional variances at time t of each of the $N-1$ excess returns at $t+k$. We then have

$$V_t = e^{A + (b_t^I)'B + (b_t^I)'Cb_t^I}/(1-\gamma) \tag{215}$$

where

$$A = (1-\gamma)\bar{r}T \tag{216}$$

$$B = (1-\gamma)E_t q_{t,t+T} + 0.5(1-\gamma)\sum_{k=1}^{T} v\bar{a}r_k(q) \tag{217}$$

$$\begin{aligned} C &= 0.5(1-\gamma)^2 var_t(q_{t,t+T}) - 0.5(1-\gamma)\sum_{k=1}^{T} var_t(q_{t+k}) \\ &= -0.5\gamma(1-\gamma)\sigma_I^2 \end{aligned} \tag{218}$$

where

$$\sigma_I^2 = \left(1 - \frac{1}{\gamma}\right) var_t(q_{t,t+T}) + \frac{1}{\gamma}\sum_{k=1}^{T} var_t(q_{t+k}) \tag{219}$$

The optimal portfolio is

$$b_t^I = -0.5C^{-1}B = b^I + (\sigma_I^2)^{-1}\frac{E_t q_{t,t+T}}{\gamma} \tag{220}$$

where

$$b^I = (\sigma_I^2)^{-1}\frac{0.5\sum_{k=1}^{T} v\bar{a}r_k(q)}{\gamma} \tag{221}$$

Writing $\tilde{q}_{t,t+T}$ as the excess returns in deviation from steady state, we have

$$b_t^I = \bar{b}^I + (\sigma_I^2)^{-1}\frac{E_t \tilde{q}_{t,t+T}}{\gamma} \tag{222}$$

where

$$\bar{b}^I = (\sigma_I^2)^{-1}\frac{0.5\sum_{k=1}^{T} v\bar{a}r_k(q) - \bar{r}T\iota}{\gamma} \tag{223}$$

Substituting the optimal portfolio, the value function becomes

$$V_t = e^{A-(1/4)B'\left(C^{-1}\right)'B}/(1-\gamma) \tag{224}$$

We will derive below that $E_t q_{t,t+T} = M_1 Y_t$, where M_1 is a $N-1$ by $(\bar{T}+1)(N-1)+1$ matrix with $M_1(1:N-1, 1+v(N-1):(v+1)(N-1)) = \delta_u(v,T)$ for $v = 0,..,\bar{T}-1$, $M_1(1:N-1,(N-1)\bar{T}+1:(N-1)(\bar{T}+1)) = (\rho_x^T - 1)I_{N-1,N-1}$ and $M_1(1:N-1,(N-1)(\bar{T}+1)+1) = -\bar{r}T\iota$. We have

$$A - (1/4)B'\left(C^{-1}\right)'B = G_1 + G_2 Y_t + Y_t' G_3 Y_t \tag{225}$$

where

$$G_1 = (1-\gamma)\bar{r}T + (1/8)\frac{1-\gamma}{\gamma}\left(\sum_{k=1}^{T} v\bar{a}r_k(q)'\right)(\sigma_I^2)^{-1}\left(\sum_{k=1}^{T} v\bar{a}r_k(q)\right) \tag{226}$$

$$G_2 = 0.5\frac{1-\gamma}{\gamma}\sum_{k=1}^{T} v\bar{a}r_k(q)'(\sigma_I^2)^{-1}M_1 \tag{227}$$

$$G_3 = 0.5\frac{1-\gamma}{\gamma}(M_1)'(\sigma_I^2)^{-1}M_1 \tag{228}$$

We can write

$$G_1 + G_2 Y_t + Y_t' G_3 Y_t = Y_t' G Y_t \tag{229}$$

where $G = G_3 + G_4 + G_5$. G^4 is a square $(\bar{T}+1)(N-1)+1$ matrix with the vector G^2 in the last row and otherwise zeros and G^5 is a square $(\bar{T}+1)(N-1)+1$ matrix with the element $((\bar{T}+1)(N-1)+1, (\bar{T}+1)(N-1)+1)$ equal to G^1 and all other elements zero. The time t value function is then

$$V_t = e^{Y_t' G Y_t}/(1-\gamma) \tag{230}$$

2.3 Linearization of Market Equilibrium Conditions

The market equilibrium condition for currency n bonds is

$$n_F \sum_{k=1}^{T} b_{t-k+1,t}^{n,F} W_{t-k+1,t}^{F} + n_I \sum_{k=1}^{T} b_{t-k+1}^{n,I} W_{t-k+1,t}^{I} + (\bar{x} + x_t^n)\bar{W} = Be^{s_t^n} \tag{231}$$

where

$$W_{t-k+1,t}^{F} = \prod_{i=1}^{k-1} R_{t-k+i+1}^{p} (1-\tau)^{k-1} \tag{232}$$

$$R_{t-k+i+1}^{p} = \left(1 - \sum_{n=1}^{N-1} b_{t-k+1,t-k+i}^{n,F}\right) e^{\bar{r}} + \sum_{n=1}^{N-1} b_{t-k+1,t-k+i}^{n,F} e^{s_{t-k+i+1}^n - s_{t-k+i}^n + i_{t-k+i}^n} \tag{233}$$

and

$$W_{t-k+1,t}^{I} = \prod_{i=1}^{k-1} R_{t-k+i+1}^{p} \tag{234}$$

$$R_{t-k+i+1}^{p} = (1 - \sum_{n=1}^{N-1} b_{t-k+1}^{n,I}) e^{\bar{r}} + \sum_{n=1}^{N-1} b_{t-k+1}^{n,I} e^{s_{t-k+i+1}^n - s_{t-k+i}^n + i_{t-k+i}^n} \tag{235}$$

We differentiate this budget constraint around the point where exchange rates and asset returns are zero, $\tau = 0$, and portfolio shares are equal to their steady state values. The first order Taylor approximation of the term $b_{t-k+1,t}^{n,F} W_{t-k+1,t}^{F}$ is:

$$b_{t-k+1,t}^{n,F} W_{t-k+1,t}^{F} = b_{t-k+1,t}^{n,F} + \bar{b}^{n,F}(k-1) \sum_{i=1}^{k-1} \left[\bar{r} - \tau + \sum_{m=1}^{N-1} \bar{b}^{m,F}(i-1) q_{t-k+i+1}^{m}\right] \tag{236}$$

therefore

$$\sum_{k=1}^{T} b^{n,F}_{t-k+1,t} W^{F}_{t-k+1,t} = \sum_{k=1}^{T} b^{n,F}_{t-k+1,t} + \bar{k}^{n,F} + \sum_{m=1}^{N-1} \sum_{k=1}^{T-1} k^{n,m,F}(k) q^{m}_{t-k+1} \tag{237}$$

where

$$\bar{k}^{n,F} = \sum_{k=1}^{T-1} \bar{b}^{n,F}(k) k(\bar{r} - \tau)$$
$$k^{n,m,F}(k) = \sum_{i=1}^{T-k} \bar{b}^{n,F}(i-1) \bar{b}^{m,F}(i+k-1)$$

Similarly

$$\sum_{k=1}^{T} b^{n,I}_{t-k+1} W^{I}_{t-k+1,t} = \sum_{k=1}^{T} b^{n,I}_{t-k+1} + \bar{k}^{n,I} + \sum_{m=1}^{N-1} \sum_{k=1}^{T-1} k^{n,m,I}(k) q^{m}_{t-k+1} \tag{238}$$

where

$$\bar{k}^{n,I} = \sum_{k=1}^{T-1} \bar{b}^{n,I} k \bar{r}$$
$$k^{I}(n,m,k) = (T-k) \bar{b}^{n,I} \bar{b}^{m,I}$$

Finally, a first order Taylor approximation of the right hand side of the budget constraint is $Be^{s_t} = B + Bs^n_t$. The budget constraint can then be rewritten as

$$n_F \sum_{k=1}^{T} b^{n,F}_{t-k+1,t} + n_I \sum_{k=1}^{T} b^{n,I}_{t-k+1} + n_F \bar{k}^{n,F} + n_I \bar{k}^{n,I} +$$
$$\sum_{m=1}^{N-1} \sum_{k=1}^{T-1} (n_F k^{n,m,F}(k) + n^I k^{n,m,I}(k)) q^{m}_{t-k+1} + (\bar{x} + x^n_t) \bar{W} = B + Bs^n_t \tag{239}$$

We define $\bar{W}$ as equal to total steady state wealth when evaluated at returns on Home and foreign bonds equal to their steady state levels ($\bar{r}$ for Home bonds and 0 for Foreign bonds), the fraction invested in each Foreign bonds is b for all investors (the steady state relative supply of each Foreign bond) and a zero portfolio decision cost for investors making frequent portfolio decisions.

Based on that definition we have

$$\bar{W} = wnT \tag{240}$$

where

$$w = \sum_{k=1}^{T} \left(\bar{R}^p\right)^{k-1} / T \tag{241}$$

$$\bar{R}^p = (1-(N-1)b)e^{\bar{r}} + (N-1)b \tag{242}$$

Dividing (239) by nT yields

$$f\frac{1}{T}\sum_{k=1}^{T} b^{n,F}_{t-k+1,t} + (1-f)\frac{1}{T}\sum_{k=1}^{T} b^{n,I}_{t-k+1} + \frac{1}{T}(f\bar{k}^{n,F} + (1-f)\bar{k}^{n,I}) + \sum_{m=1}^{N-1}\sum_{k=1}^{T-1}\frac{1}{T}(fk^{n,m,F}(k) + (1-f)k^{n,m,I}(k))q^m_{t-k+1} + w\bar{x} + wx^n_t = wb + wbs^n_t \tag{243}$$

where $b = B/\bar{W}$ is the steady state supply of each foreign bond divided by steady state wealth, which is the steady state fraction invested in each foreign bond.

Let $\tilde{q}$ again denote the excess return in deviation from its steady state. Then subtracting the steady state from both the right and left hand sides of (243), and substituting the expressions for optimal portfolios, gives (for the entire set of $N-1$ market clearing conditions)

$$f(\sigma^2)^{-1}\frac{E_t\tilde{q}_{t+1}}{\gamma} + fDY_t + (1-f)\frac{1}{T}(\sigma_I^2)^{-1}\sum_{k=1}^{T}\frac{E_{t-k+1}\tilde{q}_{t-k+1,t-k+1+T}}{\gamma} + \sum_{k=1}^{T-1}\frac{1}{T}A_k\tilde{q}_{t-k+1} + wx_t = wbs_t \tag{244}$$

where A_k is a $N-1$ by $N-1$ matrix such that $A_k(n,m) = fk^{n,m,F}(k) + (1-f)k^{n,m,I}(k)$ and

$$D = \frac{1}{T}\sum_{k=1}^{T}(\bar{C}^{k-1})^{-1}D_1^{k-1}$$

$$(\sigma^2)^{-1} = \frac{1}{T}\sum_{k=1}^{T}\gamma(\bar{C}^{k-1})^{-1} \tag{245}$$

The constant term in the portfolio of liquidity traders is again set such that the market clearing conditions holds in steady state for a given real interest rate $\bar{r}$. The steady state $\bar{x}$ plays no role in the analysis below.

2.4 Solution Equilibrium Exchange Rate

In this section we will discuss how the equilibrium exchange rate is solved for a given fraction f of investors making frequent portfolio decisions. We will assume symmetry of the innovations in ϵ_t^x (we already assumed symmetry in the interest rate innovations). In particular, all the diagonal elements of $\Sigma_x = var(\epsilon_t^x)$ are equal to σ_x^2. All off-diagonal elements are $\rho_{xx}\sigma_x^2$. Write $s_t = \left(s_t^1, .., s_t^{N-1}\right)'$. We conjecture that equilibrium exchange rates at time t depend linearly on present and past innovations:

$$s_t = A(L)\epsilon_t^u + B(L)\epsilon_t^x \tag{246}$$

where $\epsilon_t^u = \left(\epsilon_t^1, .., \epsilon_t^{N-1}\right)'$ and $\epsilon_t^x = \left(\epsilon_t^{1,x}, .., \epsilon_t^{N-1,x}\right)'$

$$A(L) = a_1 + a_2 L + a_3 L^2 + ... \tag{247}$$

$$B(L) = b_1 + b_2 L + b_3 L^2 + ... \tag{248}$$

Here a_i and b_i are $N-1$ by $N-1$ matrices. The number of lags is infinite in both lag operators. Substituting (246) into the market equilibrium condition (244), we obtain an equilibrium exchange rate equation. We then need to equate the conjectured to the equilibrium exchange rate equation. We choose the process

$$x_t = C(L)\epsilon_t^x = (c_1 + c_2 L + c_3 L^2 + ...)\epsilon_t^x \tag{249}$$

such that $b_{k+1} = \rho_x b_k$ for $k \geq 1$. Assume that $c_1 = I_{N-1,N-1}$. Below we will show how to solve $A(L)$, $C(L)$ and b_1 by imposing foreign bond market equilibrium. We will write $\hat{x}_t = B(L)\epsilon_t^x$. Therefore

$$s_t = A(L)\epsilon_t^u + \hat{x}_t \tag{250}$$

$$\hat{x}_t = \rho_x \hat{x}_{t-1} + b_1 \epsilon_t^x \tag{251}$$

In order to solve for the equilibrium exchange rate equation we need to write both sides of the market equilibrium equation as a function of the underlying innovations and then equate the coefficients multiplying these innovations on the right and left side of the equation. The overall approach is rather straightforward, but the algebra is a bit lengthy.

We first need to compute first and second order moments in the market equilibrium condition. Starting with the second moments, we need to compute (i)

$var_t(q_{t+k}),\ \ k=1,..T$; and (ii) $var_t(q_{t,t+T})$. Starting with the former,

$$\begin{aligned}\tilde{q}_{t+k} &= s_{t+k}-s_{t+k-1}-u_{t+k-1}\equiv \\ &\sum_{v=1}^{k}\left(\bar{\lambda}_u(v,k)\epsilon^u_{t+v}+\bar{\lambda}_x(v,k)\epsilon^x_{t+v}\right)+ \\ &\sum_{v=0}^{\infty}\bar{\delta}_u(v,k)\epsilon^u_{t-v}+\rho_x^{k-1}(\rho_x-1)\hat{x}_t\end{aligned} \tag{252}$$

where (I a $N-1$ by $N-1$ matrix)

$$\begin{aligned}\bar{\lambda}_u(k,k) &= a_1 \\ \bar{\lambda}_u(v,k) &= a_{k+1-v}-a_{k-v}-\rho^{k-v-1}I \quad v=1,..,k-1 \\ \bar{\delta}_u(v,k) &= a_{k+1+v}-a_{k+v}-\rho^{k-1+v}I \quad v\geq 0\end{aligned}$$

and

$$\begin{aligned}\bar{\lambda}_x(k,k) &= b_1 \\ \bar{\lambda}_x(v,k) &= b_{k+1-v}-b_{k-v} \quad v=1,..,k-1\end{aligned}$$

We will also write

$$\rho_x^{k-1}(\rho_x-1)\hat{x}_t=\sum_{v=0}^{\infty}\bar{\delta}_x(v,k)\epsilon^x_{t-v}$$

where $\bar{\delta}_x(v,k)=b_{k+1+v}-b_{k+v}$ for $v\geq 0$. Therefore

$$var_t(q_{t+k})=\sum_{v=1}^{k}\left(\bar{\lambda}_u(v,k)\Sigma_u\bar{\lambda}_u(v,k)'+\bar{\lambda}_x(v,k)\Sigma_x\bar{\lambda}_x(v,k)'\right) \tag{253}$$

Now consider the variance of $q_{t,t+T}$. We have

$$\tilde{q}_{t,t+T}=\sum_{k=1}^{T}\tilde{q}_{t+k}=s_{t+T}-s_t-\sum_{v=1}^{T}u_{t+v-1} \tag{254}$$

We can write

$$\begin{aligned}\textstyle\sum_{v=1}^{T}u_{t+v-1} = &\left(\sum_{v=1}^{T}\rho^{v-1}\right)\sum_{v=0}^{\infty}\rho^v\epsilon^u_{t-v}+\sum_{v=1}^{T-1}\left(\sum_{k=1}^{T-v}\rho^{k-1}\right)\epsilon^u_{t+v}= \\ &\frac{1-\rho^T}{1-\rho}\sum_{v=0}^{\infty}\rho^v\epsilon^u_{t-v}+\sum_{v=1}^{T-1}\frac{1-\rho^{T-v}}{1-\rho}\epsilon^u_{t+v}\equiv \\ &\rho(0,T)\sum_{v=0}^{\infty}\rho^v\epsilon^u_{t-v}+\sum_{v=1}^{T-1}\rho(v,T)\epsilon^u_{t+v}\end{aligned} \tag{255}$$

where $\rho(v,T) = (1-\rho^{T-v})/(1-\rho)$.

Substituting (246) and (255) into (254) we get

$$\tilde{q}_{t,t+T} = a_1\epsilon^u_{t+T} + \sum_{v=1}^{T-1}\left(-\rho(v,T)I_{N-1,N-1} + a_{T-v+1}\right)\epsilon^u_{t+v} + \tag{256}$$
$$b_1\epsilon^x_{t+T} + \sum_{v=1}^{T-1} b_{T-v+1}\epsilon^x_{t+v} +$$
$$\sum_{v=0}^{\infty}\left(-\rho(0,T)\rho^v I_{N-1,N-1} + a_{T+v+1} - a_{v+1}\right)\epsilon^u_{t-v} + (\rho^T_x - 1)\hat{x}_t \equiv$$
$$\sum_{v=1}^{T}\left(\lambda_u(v,T)\epsilon^u_{t+v} + \lambda_x(v,T)\epsilon^x_{t+v}\right) + \sum_{v=0}^{\infty}\delta_u(v,T)\epsilon^u_{t-v} + (\rho^T_x - 1)\hat{x}_t$$

At times we will also write

$$(\rho^T_x - 1)\hat{x}_t = \sum_{v=0}^{\infty}\delta_x(v,T)\epsilon^x_{t-v}$$

It follows that

$$var_t(q_{t,t+T}) = \sum_{v=1}^{T}\left(\lambda_u(v,T)\Sigma_u\lambda_u(v,T)' + \lambda_x(v,T)\Sigma_x\lambda_x(v,T)'\right) \tag{257}$$

Next consider the first moments. Using (256) for $T=1$ we have

$$E_t\tilde{q}_{t+1} = E_t\tilde{q}_{t,t+1} = \sum_{v=0}^{\infty}\delta_u(v,1)\epsilon^u_{t-v} + (\rho_x - 1)\hat{x}_t =$$
$$\sum_{v=0}^{\infty}\delta_u(v,1)\epsilon^u_{t-v} + \sum_{v=0}^{\infty}\delta_x(v,1)\epsilon^x_{t-v} \tag{258}$$

and

$$\sum_{k=1}^{T} E_{t-k+1}q_{t-k+1,t-k+1+T} = \left(1 + L + ... + L^{T-1}\right)E_t q_{t,t+T} =$$
$$\left(1 + L + ... + L^{T-1}\right)\sum_{v=0}^{\infty}\left(\delta_u(v,T)\epsilon^u_{t-v} + \delta_x(v,T)\epsilon^x_{t-v}\right) =$$
$$\sum_{v=0}^{\infty}\sum_{k=0}^{min(v,T-1)}\delta_u(v-k,T)\epsilon^u_{t-v} + \sum_{v=0}^{\infty}\sum_{k=0}^{min(v,T-1)}\delta_x(v-k,T)\epsilon^x_{t-v} \equiv$$
$$\sum_{v=0}^{\infty}\eta^T_u(v)\epsilon^u_{t-v} + \sum_{v=0}^{\infty}\eta^T_x(v)\epsilon^x_{t-v} \tag{259}$$

Therefore

$$f(\sigma^2)^{-1}\frac{E_t\tilde{q}_{t+1}}{\gamma}+(1-f)\frac{1}{T}(\sigma_I^2)^{-1}\sum_{k=1}^{T}\frac{E_{t-k+1}\tilde{q}_{t-k+1,t-k+1+T}}{\gamma}\equiv$$
$$\sum_{v=0}^{\infty}\eta_u(v)\epsilon_{t-v}^u+\sum_{v=0}^{\infty}\eta_x(v)\epsilon_{t-v}^x \quad (260)$$

where

$$\eta_u(v)=f(\sigma^2)^{-1}\eta_u^1(v)\frac{1}{\gamma}+(1-f)\frac{1}{T}(\sigma_I^2)^{-1}\eta_u^T\frac{1}{\gamma} \quad (261)$$
$$\eta_x(v)=f(\sigma^2)^{-1}\eta_x^1(v)\frac{1}{\gamma}+(1-f)\frac{1}{T}(\sigma_I^2)^{-1}\eta_x^T\frac{1}{\gamma} \quad (262)$$

We finally need to compute, as a function of model innovations, the expression

$$\frac{1}{T}\sum_{k=1}^{T-1}A_k\tilde{q}_{t-k+1} \quad (263)$$

in the linearized market clearing condition. Using (256) with t replaced by $t-k$ and T by 1, we have for $k>1$

$$\begin{aligned}\tilde{q}_{t+1-k}= \;& \lambda_u(1,1)\epsilon_{t+1-k}^u+\lambda_x(1,1)\epsilon_{t+1-k}^x+ \\ & \sum_{v=0}^{\infty}\Big(\delta_u(v,1)\epsilon_{t-k-v}^u+\delta_x(v,1)\epsilon_{t-k-v}^x\Big)\equiv \\ & \sum_{v=0}^{\infty}\Big(\omega_u(v,k)\epsilon_{t-v}^u+\omega_x(v,k)\epsilon_{t-v}^x\Big)\end{aligned} \quad (264)$$

where

$$\begin{aligned}\omega_u(v,k) &= 0_{N-1,N-1} \quad v=0,..,k-2\\ \omega_u(v,k) &= \lambda_u(1,1) \quad v=k-1\\ \omega_u(v,k) &= \delta_u(v-k,1) \quad v\geq k\\ \omega_x(v,k) &= 0_{N-1,N-1} \quad v=0,..,k-2\\ \omega_x(v,k) &= \lambda_x(1,1) \quad v=k-1\\ \omega_x(v,k) &= \delta_x(v-k,1) \quad v\geq k\end{aligned}$$

It follows that

$$\frac{1}{T}\sum_{k=1}^{T-1}A_k\tilde{q}_{t-k+1}\equiv$$
$$\sum_{v=0}^{\infty}\Big(\mu_u(v)\epsilon_{t-v}^u+\mu_x(v)\epsilon_{t-v}^x\Big) \quad (265)$$

where

$$\mu_u(v) = \frac{1}{T}\sum_{k=1}^{T-1} A_k \omega_u(v,k)$$
$$\mu_x(v) = \frac{1}{T}\sum_{k=1}^{T-1} A_k \omega_x(v,k)$$

Substituting (260) and (265) into (244) we get

$$\begin{aligned} wbs_t = \quad & \sum_{v=0}^{\infty} \left(\eta_u(v) + \mu_u(v)\right) \epsilon^u_{t-v} + \\ & \sum_{v=0}^{\bar{T}-1} fD(1:N-1, 1+v(N-1):(v+1)(N-1))\epsilon^u_{t-v} \\ & + fD((N-1)\bar{T}+1:(N-1)(\bar{T}+1))\hat{x}_t + \sum_{v=0}^{\infty} \left(\eta_x(v) + \mu_x(v) + wc_{v+1}\right) \epsilon^x_{t-x} \end{aligned}$$

Equating the conjectured to the equilibrium exchange rate equation then gives

$$\begin{aligned} wba_{v+1} = \quad & \eta_u(v) + \mu_u(v) + \\ & fD(1:N-1, 1+v(N-1):(v+1)(N-1)) \quad \forall v = 0,..,\bar{T}-1 \end{aligned} \tag{266}$$
$$wba_{v+1} = \quad \eta_u(v) + \mu_u(v) \quad \forall v \geq \bar{T} \tag{267}$$
$$\begin{aligned} wbb_{v+1} = \quad & \eta_x(v) + \mu_x(v) + wc_{v+1} + \\ & fD(1:N-1, (N-1)\bar{T}+1:(N-1)(\bar{T}+1))b_1\rho^v_x \quad \forall v \geq 0 \end{aligned} \tag{268}$$

We can solve for $A(L)$ and b_1 using (266) for $v = 0,..,\bar{T}-1$ and (268) for $v = 0$ (using $c_1 = I_{N-1,N-1}$). We therefore omit equation (267). This is equivalent to truncating the state space after $\bar{T}$ periods. We only solve for $a_1,..,a_{\bar{T}}$ and set $a_{\bar{T}+k} = 0$ for $k > 0$. This is a very close approximation for large $\bar{T}$ as the impact of interest rate shocks on the exchange rate dies out over time. Once $A(L)$ and b_1 have been solved, $c(v+1)$ for $\forall v > 0$ can be solved from (268). But these parameters are inconsequential for the analysis.

The solution is simplified by the fact that we have assumed symmetry. This means that each a_v has really only two unknown parameters. All elements on the diagonal are the same and all elements off the diagonal are the same. The same is the case for b_1. So the total number of parameters to solve is $2(\bar{T}+1)$, twice the number as with 2 currencies.

2.5 Computation of the Threshold Cost

In this section we compute the threshold cost such that investors break even between making frequent or infrequent portfolio decisions. Consider an investor born at time t. If the investor makes frequent portfolio decisions the time t value function is

$$V_t = e^{Y_t' H_0 Y_t}(1-\tau)^{(1-\gamma)T}/(1-\gamma) \tag{269}$$

The decision whether to make frequent portfolio decisions or not is made when born before the state is observed. Therefore we need to compare the unconditional value functions at time t, before observing the state vector Y_t. We therefore need to compute the unconditional expectation of

$$e^{Y_t' H_0 Y_t} \tag{270}$$

Define

$$H_0 = \begin{pmatrix} \bar{H} & \bar{h}_1 \\ \bar{h}_2' & h \end{pmatrix} \tag{271}$$

where $\bar{H}$ is a $(N-1)(\bar{T}+1)$ by $(N-1)(\bar{T}+1)$ matrix, $\bar{h}_1$ and $\bar{h}_2$ are $(N-1)(\bar{T}+1)$ by 1 vectors and h is a scalar. Also write

$$Y_t = \begin{pmatrix} \widehat{Y}_t \\ 1 \end{pmatrix} \tag{272}$$

Then

$$V_t = e^{h + 2\widehat{Y}_t'\bar{h} + \widehat{Y}_t'\bar{H}\widehat{Y}_t}/(1-\gamma) \tag{273}$$

where

$$\bar{h} = \frac{\bar{h}_1 + \bar{h}_2}{2} \tag{274}$$

We have $\widehat{Y}_t \sim N(0, \Sigma_y)$, where $\Sigma_y((N-1)(k-1)+1 : (N-1)k, (N-1)(k-1)+1 : (N-1)k) = \Sigma_u$ for $k = 1, .., \bar{T}$, $\Sigma_y((N-1)\bar{T}+1 : (N-1)(\bar{T}+1), (N-1)\bar{T}+1 : (N-1)(\bar{T}+1)) = b_1 \Sigma_x b_1'/(1-\rho_x^2)$. The other elements of Σ_y are zero.

Therefore

$$\begin{aligned} & E_t e^{h+2\widehat{Y}_t'\bar{h}+\widehat{Y}_t'\bar{H}\widehat{Y}_t} = \\ & \frac{1}{|\Sigma_y|^{0.5}(2\pi)^{0.5\bar{T}+0.5}} \int e^{h+2\bar{h}'\widehat{Y}_t+\widehat{Y}_t'\bar{H}\widehat{Y}_t-0.5\widehat{Y}_t'\Sigma_y^{-1}\widehat{Y}_t} d\widehat{Y}_t = \\ & \frac{1}{|\Sigma_y|^{0.5}(2\pi)^{0.5\bar{T}+0.5}} \int e^{h+2\bar{h}'\widehat{Y}_t+\widehat{Y}_t'(\bar{H}-0.5\Sigma_y^{-1})\widehat{Y}_t} d\widehat{Y}_t = \frac{|\Omega_y|^{0.5}}{|\Sigma_y|^{0.5}} e^{h+2\bar{h}'\Omega_y\bar{h}} \end{aligned} \tag{275}$$

where

$$\Omega_y = (\Sigma_y^{-1} - 2\bar{H})^{-1} \tag{276}$$

The unconditional expectation of the value function for investors making frequent portfolio decisions is computed in exactly the same way. If we write the expected value for investors making frequent portfolio decisions as

$$v^F(1-\tau)^{(1-\gamma)T} \tag{277}$$

and the value function for investors making infrequent portfolio decisions as v^I, then the threshold cost

$$\tau = 1 - (v_I/v_F)^{\frac{1}{(1-\gamma)T}} \tag{278}$$

2.6 Computation of Excess Return Predictability

In this section we discuss how to compute in the context of the model the coefficients β_k in the regression of the excess return q_{t+k}^n on the forward discount at time t. It is equal to

$$\frac{cov(q_{t+k}^n, fd_t^n)}{var(fd_t^n)} \tag{279}$$

We need to compute both the numerator and denominator. The forward discount is equal to

$$fd_t^n = u_t^n + \bar{r} \tag{280}$$

We have

$$cov(q_{t+k}, fd_t) = \sum_{v=0}^{\infty} \delta_u(v+k-1, 1)\rho^v \Sigma_u \tag{281}$$

The numerator of (279) is the element (n, n) of $cov(q_{t+k}, fd_t)$ (or any element on the diagonal of $cov(q_{t+k}, fd_t)$). The denominator of (279) is

$$var(fd_t^n) = \sum_{v=0}^{\infty} (\rho^v)^2 \sigma_u^2 \tag{282}$$

2.7 Expectations Conditioned on Limited Information

We now consider the case where agents only use the past M interest rates to form expectations about future exchange rates. Most of the solution procedure is the same as above. We therefore only note the changes.

First consider investors making frequent portfolio decisions. The relevant state space at $t+k$ is now

$$Y_{t+k} = \begin{pmatrix} u_{t+k} \\ \ldots \\ u_{t+k-M+1} \\ 1 \end{pmatrix} \tag{283}$$

where $u_t = (u_t^1, .., u_t^{N-1})'$. We make the same guess for the value function as before, although now the matrix H_k in the exponential of the value function is a square matrix of size $M(N-1)+1$.

Define the vector

$$\epsilon_{t+s+1} = \begin{pmatrix} \epsilon^u_{t+k+1} \\ q_{t+k+1} - E_{t+k}q_{t+k+1} \end{pmatrix} \tag{284}$$

Below we will derive

$$E_{t+k}q_{t+k+1} = \beta^1(1)u_{t+k} + .. + \beta^1(M)u_{t+k-M+1} \tag{285}$$

Below we will also derive the conditional variance $var_{t+k}(q_{t+k+1})$. Clearly $var_{t+k}(\epsilon^u_{t+k+1}) = \Sigma_u$ and $cov_{t+k}(\epsilon^u_{t+k+1}, q_{t+k+1}) = \lambda_u(1,1)\Sigma_u$. These define the elements of the matrix $\Sigma = var_{t+k}(\epsilon_{t+k+1})$. The matrix M_1^k is now $N-1$ by $M(N-1)+1$ with $M_1^k((v-1)(N-1)+1 : v(N-1)) = \beta^1_v$ for $v = 1,..,M$ and $M_1^k(1 : N-1, M(N-1)+1) = -\bar{r}\iota$. The matrix M_2^k is $N-1$ by $2(N-1)$ with $M_2^k(1 : N-1, N : 2N-2) = I_{N-1,N-1}$ and otherwise zeros.

The matrix N_1^k is now $M(N-1)+1$ by $M(N-1)+1$. We have $N_1^k(1 : N-1, 1 : N-1) = \rho I_{N-1,N-1}$, $N_1^k((v-1)(N-1)+1 : v(N-1), (v-2)(N-1)+1 : (v-1)(N-1)) = I_{N-1,N-1}$ for $v = 2,..,M$, $N_1^k(M(N-1)+1, M(N-1)+1) = 1$. All other elements are zero. N_2^k is a $M(N-1)+1$ by $2(N-1)$ matrix. It has $N_2^k(1 : N-1, 1 : N-1) = I_{N-1,N-1}$ and all other elements zero.

The computation of the unconditional value function, used to compute the threshold cost, is also very similar for when expectations are conditioned on limited information. In this case $\bar{H}$ is a $M(N-1)$ by $M(N-1)$ matrix, $\bar{h}_1$ and $\bar{h}_2$ are $M(N-1)$ by 1 vectors and h is a scalar. We have $\widehat{Y}_t = (u_t, .., u_{t-M+1})'$, so that

$$var(\widehat{Y}_t) = \begin{pmatrix} \Sigma_u & \rho\Sigma_u & \rho^2\Sigma_u & \ldots & \rho^{M-1}\Sigma_u \\ \rho\Sigma_u & \Sigma_u & \rho\Sigma_u & \ldots & \rho^{M-2}\Sigma_u \\ \ldots & & & \ldots & \ldots \\ \rho^{M-1}\Sigma_u & \ldots & & & \Sigma_u \end{pmatrix} \frac{1}{1-\rho^2} \tag{286}$$

The rest is the same as before.

The expressions for the optimal portfolio and value function for investors making infrequent portfolio decisions are still the same, except that the first and second moments of the excess return are now different. In particular, we now have $E_t q_{t,t+T} = M_1 Y_t$, where M_1 is now a vector of length $M(N-1)+1$ with $M_1((v-1)(N-1)+1 : v(N-1)) = \beta^T(v)$ for $v = 1,..,M$ and $M_1(M(N-1)+1) = -\bar{r}T$. We can still write the value function for investors making infrequent portfolio decisions as

$$V_t = e^{A-\frac{B^2}{4C}}/(1-\gamma) \tag{287}$$

with

$$A - \frac{B^2}{4C} = G_1 + G_2 Y_t + Y_t' G_3 Y_t \tag{288}$$

The matrices G_1, G_2 and G_3 are the same as before. We again have

$$G_1 + G_2 Y_{t+k} + Y_{t+k}' G_3 Y_{t+k} = Y_{t+k}' G Y_{t+k} \tag{289}$$

where $G = G_3 + G_4 + G_5$. G_4 is now a square $M(N-1)+1$ matrix with the vector G_2 in the last row and otherwise zeros and G_5 is a square $M(N-1)+1$ matrix with the element $(M(N-1)+1, M(N-1)+1)$ equal to G_1 and all other elements zero. The computation of the unconditional value function proceeds as before.

The first and second moments of excess returns, needed to compute the equilibrium exchange rate, optimal portfolios and value function, are now different. First consider the expectation $E_t \tilde{q}_{t,t+T}$. The expression (256) of $\tilde{q}_{t,t+T}$ remains correct. If $M > 1$, the expectation of $\tilde{q}^n_{t,t+T}$ can be computed by regressing $\tilde{q}^n_{t,t+T}$ on $u_t, .., u_{t-M+1}$. We can write this regression as

$$\tilde{q}^n_{t,t+T} = \widehat{Y}_t' \beta_n^T + \epsilon \tag{290}$$

The regression yields

$$\beta_n^T = (var(\widehat{Y}_t))^{-1} cov(\widehat{Y}_t, \tilde{q}^n_{t,t+T}) \tag{291}$$

where $var(\widehat{Y}_t)$ is given by (286) and

$$cov(\widehat{Y}_t, \tilde{q}^n_{t,t+T}) = \begin{pmatrix} \sum_{v=0}^{\infty} \Sigma_u \delta_u(v,T)_{1:N-1,n} \rho^v \\ ... \\ \sum_{v=M-1}^{\infty} \Sigma_u \delta_u(v,T)_{1:N-1,n} \rho^{v-M+1} \end{pmatrix} \tag{292}$$

We then have

$$E_t\widetilde{q}_{t,t+T} = \sum_{v=1}^{\infty} z_u(v,T)\epsilon_{t-v}^u \tag{293}$$

where

$$z_u(v,T) = \begin{pmatrix} z_u^1(v,T) \\ ... \\ z_u^{N-1}(v,T) \end{pmatrix} \tag{294}$$

with

$$z_u^n(v,T) = \sum_{k=1}^{v+1} \beta_n^T((k-1)(N-1)+1:k(N-1))'\rho^{v+1-k} \quad v = 0,..,M-1$$
$$z_u^n(v,T) = \rho z_u^n(v-1,T) \quad \forall v > M-1$$

We now have

$$\sum_{k=1}^{T} E_{t-k+1}\widetilde{q}_{t-k+1,t-k+1+T} = \zeta_u^T(v)\epsilon_{t-v}^u \tag{295}$$

where

$$\zeta_u^T(v) = \sum_{k=0}^{min(v,T-1)} z_u(v-k,T) \tag{296}$$

The following change now needs to be made to (261)-(262). If investors making frequent portfolio decisions condition expectations on limited information we need to replace $\eta_u^1(v)$ with $\zeta_u^1(v)$ and $\eta_x^1(v)$ with 0. If investors making infrequent portfolio decisions condition expectations on limited information we need to replace $\eta_u^T(v)$ with $\zeta_u^T(v)$ and $\eta_x^T(v)$ with 0.

If investors making infrequent portfolio decisions condition expectations on limited information we also need to compute $var_t(\widetilde{q}_{t,t+T})$ and $var_t(\widetilde{q}_{t+k})$ $(k = 1,..,T)$. The former is

$$\begin{aligned} var_t(\widetilde{q}_{t,t+T}) = \sum_{v=1}^{T} (\lambda_u(v,T)\Sigma_u\lambda_u(v,T)' + \lambda_x(v,T)\Sigma_x\lambda_x(v,T)') + \\ \sum_{v=0}^{\infty} ((\delta_u(v,T) - z_u(v,T))\Sigma_u(\delta_u(v,T) - z_u(v,T))') + \\ \frac{(\rho_x^T - 1)^2 b_1\Sigma_x b_1'}{1-\rho_x^2} \end{aligned} \tag{297}$$

$var_t(\widetilde{q}_{t+k})$ can be computed similarly as the conditional variance of $\widetilde{q}_{t,t+T}$. Following the same procedure as above with $\widetilde{q}_{t+k}$ on the left-hand side we get

$$E_t\widetilde{q}_{t+k} = \sum_{v=1}^{\infty} \bar{z}_u(v,k)\epsilon_{t-v}^u \tag{298}$$

Using (252) it follows that analogous to $\widetilde{q}_{t,t+T}$ we have

$$\widetilde{q}_{t+k} = \sum_{v=1}^{k} \left(\bar{\lambda}_u(v,k)\epsilon_{t+v}^u + \bar{\lambda}_x(v,k)\epsilon_{t+v}^x \right) + \sum_{v=0}^{\infty} \left(\bar{\delta}_u(v,k)\epsilon_{t-v}^u + \bar{\delta}_x(v,k)\epsilon_{t-v}^x \right) \tag{299}$$

where

$$\begin{aligned}
\bar{\lambda}_u(k,k) &= a_1 \\
\bar{\lambda}_u(v,k) &= a_{k+1-v} - a_{k-v} - \rho^{k-v-1} I_{N-1,N-1} \quad v = 1,..,k-1 \\
\bar{\delta}_u(v,k) &= a_{k+1+v} - a_{k+v} - \rho^{k-1+v} I_{N-1,N-1} \quad v \geq 0
\end{aligned}$$

and

$$\begin{aligned}
\bar{\lambda}_x(k,k) &= b_1 \\
\bar{\lambda}_x(v,k) &= b_{k+1-v} - b_{k-v} \quad v = 1,..,k-1 \\
\bar{\delta}_x(v,k) &= b_{k+1+v} - b_{k+v}
\end{aligned}$$

It therefore follows that

$$var_t(\widetilde{q}_{t+k}) = \sum_{v=1}^{k} \left(\bar{\lambda}_u(v,k)\Sigma_u\bar{\lambda}_u(v,k)' + \bar{\lambda}_x(v,k)\Sigma_x\bar{\lambda}_x(v,k)' \right) + \sum_{v=0}^{\infty} \left((\bar{\delta}_u(v,k) - \bar{z}_u(v,k))\Sigma_u(\bar{\delta}_u(v,k) - \bar{z}_u(v,k))' \right) + \frac{(\rho_x^k - \rho_x^{k-1})^2 b_1 \Sigma_x b_1'}{1-\rho_x^2} \tag{300}$$

Finally, it should be pointed out that when investors making frequent portfolio decisions condition expectations on limited information the matrix D in the hedge term is now of length $M(N-1)+1$ and $D(1:N, M(N-1)+1) = 0$. Therefore in equations (266)-(268) we now have that the elements $D(1:N-1,(k-1)(N-1)+1: k(N-1))$ for k larger than M are zeros.

We finally need to discuss the case where $M = 0$, so that no information is processed to compute the expected change of the exchange rate. The exchange rate is assumed to be a random walk. We only consider this when all agents make infrequent portfolio decisions. We assume that there is knowledge of the current interest rate and projected future interest rates. In that case

$$E_t \widetilde{q}_{t,t+T} = -\sum_{v=0}^{\infty} \rho(0,T)\rho^v \epsilon^u_{t-v} \tag{301}$$

Therefore $z_u(v,T) = -\rho(0,T)\rho^v I_{N-1,N-1}$. Everything else follows as before. In this case we also have

$$E_t \widetilde{q}_{t+k} = -\sum_{v=0}^{\infty} \rho^{k-1+v} \epsilon^u_{t-v} \tag{302}$$

so that $\bar{z}_u(v,k) = -\rho^{k-1+v} I_{N-1,N-1}$.

3 Alternative Modeling Strategies

3.1 Nominal Rigidities

In this subsection we examine the Foreign bond market clearing condition (which determines the equilibrium exchange rate) in a more general setup that allows for the possibility of nominal rigidities. Let P be the price index in the Home country in the Home currency and P^* the price index in the Foreign country in the Foreign currency. While PPP holds in the flexible-price model, so that $P = SP^*$, we now allow for the possibility that this is not the case. We do not develop an entirely new model based on nominal rigidities, but instead consider the asset market equilibrium for any given process for P and P^*. This exercise is motivated by nominal rigidities, in which case P and P^* change only very gradually. This leads to real exchange rate fluctuations that are closely correlated with the nominal exchange rate.

Allowing for such real exchange rate volatility in the context of nominal rigidities changes the asset market equilibrium in two ways. First, Home and Foreign investors will generally choose different portfolios as they now face different price indices and therefore different real asset returns. Second, for given real endowments when agents are born (assumed equal to 1 in the paper), the relative value of the initial endowments now varies with the real exchange rate. These changes do not fundamentally change the equilibrium though.

Skipping the algebra, the optimal portfolio shares of Home and Foreign investors born at time t are

$$b_t^H = \bar{b}^H + \frac{E_t q_{t,t+T}}{\gamma \sigma_I^2} \tag{303}$$

$$b_t^F = \bar{b}^F + \frac{E_t q_{t,t+T}}{\gamma \sigma_I^2} \tag{304}$$

where

$$\bar{b}^H = \frac{0.5 \sum_{k=1}^{T} var_t(q_{t+k}) + (1-\gamma) cov_t(i_{t,t+T} - p_{t+T}, q_{t,t+T})}{\gamma \sigma_I^2} \tag{305}$$

$$\bar{b}^F = \frac{0.5 \sum_{k=1}^{T} var_t(q_{t+k}) + (1-\gamma) cov_t(i_{t,t+T} - s_{t+T} - p_{t+T}^*, q_{t,t+T})}{\gamma \sigma_I^2} \tag{306}$$

where the definition of the excess return $q_{t,t+T}$ remains exactly the same as in the paper, as well as the expression for σ_I^2. The constant terms $\bar{b}^H$ and $\bar{b}^F$ are now generally different. The differences is proportional to the covariance between the real exchange rate and the excess return: $cov(s_{t+T} + p_{t+T}^* - p_{t+T}, q_{t,t+T})$. This is a familiar term as it is well-known that real exchange rate fluctuations give rise to differences in portfolios as they imply different real returns from the perspective of Home and Foreign agents. If we adopt a completely symmetric modeling setup (symmetry between Home and Foreign country), then it is straightforward to show that the constant terms will add up to 1: $\bar{b}^H + \bar{b}^F = 1$. The important point though is that apart from these constant terms, portfolio shares are exactly the same as in the paper where we adopted the PPP assumption. Real exchange rate fluctuations only affect steady state hedge terms that go into the constant terms.

We now turn to the Foreign bond market clearing condition. When prices are sticky it makes little difference whether the Foreign bond supply is constant in real or nominal terms. Let $\delta = 1$ when B^F is the constant Foreign bond supply in real terms and $\delta = 0$ when B^F is the constant Foreign bond supply in nominal terms. Then, abstracting from liquidity traders, the Foreign bond market clearing condition is (in the Home currency)

$$\sum_{k=1}^{T} b_{t-k+1}^H W_{t-k+1,t}^H + \sum_{k=1}^{T} b_{t-k+1}^F W_{t-k+1,t}^F = B^F (P_t^*)^\delta S_t \tag{307}$$

where $W_{t-k+1,t}^H$ and $W_{t-k+1,t}^F$ are wealth at time t of Home and Foreign investors born at $t - k + 1$.

After dividing by $(P_t^*)^\delta S_t$, linearizing and dividing by $\bar{W}$, this becomes

$$\frac{1}{2T}\left(\sum_{k=1}^{T}(b_{t-k+1}^H-\bar{b}^H)+\sum_{k=1}^{T}(b_{t-k+1}^F-\bar{b}^F)\right)+ \tag{308}$$
$$\bar{b}^H\left(\frac{W_t^H}{S_t(P_t^*)^\delta}-0.5\bar{W}\right)/\bar{W}+\bar{b}^F\left(\frac{W_t^F}{S_t(P_t^*)^\delta}-0.5\bar{W}\right)/\bar{W}=0$$

where $W_t^H=\sum_{k=1}^T W_{t-k+1,t}^H$ and $W_t^F=\sum_{k=1}^T W_{t-k+1,t}^F$ are aggregate Home and Foreign wealth at time t.

The first term of (308) remains exactly the same as in the flexible price case. This term, which has portfolio shares in deviation from steady states, reflects gradual portfolio reallocation. These portfolio shares are still the same as in the PPP case. The sum of the last two terms reflects passive portfolio rebalancing. When P and P^* change very gradually, rebalancing remains very similar to the flexible price case in the paper. Intuitively, a rise in the Foreign interest rate leads to a rise in demand for Foreign bonds (first term rises) and a depreciation of the Home currency. In order to buy more Foreign bonds, others must be selling them. This happens through passive portfolio rebalancing. A depreciation of the Home currency reduces the relative share invested in Home bonds as a result of a valuation effect. This leads to a sale of Foreign bonds for the purpose of portfolio rebalancing. In (308) this shows up as a drop in the last two terms as the value of wealth in the Foreign currency, W_t^H/S_t and W_t^F/S_t, falls when some of the wealth is denominated in the Home currency.

While this mechanism is pervasive and similarly applies to the PPP case and the case with nominal rigidities, the last two terms are somewhat different for the case of sticky prices. In particular, omitting the algebra, the sum of the last two terms in (308) can be written as

$$0.5\frac{1}{T}\sum_{k=1}^{T}\left(\bar{b}^H p_{t-k+1}+\bar{b}^F(s_{t-k+1}+p_{t-k+1}^*)\right)+ \tag{309}$$
$$\left(\frac{1}{4}+(\bar{b}^H-0.5)^2\right)\frac{1}{T}\sum_{k=1}^{T-1}(T-k)q_{t-k+1}-0.5(s_t+\delta p_t^*)$$

In the PPP case (where $\delta=0$, $p_t=s_t+p_t^*=0$), (309) becomes

$$\frac{1}{4}\frac{1}{T}\sum_{k=1}^{T-1}(T-k)q_{t-k+1}-0.5s_t \tag{310}$$

There are two differences between (309) and (310). First, the steady state portfolio shares of Home and Foreign investors are generally different under real exchange rate fluctuations. This implies that $\bar{b}^H$ can be different from 0.5, which affects somewhat the term involving excess returns (only the magnitude, not the sign). Second, the relative value of the initial endowments varies with the real exchange rate. This is captured by the first term in (309).

Neither of these changes fundamentally affect the passive portfolio rebalancing mechanism, which is reflected in the negative relationship between these portfolio rebalancing terms and the exchange rate. In the PPP case the derivative with respect to s_t is

$$-0.5 + \frac{T-1}{4T} = -\frac{1}{4}(1 + \frac{1}{T})$$

In the model with nominal rigidities, assuming that changes in s_t do not immediately affect p_t and p_t^*, it is

$$-\bar{b}^H \left(1 - \bar{b}^H + (\bar{b}^H - 0.5)\frac{1}{T}\right)$$

It is easily seen that this term remains negative as long as $\bar{b}^H$ is between 0 and 1, which the algebra shows is indeed the case.

3.2 A Real Model with Equity

We show that the exact same mechanisms as in the paper hold when we introduce equity rather than bonds. This gives rise to excess return predictability for Foreign equity. The model is virtually identical to that in the paper, with some straightforward relabeling of the variables.

One could assume that there are both Home and Foreign trees, but for closest analogy to the model in the paper we assume that there is only a fixed supply of Foreign trees and that there is an elastic supply of Home capital with a constant risk-free return $e^{\bar{r}}$. The supply of Foreign trees is equal to K and delivers a random dividend of D_t. While not essential in any way, for further analogy to the model in the paper assume that D_t is the dividend paid at $t+1$ but known at time t. This is analogous to the interest payments on bonds at $t+1$ being known at time t.

The gross return on Foreign equity is then

$$\frac{P_{t+1} + D_t}{P_t}$$

where P_t is the price of Foreign equity (in terms of the consumption good). The log-linearized Foreign equity return is

$$r^*_{t+1} = k + \frac{1}{1+\kappa} p_{t+1} - p_t + \frac{\kappa}{1+\kappa}(d_t - ln(\kappa)) \tag{311}$$

where κ is the steady state dividend yield and $k = ln(1+\kappa) = \bar{r}$. The excess return is therefore

$$q_{t+1} = \frac{1}{1+\kappa} p_{t+1} - p_t + \frac{\kappa}{1+\kappa}(d_t - ln(\kappa)) \tag{312}$$

Let $u_t = \kappa(d_t - ln(\kappa))$ follow the same AR process as in the paper. The above equation becomes:

$$(1+\kappa) q_{t+1} = p_{t+1} - p_t + u_t - \kappa p_t \tag{313}$$

This can be compared to the excess return in the paper :

$$q_{t+1} = s_{t+1} - s_t + i^*_t - i_t = s_{t+1} - s_t + u_t - \psi s_t \tag{314}$$

This excess return is analogous to that for equity when we replace s_t by p_t and ψ by κ. The only small difference is that the excess return is multiplied by $1/(1+\kappa)$ for equity.

The asset market clearing condition is the same as well. For Foreign equity it is

$$n \sum_{k=1}^{T} b^I_{t-k+1} W^I_{t-k+1,t} = K P_t \tag{315}$$

where the optimal portfolios are derived as in the paper and the wealth accumulates as in the paper. This is the same as equation (8) in the paper, with the Foreign bond supply BS_t now replaced by the equity supply KP_t. The same analysis as in the paper therefore carries through, with s_t now replaced by p_t.

3.3 Government Tax Policy

In the paper, the governments of both countries adopt a balanced budget rule so that their nominal debt is constant. They receive a return on the risk-free capital they hold, pay interest on their nominal debt and use the remainder for consumption. The holding of risk-free capital by the government is just a device to generate some income to finance government spending and interest payments. Instead here we consider a setup where the government receives revenue from direct taxation rather than from asset income and where the budget is balanced through adjustment in this direct taxation while holding government spending constant. We assume that each country has the same constant real level of government consumption G and the same constant bond supply in local currency B. Taxation by the Home and Foreign government is then equal to

$$Tax_t^H = G + (e^{i_t} - 1)B \tag{316}$$

$$Tax_t^F = G + (e^{i_t^*} - 1)BS_t \tag{317}$$

The impact of this change on the model depends on which agents the government will tax. We consider two extremes: i) newborn agents; ii) agents in the last period of their life, when they consume.

We now need to distinguish between Home and Foreign agents. When the government taxes newborn agents, the wealth of the newborn agents is reduced by the tax. This makes little difference for Home agents since the tax is constant as $i_t = \bar{r}$ is constant. For Foreign agents it will reduce their initial wealth by a time-varying tax, which in linearized form is equal to $Tax_t^F = G + i_t^* B$. The initial wealth is therefore reduced from 1 to $1 - G - i_t^* B$. This change is small when $i_t^* B$ is small relative to $1 - G$. It will affect the Foreign bond market equilibrium condition only through this change in the initial wealth of agents. Importantly, it does not change the passive portfolio rebalancing mechanism as it does not impact the effect of the exchange rate on relative asset holdings. It also does not change optimal portfolio allocation as a function of the expected excess return.

Now consider the case where the government instead taxes agents in their last period of life. The final wealth that is consumed is now equal to

$$W_{t+T} = \prod_{j=0}^{T-1} R_{t+T-j}^p - Tax_{t+T} \tag{318}$$

where the tax Tax_{t+T} is the Home tax for Home agents and the Foreign tax for Foreign agents.

In this case, the only impact is on portfolio allocation. We can write the linearized log wealth for respectively Home and Foreign agents as

$$w^H_{t+T} = ln(1-G) + \frac{1}{1-G}(r^p_{t+1} + ...r^p_{t+T}) \tag{319}$$

$$w^F_{t+T} = ln(1-G) + \frac{1}{1-G}(r^p_{t+1} + ...r^p_{t+T}) - \frac{1}{1-G}i^*_{t+T}B \tag{320}$$

This changes portfolio in two ways. First, there is an extra constant term $1/(1-G)$ in front of the log portfolio returns. In addition, it leads to an additional hedge term in the optimal portfolio of Foreign agents as they hedge future interest rate risk. The additional hedge term is

$$-\frac{Bcov(q_{t,t+T}, i^*_{t+T})}{\bar{\gamma}\sigma^2_I}$$

Similar to the discussion about nominal rigidities, where additional hedge terms affect the steady state portfolio shares, this alters slightly the quantitative magnitude of the passive portfolio rebalancing term without altering the basic mechanism. It remains the case that agents on average will sell Foreign bonds for portfolio balance reasons when the Foreign currency appreciates.