Session 7: Economic Growth

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December 2010

Solow: The Production Function

$$Y_t = F(K_t, A_t L_t)$$

- Y is GDP, K is capital and L is labour.
- A is labour augmenting technology, shifter of production. **Free**, non rival, non excludable.
- $F[.]$ is increasing and concave in both K and L. Diminishing marginal returns
- $F[.]$ has constant returns to scale, i.e

$$F(\lambda K_t, \lambda A_t L_t) = \lambda F(K_t, A_t L_t)$$

- In particular, $\frac{1}{A_t L_t} F(K_t, A_t L_t) = F\left(\frac{K_t}{A_t L_t}, 1\right)$.
- Classic case is Cobb-Douglas:

$$F(K_t, A_t L_t) = A_t K_t^{\alpha} L_t^{1-\alpha}$$

Representative Firm Profits

- Representative firm chooses K_t and L_t to maximize

$$F\left(K_t, A_t L_t\right) - w_t L_t - R_t K_t$$

- Notice price of final good is normalized to one, and wages and interest rates are given to the firm. Perfect competition.
- Easy to obtain

$$\begin{aligned} w_t &= F_L\left[K_t, A_t L_t\right] \\ R_t &= F_K\left[K_t, A_t L_t\right] \end{aligned}$$

- Inputs are paid their marginal product. Standard result under perfect competition: there are no profits.

Useful Notation for Dynamics

- Capital accumulates according to the following:

$$\dot{K}_t = I_t - \delta K_t$$

- Since we are in a closed economy without government spending:

$$Y_t = C_t + I_t$$

- So:

$$\dot{K}_t = F\left(K_t, A_t L_t\right) - \delta K_t - C_t$$

- Closed economy also means that whatever is invested domestically has to come from domestic savings. The current (and capital) accounts are zero. So:

$$I_t = S_t$$

- Everything so far is true in general.

Fundamental Law of Motion

- The **KEY** assumption in the Solow/Swan model is that of constant, given savings rate. The representative consumer is "forced" to save a constant proportion s of her income, each and every period. Of course eminently implausible. Still, implies:

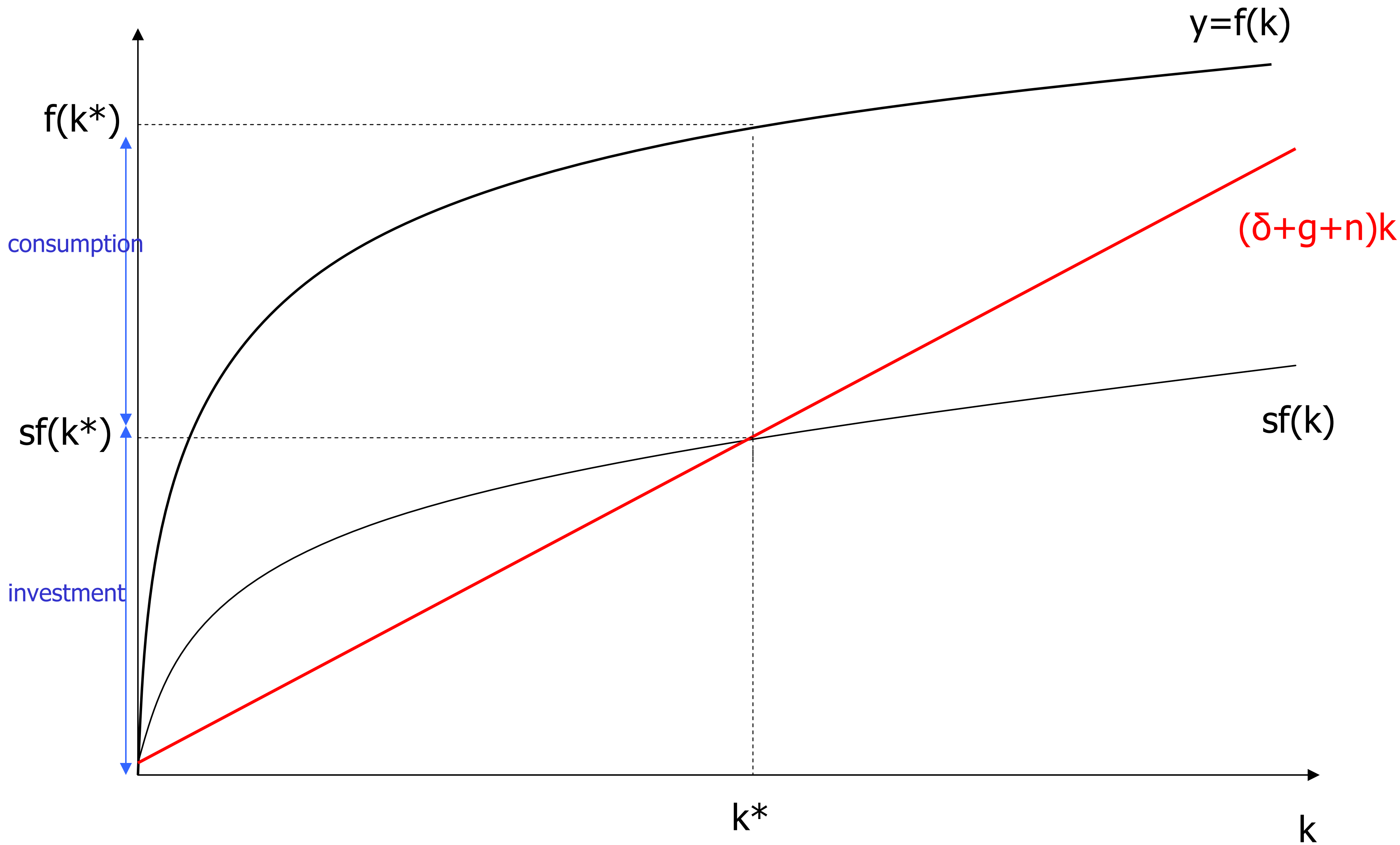
$$S_t = sY_t, \text{ that is } C_t = Y_t - I_t = (1 - s)Y_t$$

- So finally:

$$\dot{K}_t = sF(K_t, A_t L_t) - \delta K_t$$

- This is the fundamental law of motion of capital in Solow/Swan. A differential equation in K_t.
- Important to note that many other models have an exogenous assumption on A_t, the "true" engine of long run growth. The one important difference in the Solow model relative to its successors is the assumption on s, not on A_t.

The Steady State with Population and Technology Growth



Steady State - Algebra

- The model must assume exogenous growth rate in technology for long run growth. In addition, assume a population growth rate as well. In notation:

$$\frac{\dot{A}}{A} = g \text{ and } \frac{\dot{L}}{L} = n$$

- Both are percentage growth rates. Time t subscripts omitted for clarity.
- A steady state will not happen in K, or Y or C, since presumably all of these grow with A (and L). Normalize all variables by $A_t L_t$:

$$\frac{\dot{K}_t}{A_t L_t} = sF\left(\frac{K_t}{A_t L_t}, 1\right) - \delta \frac{K_t}{A_t L_t}$$

- Now denote normalized variables with lower case (and use the "product rule" to solve the left-hand side):

$$\dot{k}_t = sf(k_t) - (\delta + g + n)k_t$$

- A steady state will obtain when capital (per unit of $A_t L_t$) ceases to grow, which happens for $\dot{k}_t = 0$. Defines previous graph. Comparative statics.

Golden Rule - Algebra

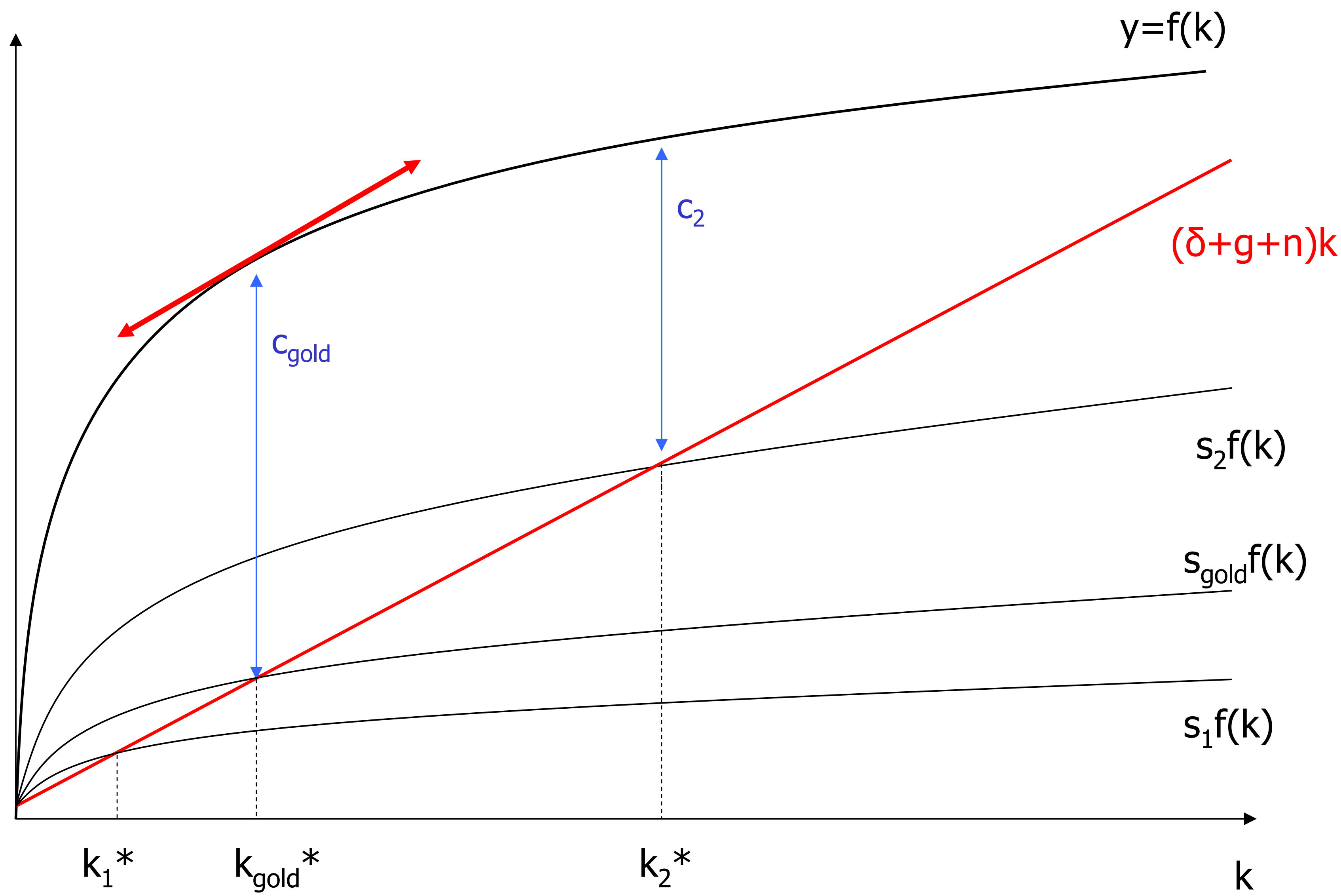
- With * denoting steady state values, we know that

$$c^* = (1 - s) f(k^*) = f(k^*) - (\delta + g + n) k^*$$

since by definition $sf(k^*) = (\delta + g + n) k^*$ (in words: at the steady state, investment is exactly enough to replenish capital per unit of $A_t L_t$.)

- So what is the level of s that maximizes consumption? The one that sets $\frac{\partial c^*}{\partial s} = 0$, i.e. $f'(k^*) = (\delta + g + n)$
- Tangency property.
- Suppose saving above golden rule level (s_2). Then reducing saving means immediately higher consumption, and higher consumption at the steady state. This is called "dynamic inefficiency" because savings are "too high". A clear abuse of language, however. Savings are not optimal in the model
- Suppose saving below golden rule level (s_1). Then increasing saving will increase consumption at the steady state, but will decrease consumption on impact. Whether this is desirable (optimal) depends on consumption/saving optimal choice. Again, absent from here.

The Golden Rule rate of Saving



Empirical Implications

This (simple?) framework has wide-ranging testable implications. Now review them, and their link with theory

- Growth Accounting
- Convergence
- Cross-Country Disparities
- Level Accounting

Growth Accounting

- Consider the production function in its general form:

$$Y_t = F\left(K_t, A_t L_t\right)$$

- Take a total differentiation and divide through by Y:

$$\frac{\dot{Y}}{Y} = \frac{F_K K}{Y}\,\frac{\dot{K}}{K} + \frac{F_L L}{Y}\,\frac{\dot{L}}{L} + \frac{F_A A}{Y}\,\frac{\dot{A}}{A}$$

- Now denote growth rates by $g_Y = \frac{\dot{Y}}{Y}$, $g_K = \frac{\dot{K}}{K}$ and $g_L = \frac{\dot{L}}{L}$.
- Introduce $x = \frac{F_A A}{Y}\,\frac{\dot{A}}{A}$ the contribution of technology to growth. With competition, recall $R = F_K$ and $w = F_L$. So

$$g_Y = \frac{RK}{Y}\,g_K + \frac{wL}{Y}\,g_L + \frac{F_A A}{Y}\,\frac{\dot{A}}{A}$$

Growth Accounting

- Define factor shares $\alpha_K = \frac{RK}{Y}$ and $\alpha_L = \frac{wL}{Y}$ to obtain well known estimates of Total Factor Productivity:

$$x = g_Y - \alpha_K \; g_K + \alpha_L \; g_L$$

- Generalization of well known Cobb Douglas case. Crucial assumption is perfect competition.
- With observed (time-varying) factor shares and observed (time varying) growth rates in Y, K and L, can get x.

Pitfalls

- Both growth rates and factor shares are varying over time. When computing x period-by-period, which measure to use? Beginning of period? End of period?
- Approximation to exact continuous time formulae fairly good when difference between t and $t+1$ small. But high frequency data typically not available for these computations
- Mismeasurement of g_K and g_L as well:
    - Effective labour hours?

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- Mismeasurement of g_K and g_L as well:
  - Effective labour hours?
  - Adjustment for skill of workers?
  - Capital in theory is final good used as input to produce more goods. In practice, machinery. Does price of machinery vary over time? Are machines not becoming cheaper over time? Using observed capital expenditures severely underestimate g_K.

Convergence

- Barro's Growth Regressions. From normalized Solow model, with y output *per capita*:

$$\begin{aligned} y_t &= A_t \; f(k_t) \\ \frac{\dot{k}_t}{k_t} &= \frac{sf(k_t)}{k_t} - \delta - g - n \end{aligned}$$

- Differentiate first relation with respect to time t and divide through by y_t:

$$\begin{aligned} \frac{\dot{y}_t}{y_t} &= \frac{\dot{A}_t \; f(k_t)}{y_t} + \frac{A_t \; f'(k_t)\, \dot{k}_t}{y_t} \\ &= \frac{\dot{A}_t}{A_t} + \frac{f'(k_t)\, k_t}{f(k_t)} \frac{\dot{k}_t}{k_t} \\ &= g + \varepsilon_f(k_t) \frac{\dot{k}_t}{k_t} \end{aligned}$$

where $\varepsilon_f(k_t)$ is the elasticity of $f(.)$. For example, with Cobb-Douglas, this is α. In general it will be between 0 and 1. 1 for linear case.

Convergence (2)

- Now focus on the other equation, the law of motion for capital. In particular, take a first-order Taylor expansion around $k_t = k^*$:

$$\begin{aligned}\frac{\dot{k}_t}{k_t} &\simeq \left(\frac{sf(k^*)}{k^*} - \delta - g - n\right) \\ &\quad + \frac{sf(k^*)}{k^*}\frac{1}{k^*}\left(\frac{f'(k^*)\,k^*}{f(k^*)} - 1\right)(k_t - k^*) \\ &\simeq (\delta + g + n)\left[\varepsilon_f(k^*) - 1\right](\ln k_t - \ln k^*)\end{aligned}$$

where we made use of the facts that $\frac{sf(k^*)}{k^*} = \delta + g + n$ and $\ln \frac{k_t}{k^*} \simeq \frac{k_t}{k^*} - 1$.

- So putting all together:

$$\frac{\dot{y}_t}{y_t} \simeq g - \varepsilon_f(k_t)(\delta + g + n)\left[1 - \varepsilon_f(k^*)\right](\ln k_t - \ln k^*)$$

Convergence (3)

- Almost there. Now take a Taylor expansion of $y_t = A_t f(k_t)$ around its "steady state value" $y_t^* = A_t f(k^*)$:

$$\begin{aligned} y_t &\simeq A_t f(k^*) + A_t f'(k^*)(k_t - k^*) \\ &\simeq y_t^* + A_t f(k^*) \frac{f'(k^*)k^*}{f(k^*)} \frac{(k_t - k^*)}{k^*} \\ &\simeq y_t^* + y_t^* \varepsilon_f(k^*) \frac{(k_t - k^*)}{k^*} \end{aligned}$$

- So using the same approximation that $\ln \frac{k_t}{k^*} \simeq \frac{k_t}{k^*} - 1$ and $\ln \frac{y_t}{y^*} \simeq \frac{y_t}{y^*} - 1$,

$$\ln y_t - \ln y^* = \varepsilon_f(k^*)(\ln k_t - \ln k^*)$$

- And finally the growth regression:

$$\frac{\dot{y}_t}{y_t} \simeq g - [1 - \varepsilon_f(k^*)](\delta + g + n)(\ln y_t - \ln y^*)$$

- Two sources of growth: g, the rate of technological progress and "convergence".

Convergence (4)

- Convergence: Since $0 < \varepsilon_f(k^*) < 1$, negative impact of gap between current level and steady-state level of output per capita on growth.
- Speed of convergence measured by the term $[1 - \varepsilon_f(k^*)](\delta + g + n)$, which depends on:
  - $\delta + g + n$: determines rate at which capital (by unit of $A_t L_t$) needs to be replenished
  - $\varepsilon_f(k^*)$: when elasticity close to 1 (i.e. a linear technology in capital), convergence will be slow
- In the Cobb-Douglas case, $\varepsilon_f(k^*) = \alpha$. Can do a calibration exercise. Focus on advanced economies:
  - $g \simeq 0.02$ for approximately 2% per year output per capita growth
  - $n \simeq 0.01$ for approximately 1% population growth, and
  - $\delta \simeq 0.05$ for about 5% per year depreciation
  - In addition, $\alpha \simeq 1/3$ is the share of capital in national income
- So convergence coefficient around 0.67 x 0.08 = 0.054. Very rapid rate of convergence - halves incomes gap in little more than 10 years. At odds with reality.

Conditional Convergence?

- Approximate growth regression in discrete time (Barro (1991))

$$g_{i,t,t-1} \simeq b^0 + b^1 \ln y_{i,t-1} + \varepsilon_{i,t}$$

- $\varepsilon_{i,t}$ is stochastic term capturing all omitted influences
- In the OECD, get b^1 negative and significant. But not for the whole world. No unconditional convergence.
- May be too demanding. If countries do differ (e.g. technological opportunities, investment, policies, etc) the Solow model does not predict that they should converge in income level. They should converge to their own steady state.

Conditional Convergence?

- If countries differ, a more appropriate regression may be

$$g_{i,t,t-1} \simeq b_i^0 + b^1 \ln y_{i,t-1} + \varepsilon_{i,t}$$

- Constant term is country-specific. (Slope term should actually also be...). May then model b_i^0 as a function of country characteristics.
- If true equation is b_i^0, having just b^0 would not be a good fit of the data, and so no guarantee that b^1 would be negative.
- For instance, it is natural to expect that $cov(b_i^0, y_{i,t-1}) < 0$: economies with growth reducing characteristics also have low income level. Implies an attenuating bias on b^1.
- Reason for "conditional convergence" in Barro and Sala-i-Martin (2004).

Conditional Convergence

- Barro and Sala-i-Martin (2004) estimate models where b_i^0 is assumed to be a function of: male schooling rate, female schooling rate, fertility rate, investment rate, government-consumption ratio, inflation rate, changes in terms of trade, openness and institutions such as the rule of law and democracy.
- In regression form

$$g_{i,t,t-1} = \mathbf{X}'_{i,t}\beta + b^1 \ln y_{i,t-1} + \varepsilon_{i,t}$$

  $X_{i,t}$ is a (column) vector including all variables (and a constant).
- These regressions tend to show negative estimates of b^1, but with much lower magnitudes than what is implied by Cobb-Douglas calibration.

Pitfalls

- These regressions not only used to support "conditional convergence", but also to estimate "determinants of economic growth".
- In other words, draw causal inferences from estimates of β. Problematic for several reasons.
  - Variables in $X_{i,t}$, and $y_{i,t-1}$ are econometrically endogenous, i.e. jointly determined with $g_{i,t,t-1}$. Creates inconsistency (unless $X_{i,t}$ is independent from $y_{i,t-1}$)
  - In the model, the only channel for convergence is investment. Poor countries catch up because gross investment generates much more installed capital than in rich countries. So why should $y_{i,t-1}$ continue to be significant after controls for investment are included?
  - Measurement error in income per capita will be problematic. Suppose we only observe $\ln \hat{y}_{i,t} = \ln y_{i,t} + u_{i,t}$. Then $u_{i,t-1}$ is present on the right hand side, and $-u_{i,t-1}$ on the left hand side. Negative bias, which can explain negative values for b^1.

Cross-Country Evidence

- Augment the model with human capital. Focus on the simple Cobb-Douglas case:

$$Y_t = K_t^{\alpha} \; H_t^{\beta} \; (A_t \; L_t)^{1-\alpha-\beta}$$

- Normalizations:

$$\begin{aligned} k_t &= \frac{K_t}{A_t L_t} \text{ and } h_t = \frac{H_t}{A_t L_t} \\ y_t &= \left(\frac{K_t}{A_t L_t}\right)^{\alpha} \left(\frac{H_t}{A_t L_t}\right)^{\beta} = k_t^{\alpha} h_t^{\beta} \end{aligned}$$

- Still maintain the key assumption that savings are exogenous and constant. In particular,

$$\begin{aligned} \dot{k}_t &= s_K \; k_t^{\alpha} h_t^{\beta} - (\delta_K + g + n) \, k_t \\ \dot{h}_t &= s_H \; k_t^{\alpha} h_t^{\beta} - (\delta_H + g + n) \, h_t \end{aligned}$$

- Setting $\dot{k}_t = 0$ and $\dot{h}_t = 0$ defines steady state.

Cross Country Evidence (2)

- Substitute steady state values for both inputs into steady state income per $A_t L_t$:

$$y^* = \left(\frac{s_K}{\delta_K + g + n}\right)^{\frac{\beta}{1-\alpha-\beta}} \left(\frac{s_H}{\delta_H + g + n}\right)^{\frac{\alpha}{1-\alpha-\beta}}$$

- Mankiw, Romer and Weil (1992) take this model to data. Each country is an island thus defined. Each country $j = 1...N$ differ in terms of their saving rates $s_{K,j}$ and $s_{H,j}$, population growth rates n_j and technology growth rates $\dot{A}_{j,t}/A_{j,t} = g_j$.
- Then "steady state" income per capita in country j is given by

$$\hat{y}_{j,t} = \frac{Y_{j,t}}{L_{j,t}} = A_{j,t} \left(\frac{s_{K,j}}{\delta_K + g_j + n_j}\right)^{\frac{\beta}{1-\alpha-\beta}} \left(\frac{s_{H,j}}{\delta_H + g_j + n_j}\right)^{\frac{\alpha}{1-\alpha-\beta}}$$

- Income can diverge if g_j are different across countries.

Cross Country Evidence (3)

- MRW impose: $A_{j,t} = \bar{A}_j \exp(gt)$. Countries can differ according to (initial) technology level, but they have the same growth rate.
- Then taking simple logarithms:

$$\ln \hat{y}_{j,t} = \ln \bar{A}_j + gt + \frac{\beta}{1-\alpha-\beta} \ln \left(\frac{s_{K,j}}{\delta_K + g + n_j} \right) + \frac{\alpha}{1-\alpha-\beta} \ln \left(\frac{s_{H,j}}{\delta_H + g + n_j} \right)$$

- Then calibrate:
  - $\delta_K = \delta_H = \delta$ and $\delta + g = 0.05$
  - $s_{K,j}$ is average investment rate (investment/GDP)
  - $s_{H,j}$ is fraction of the school age population enrolled in secondary school.

Cross Country Evidence (4)

- Still a remaining problem: $\bar{A}_j$ not directly observable. And so must be bunched in error term
- But not innocuous. Difficult to think of a model where the level of technology $\bar{A}_j$ is not correlated with investment rate.
- Serious endogeneity problem, and inconsistency. MRW make crucial assumption that $\bar{A}_j = \varepsilon_j A$ is orthogonal to all other variables.
- First MRW estimate regression without human capital in cross-section:

$$\ln \hat{y}_j = constant + \frac{\beta}{1-\beta} \ln s_{K,j} - \frac{\beta}{1-\beta} \ln \left(\delta_K + g + n_j\right) + \varepsilon_j$$

Cross Country Evidence (5)

	MRW	Updated Data	
	1985	1985	2000
$\ln s_{K,j}$	1.42	1.01	1.22
	(.14)	(.11)	(.13)
$\ln(\delta_K + g + n_j)$	-1.97	-1.12	-1.31
	(.56)	(.55)	(.36)
Adj R^2	.59	.49	.49
Implied β	.59	.50	.55
Obs	98	98	107

Cross Country Evidence (6)

- Estimate of β too high. Should be around $1/3$.
- One reason is their assumptions are not warranted (e.g. orthogonal technology)
- Another is omitted variable problem with human capital missing.
- Estimate instead:

$$\begin{aligned} \ln \hat{y}_j &= cst + \frac{\beta}{1-\alpha-\beta} \ln s_{K,j} - \frac{\beta}{1-\alpha-\beta} \ln \left(\delta_K + g + n_j\right) \\ &+ \frac{\alpha}{1-\alpha-\beta} \ln s_{H,j} - \frac{\alpha}{1-\alpha-\beta} \ln \left(\delta_H + g + n_j\right) + \varepsilon_j \end{aligned}$$

Cross Country Evidence (7)

	MRW	Updated Data	
	1985	1985	2000
$\ln s_{K,j}$	.69	.65	.96
	(.13)	(.11)	(.13)
$\ln(\delta + g + n_j)$	-1.73	-1.02	-1.06
	(.41)	(.45)	(.33)
$\ln s_{H,j}$	.66	.47	.70
	(.07)	(.07)	(.13)
Adj R^2	.78	.65	.60
Implied β	.30	.31	.36
Implied α	.28	.22	.26
Obs	98	98	107

Cross Country Evidence (8)

- Victory for the model! Three quarters of international differences in income per capita can be explained by difference sin their physical and human capital investment
- Immediate implication is TFP has a somewhat limited role.
- But....

Pitfalls

- Technology is not orthogonal. $\bar{A}_j$ may be correlated with $s_{K,j}$ and $s_{H,j}$:
  - Societies with high $\bar{A}_j$ are probably ones that have invested in technology in the past. Presumably did the same for K and H.
  - Technologies with high $\bar{A}_j$ will find it more beneficial to increase stocks of K and H.
  - Biases upwards estimates of α and β. And R^2.
- α is too large:
  - In the data, $s_{H,j}$ ranges from 0.4% to 12% - measured as working age population enrolled in school
  - Predicted log income differences between these two countries is $\frac{\alpha}{1-\alpha-\beta}(\ln 12 - \ln 0.4) = 0.66 \times (\ln 12 - \ln 0.4) = 2.24$
  - So country should be $\exp(2.24) = 8.5$ times richer
  - Is that plausible?

Pitfalls (2)

- Microeconomic wage regressions suggest the (private) returns to education are between 0.06 and 0.10.
- Much too small to explain a factor of 8.5 using such observed differences in $s_{H,j}$.
- Either there are human capital externalities
- Or α and β are overestimated.

Level Accounting

- Consider a simplified production function:

$$Y_j = K_j^{1-\alpha} \; (A_j \; H_j)^{\alpha}$$

- Get cross country information on K_j (permanent inventory method) and H_j (years of schooling multiplied by labour input). Choose $\alpha = 1/3$.
- Construct "predicted" income at a point in time using:

$$\hat{Y}_j = K_j^{1/3} \; (A_{US} \; H_j)^{2/3}$$

- A_{US} is computed so that $\hat{Y}_{US} = K_{US}^{1/3} \; (A_{US} \; H_{US})^{2/3}$.
- Then one can compare each $\hat{Y}_j$ with the actual series. Gap between the two represents the contribution of technology.

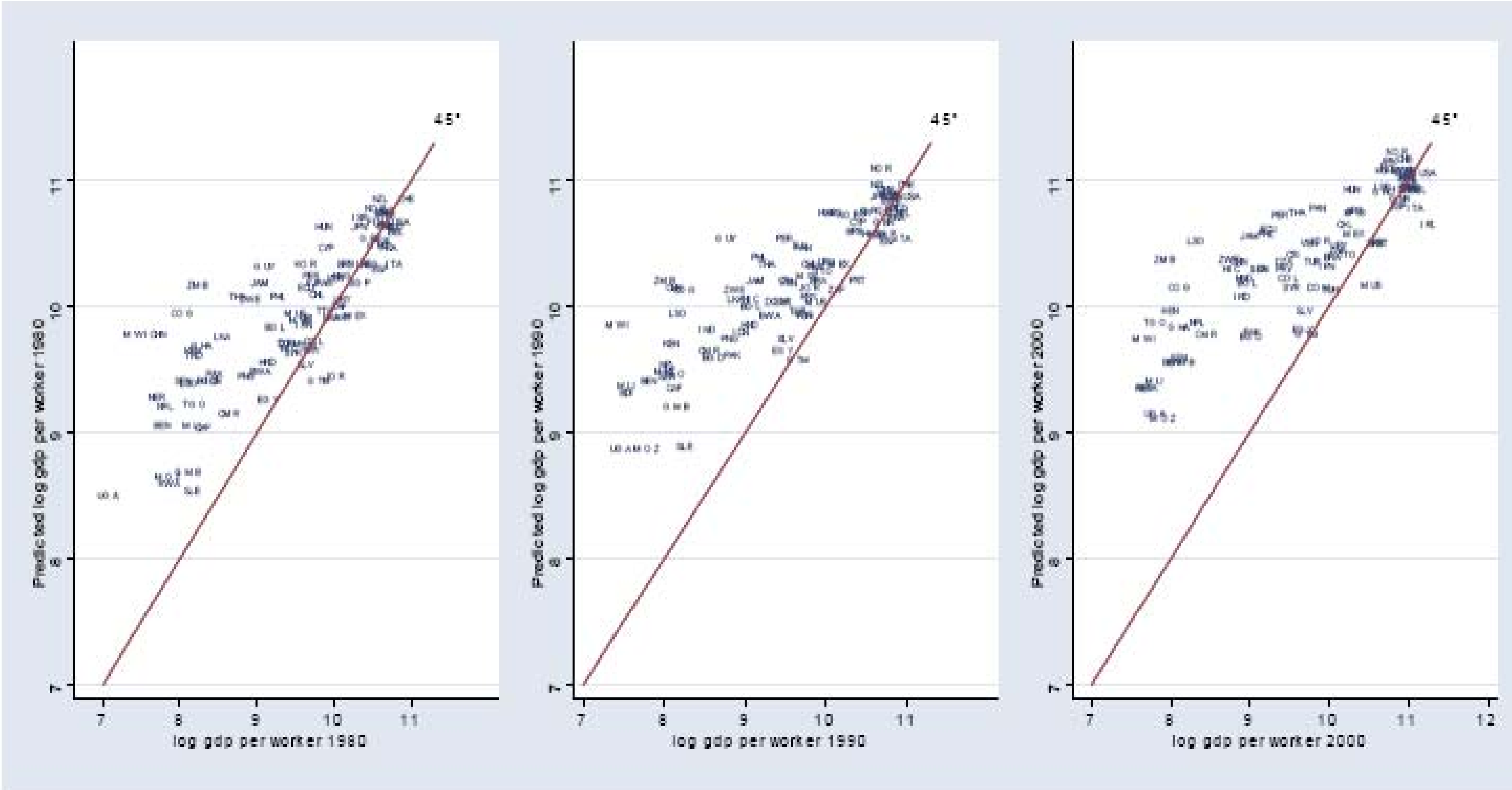

Figure: Calibrated technology levels relative to the US technology (from the Solow growth model with human capital) versus log GDP per worker, 1980, 1990 and 2000.

Level Accounting - Results

- Differences in physical and human capital still matter a lot
- However, unlike what was suggested by regression analysis, also shows significant technology (productivity) differences
- Fit of the model seems to deteriorate over time.
- Based on many assumptions: Cobb-Douglas, identical factor shares, perfect competition, no human capital externalities, range of assumptions to compute K and H.

Conclusions

- Evidence in favour of convergence overwhelming - even with econometric caveats. Supports diminishing marginal returns, i.e. a steady state
- If so, policy recommendations clear:
  - Saving, Investment, Human Capital have only level effects, not growth effects. Consistent with level accounting.
  - Long Run growth derives from TFP growth. About which model is silent.
  - Does NOT mean one "should" target golden rule saving level. No optimal consumption/investment decision.
- Need for a theory of TFP (not necessarily only "technology" - institutions maybe as well)
- Need for a framework with optimal, forward looking behaviour.
- Take these two points in turn

Two Items

- Optimal behaviour. Key for policy analysis. Demographics, taxation policy, institutions surely all affect individual behaviour. And Consumption/Saving decision.
- Develop theories of endogenous growth

Dynamic Optimization in Continuous Time

- Problem writes

$$\begin{aligned} Max \;\; V_t &= \int_{s=0}^{\infty} \exp(-\rho s)\, U(C_{t+s})\, ds \\ s.t \;\; \dot{K}_t &= (R_t - \delta) K_t + w_t L_t - C_t \end{aligned}$$

- Introduce the Hamiltonian

$$H(K_t, C_t) = \exp(-\rho t)\, U(C_t) + \lambda_t \left[(R_t - \delta) K_t + w_t L_t - C_t \right]$$

- An optimum must verify:

$$\begin{aligned} \frac{dH(K_t, C_t)}{dC_t} &= 0 \\ \frac{dH(K_t, C_t)}{dK_t} &= -\dot{\lambda}_t \end{aligned}$$

Dynamic Optimization in Continuous Time

- Tranversality condition:

$$\lim_{T\to\infty} K_T \exp\left(-\int_0^T R_s ds\right) = 0$$

- The two conditions rewrite

$$\begin{aligned} \lambda_t &= \exp(-\rho t)\, U'(C_t) \\ -\frac{\dot{\lambda}_t}{\lambda_t} &= R_t - \delta \end{aligned}$$

- Differentiate the first expression with respect to time, and divide through by λ_t:

$$\frac{\dot{\lambda}_t}{\lambda_t} = -\rho + \frac{U''(C_t)}{U'(C_t)} \dot{C}_t = -(R_t - \delta)$$

Dynamic Optimization in Continuous Time

- Rearrange:

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\varepsilon_U(C_t)} (R_t - \delta - \rho)$$

with $\varepsilon_U(C_t) = -\frac{U''(C_t)C_t}{U'(C_t)}$, the elasticity of marginal utility.

- Consumption grows as long as return on assets larger than discount rate. If so, save today and consume more tomorrow.
- Also related to $\varepsilon_U(C_t)$. Regulates willingness to substitute consumption over time.
- In fact, one can show that the intertemporal elasticity of substitution $\sigma_U(t) = 1/\varepsilon_U(C_t)$.

Dynamic Optimization in Continuous Time

- Need to solve for R_t. With competitive factor markets,

$$R_t = \frac{dF\left(K_t, A_t\right)}{dK_t} = \frac{dF\left(\frac{K_t}{A_t}, 1\right)}{dK_t} = f'(k_t)$$

- (where we made use of theorem that $F(.)$ homogeneous of degree 1 implies $F'(.)$ homogeneous of degree 0.)
- So:

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\varepsilon_U(C_t)}\left(f'(k_t) - \delta - \rho\right)$$

Steady State and Normalization

- k_t is normalized. But not C_t. Define $c_t = \frac{C_t}{A_t}$. Then, use product rule:

$$\frac{\dot{C}_t}{C_t} = \frac{\frac{\dot{C}_t}{A_t}}{c_t} = \frac{\dot{c}}{c} + g$$

- So finally:

$$\frac{\dot{c}}{c} = \frac{1}{\varepsilon_U(C_t)}[f'(k_t) - \delta - \rho - g\varepsilon_U(C_t)]$$

- Now everything is normalized, EXCEPT $\varepsilon_U(C_t)$. Define "balanced growth": corresponds to Kaldor fact that growth is stationary in the long run.
- Balanced growth is only possible with utilities that have constant $\varepsilon_U(C_t)$. Most prominent example is CRRA:

$$U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}$$

- Here, the elasticity of intertemporal substitution σ is constant, and equal to the inverse of the coefficient of relative risk aversion.

Steady State

- CRRA case:
$$\frac{\dot{c}}{c} = \frac{1}{\varepsilon_U}[f'(k_t) - \delta - \rho - g\varepsilon_U] \tag{1}$$
- High values for ε_U (i.e. low intertemporal elasticities of substitution) may prevent growth. This will happen if agents are totally unwilling to substitute consumption between periods. Positive consumption growth requires relatively small ε_U, i.e. some willingness to save and consume later.
- Key is that preference parameter now governs consumption-saving decisions. In the end, will determine equilibrium capital-labour ratio and ultimately output per capita.

Steady State

- In addition, the (normalized) dynamics of capital are given by

$$\frac{\dot{K}_t}{A_t} = \frac{R_t K_t + w_t L_t}{A_t} - \delta k_t - c_t$$

- Under perfect competition, factor payments exhaust all output, i.e. $R_t K_t + w_t L_t = Y_t$, and

$$\frac{\dot{K}_t}{A_t} = f(k_t) - \delta k_t - c_t$$

- Finally, using once again the product rule,

$$\dot{k}_t = f(k_t) - (\delta + g)\, k_t - c_t \tag{2}$$

Steady State

- Equations (1) and (2) define the economy's optimal dynamics.
- Notice equation (2) very similar to case with exogenous savings. Difference is that c_t is now endogenous, and governed by equation (1), i.e. optimal intertemporal choice of consumption vs. saving.
- Where does this economy converge to? A steady state defined by $\dot{c} = 0$ and $\dot{k}_t = 0$.
- We must have a steady state capital-labour ratio k^* such that

$$f'(k^*) = \delta + \rho + g\varepsilon_U$$

- This pins down steady state capital-labour ratio. Note this does NOT maximize steady state consumption, which would verify

$$f'(k^*) = \delta + g$$

- But still, this is an OPTIMAL capital-labour ratio. Because earlier consumption is preferred to later consumption. A modified Golden Rule.
- Note again importance of preference parameter: steady state capital-labour ratio decreases in ε_U, i.e. increases in elasticity of intertemporal substitution.

Steady State

- Given k^*, c^* must verify

$$c^* = f(k^*) - (\delta + g)\, k^*$$

- Kaldor facts are verified at the steady state:
  - stationary growth rate: c^*constant means C/L grows with A.
  - constant interest rate $(f'(k^*))$
  - k^* constant means K/L grows with A
  - factor shares constant. a bit harder to see, but will come from usual assumption on perfect competition.
- Both k^* and c^* fall with ρ. Lower discount rate means more patience and thus greater savings.
- Transition to that steady state can be computed by computing the derivatives of $\dot{c}_t$ and $\dot{k}_t$ with respect to the levels of k and c.

Policy Analysis

- Growth of per capita consumption and output are determined exogenously (g)
- But level of income depends is endogenous: depends on ε_U, ρ, δ, n and $f(.)$
- So differences in income per capita explained by differences in these parameters. Could intertemporal elasticity of substitution or discount rate vary with cultural or geographic factors?
- But an explanation for cross-country or over-time differences in economic growth based on differences in preferences is unlikely to be very satisfactory.
- More appealing: link incentives to accumulate physical capital (and human capital and technology) to the environment.

Policy Analysis

- Simplest way: differences in policies.
- Introduce linear tax policy on capital, at rate τ. Proceeds are rebated to consumers.
- Capital accumulation remains as above:

$$\dot{k}_t = f(k_t) - (\delta + g + n)\, k_t - c_t$$

- But now interest rate is given by $(1-\tau)\left(f'(k_t) - \delta\right)$
- So growth rate of (normalized) consumption is given by

$$\frac{\dot{c}}{c} = \frac{1}{\varepsilon_U}\left[(1-\tau)\left(f'(k_t) - \delta\right) - \rho - g\varepsilon_U\right]$$

- Steady state capital per capita:

$$f'(k^*) = \delta + \frac{\rho + g\varepsilon_U}{1-\tau}$$

- Higher taxes, since $f'(.)$ is decreasing, decreases k^*. They depress capital accumulation and reduce income per capita.

Policy Analysis

- Can we explain differences in income per capita across countries $j = 1...N$ by differences in policy?
- Impose specific production function

$$Y_j = K_j^{1-\alpha} (AH_j)^{\alpha}$$

H is exogenously given stock of effective labour (human capital)

- Then impose a tax rate τ_j that can vary across countries. Model it as $\frac{1}{1+\tau_j}$ for convenience.
- Then

$$\frac{\dot{c}_j}{c_j} = \frac{1}{\varepsilon_U}[\frac{1}{1+\tau_j}(1-\alpha)\left(\frac{AH_j}{K_j}\right)^{\alpha} - \delta - \rho]$$

where all consumers are assumed to have the same ε_U and A is assumed constant (hence no g)

Policy Analysis

- A steady state verifies $\frac{\dot{c}_j}{c_j} = 0$, or

$$K_j = \frac{(1-\alpha)^{1/\alpha} AH_j}{\left[(1+\tau_j)(\delta+\rho)\right]^{1/\alpha}}$$

- Countries with higher investment tax have lower capital stock, lower capital per worker and lower capital output ratio.
- Then substituting into Y_j, and comparing two countries with different taxes (but the same human capital):

$$\frac{Y(\tau)}{Y(\tau')} = \left(\frac{1+\tau'}{1+\tau}\right)^{\frac{1-\alpha}{\alpha}}$$

- Countries with high τ are poorer. Importantly, this incorporates the endogenous effects different distortions have on income and capital accumulation.

Policy Analysis

- $\alpha = 2/3$ is plausible.
- What are plausible differences in τ? Use international differences in the relative price of equipment goods: using Penn World Tables, ranges from 8 to 1.
- So model can explain

$$\frac{Y(\tau)}{Y(\tau')} = 8^{1/2} \simeq 3$$

- Enormous differences in taxes explain very little

Policy Analysis

- Parallels discussion of Mankiw-Romer-Weil. Need to account for differences in H_j.
- Many economists have tried (and still do) to salvage this version of the neo-classical model. Simple motivation: instead of using $\alpha = 2/3$, use $\alpha = 1/3$, then

$$\frac{Y(\tau)}{Y(\tau')} = 8^2 \simeq 64$$

- Need to increase the responsiveness of capital (or other distorted factors)
- One way is to introduce other accumulated factors, while still keeping the share of labour income in national product around $2/3$.
- Human capital will do it (an accumulated factor that enters labour income). Or introduce other types of capital - or indeed technology - responding to distortions like K does.

Dynamic Optimization - Summary

- Major contribution: opens the black box of capital accumulation
- Can draw OPTIMAL implications
- Does this generate new insights about the source of cross-country income differences? Largely, no.
- But model clarifies the nature of the economic decisions, so that we are in a better position to ask questions about fundamental causes of economic growth.
- Now, think about productivity growth in an optimal framework

Productivity and Human Capital Accumulation

- Focus on a model without exogenous technological progress, but endogenous acquisition of education.
- Complementarities between education and physical capital? Evidence that more capital increases productivity of high skill workers more than of low skilled.
- Virtuous circle of investments in physical and human capital?
- But also raises the issue of "imbalances": highest productivity when there is a balance between the two types of capital. Will a decentralized equilibrium ensure this balance?

Productivity and Human Capital Accumulation

- Consider following problem

$$\begin{aligned} Max \ V_0 &= \int_0^\infty \exp(-\rho t) U(C_t) dt \\ Y_t &= F(K_t, H_t, L) \end{aligned}$$

No population growth, no technological progress, labour supplied inelastically, i.e. $L_t = L$.

- Let $k_t = \frac{K_t}{L}$, $h_t = \frac{H_t}{L}$, and $y_t = \frac{Y_t}{L} = f(k_t, h_t)$.
- Importantly, assume complementarities between H and K, which means $\frac{df_K(k,h)}{dh} > 0$ and $\frac{df_H(k,h)}{dk} > 0$.
- Laws of motion

$$\begin{aligned} \dot{k}_t &= i_{Kt} - \delta_K k_t \\ \dot{h}_t &= i_{Ht} - \delta_H h_t \end{aligned}$$

- And finally a resource constraint:

$$c_t + i_{Kt} + i_{Ht} = f(k_t, h_t)$$

- Rewrite the Hamiltonian:

$$
\begin{aligned}
H &= \exp(-\rho t) U (f(k_t, h_t) - i_{Kt} - i_{Ht}) \\
&\quad + \lambda_{Ht} [i_{Ht} - \delta_H h_t] + \lambda_{Kt} [i_{Kt} - \delta_K k_t]
\end{aligned}
$$

- The optimality conditions are:

$$
\begin{aligned}
H_{i_K} &= -\exp(-\rho t) U'(c_t) + \lambda_{Kt} = 0 \\
H_{i_H} &= -\exp(-\rho t) U'(c_t) + \lambda_{Ht} = 0 \\
H_K &= f_K(k_t, h_t) \exp(-\rho t) U'(c_t) + \lambda_{Kt} \delta_K \\
&= -\dot{\lambda}_{Kt} \\
H_H &= f_H(k_t, h_t) \exp(-\rho t) U'(c_t) + \lambda_{Ht} \delta_H \\
&= -\dot{\lambda}_{Ht}
\end{aligned}
$$

Productivity and Human Capital Accumulation

- The first two conditions immediately imply that $\lambda_{Kt} = \lambda_{Ht} = \lambda_t$
- Combine this in the next two conditions to get

$$f_K(k_t, h_t) - f_H(k_t, h_t) = \delta_K - \delta_H$$

- Remarkable result: the gap between the two marginal products is a parametrized constant. And this holds everywhere in equilibrium.
- Now consider what happens if (say) k is shocked upwards. Then f_K falls, BUT f_H increases, because we assumed complementarities.
- To preserve equilibrium, must be the case that h increases along with k. Only way for f_H to fall in line with f_K.
- Virtuous circle we mentioned. In equilibrium H and K change hand in hand.

Productivity and Human Capital Accumulation

- Surprising: human and physical capital are always in "balance"
- May have conjectured that economy starting with high stocks of K relative to H will have relatively high K/H ratio for an extended period of time.
- Here, at the first instant this economy will experience a very high level of i_{H0}, compensated by a negative i_{K0}.
- After that, this economy is exactly identical to neoclassical model, since K and H are always moving hand in hand. They are not two separate concepts.
- Of course, if we impose $i_{Ht} \geq 0$ and $i_{Kt} \geq 0$, initial imbalances will persist for a while. But the economy sooner or later moves towards correcting this imbalance.

Productivity and Human Capital Accumulation - Policy?

- Introduce a tax on investment - affecting both types:

$$c_t + (1+\tau)(i_{Kt} + i_{Ht}) = f(k_t, h_t)$$

- Then simplify output into

$$Y_t = K_t^{\alpha_K} H_t^{\alpha_H} L^{1-\alpha_K-\alpha_H}$$

- By analogy, in equilibrium we get

$$\frac{Y(\tau)}{Y(\tau')} = \left(\frac{1+\tau'}{1+\tau}\right)^{\frac{\alpha_K+\alpha_H}{1-\alpha_K-\alpha_H}}$$

- With $\alpha_K = \alpha_H = 1/3$,

$$\frac{Y(\tau)}{Y(\tau')} = 8^2 = 64$$

- Responsiveness of human capital to distortion increases the impact of distortions.

Productivity and Human Capital Accumulation - Policy?

- BUT: driven by very elastic response of human capital accumulation. Are we talking about corporate taxes or corruption? why does this affect so significantly individual human capital decisions?
- Obvious parallel to Mankiw Romer Weil and the difficulties there.

Endogenous Technological Progress

- Difficult to model the process of knowledge accumulation in a competitive economy?
- Why? Schumpeter: incentive to conduct R&D requires property right on fruits of innovation.
- Innovation necessitates (risky) investment upfront, and non-rival, non-excludable innovation subsequently.
- Of course, patents grant (temporary?) property rights - but still model of endogenous innovation requires some monopoly power.
- Absent from perfectly competitive models.
- First shortcut: knowledge accumulation as a byproduct of capital accumulation

Endogenous Technological Progress - Externalities

- Introduce "externalities" of capital accumulation.
- No population growth (we will see why this is important)
- Rather than an aggregate function, assume production side of the economy consists of a set [0,1] of firms.
- Production in each firm $i \in [0, 1]$ is

$$Y_{it} = F\left(K_{it}, A_t L_{it}\right)$$

- K_{it} and L_{it} are capital and labor rented by firm i. Technology A_t on the other hand is common to all firms.

Endogenous Technological Progress - Externalities

- Market clearing conditions:

$$\int_0^1 K_{it} di = K_t \text{ and } \int_0^1 L_{it} di = L$$

- L is the constant level of labor supplied inelastically in this economy
- We have perfect competition in all markets, so that wages and interest rates must be equated across firms:

$$w_t = \frac{dF\left(K_{it}, A_t L\right)}{dL} \text{ and } R_t = \frac{dF\left(K_{it}, A_t L\right)}{dK_{it}}$$

- KEY assumption: firms take A_t as given, but this stock of technology advances endogenously for the economy as a whole.
- Extreme assumption: sufficiently strong externalities that A_t grows continuously at the economy level. In particular

$$A_t = BK_t$$

Endogenous Technological Progress - Externalities

- A way to model "learning by doing". Could also be a function of cumulated output up to now. Since all firms hire capital and labour to exactly the same extent, the production of the representative firm is governed by

$$Y_t = F\left(K_t, BK_t L\right)$$

- Rearrange

$$\frac{Y_t}{K_t} = F\left(1, BL\right) = \tilde{f}(L)$$

- So

$$Y_t = \frac{Y_t}{K_t} K_t = \tilde{f}(L) K_t$$

- And

$$w_t = \tilde{f}'(L) K_t$$

Endogenous Technological Progress - Externalities

- Since competition is perfect, there are no profits and factor payments exhaust available income, so that

$$\begin{aligned} Y_t &= \tilde{f}(L)K_t = w_t L + R_t K_t \\ &= \tilde{f}'(L)K_t L + R_t K_t \end{aligned}$$

- Infer from this that

$$R_t = \tilde{f}(L) - \tilde{f}'(L)L = R$$

- Note the return to capital is now CONSTANT - not diminishing as it was in all models up to now.

Endogenous Technological Progress - Externalities

- The model is otherwise standard, and consumption follows an Euler equation of the form

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\varepsilon_U} \left(\tilde{f}(L) - \tilde{f}'(L)L - \delta - \rho \right)$$

- Consumption grows at a CONSTANT rate. Permanent growth - and NOT the result of exogenous technological progress.
- But comes with a scale effect. $\tilde{f}(L) - \tilde{f}'(L)L$ is always increasing in L, by definition. So, WITH population growth, the economy will not admit a steady state, and the growth rate will increase over time (with output reaching infinity in finite time)
- That's why must assume constant population. Embarrassing prediction: GDP in China and India should have grown faster over the whole 20th century...

Externalities - Policy Implications

- This was the equilibrium implied by individual agents' decisions - so called "decentralized equilibrium"
- In the presence of externalities (or indeed of imperfect competition), does not coincide with "first-best" equilibrium, i.e. the so-called "social planner" problem. Decentralized equilibrium is not "Pareto-optimal"
- Social Planner will instead maximize the OVERALL ECONOMY's resources. The capital accumulation rule now becomes

$$\dot{k}_t = \tilde{f}(L)k_t - c_t - \delta k_t$$

rather than

$$\dot{k}_t = w_t L + R_t k_t - c_t - \delta k_t$$

- The second takes factor returns as given - and fails to recognize they are both affected by capital accumulation through the externality. The first, in contrast, internalizes the fact that A_t increases with K_t, and recognizes output is effectively LINEAR in K_t.

Externalities - Policy Implications

- Growth in this setup is given by

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\varepsilon_U}\left(\tilde{f}(L) - \delta - \rho\right)$$

  which is always greater than decentralized outcome.
- The social planner takes into account that by accumulating more capital, she is improving productivity for the future.
- But still a scale effect.
- Notice also this model implies global divergence: an ever expanding world income distribution

Linear Technology - A General Property

- A key feature of the previous model with externalities is the end result that output is linear in the accumulated factor - capital.
- This is a general property of models with endogenous growth - including ones that eschew the assumption of perfect competition.
- For instance, models with increasing number of varieties, invented by a proper, separate R&D sector who invents new varieties. (rather than happening in a "deus ex machina" manner as a by product of capital accumulation)

Models with Expanding Varieties and RD

- Basic setup: Production given by

$$Y_t = \frac{1}{1-\beta}\left[\int_0^{N_t} (x_{it})^{1-\beta}\, di\right] L^{\beta}$$

x_{it} are inputs of type i used in production. There are N_t such inputs.

- Increasing N_t amounts to process innovation.
- These inputs depreciate fully after each period. Not another state variable.
- For a given N_t, production exhibits constant returns to scale, and diminishing returns to L and to x_{it}.

Models with Expanding Varieties and RD

- A resource constraint is given by

$$C_t + X_t + Z_t = Y_t$$

  with X_t investment in inputs, and Z_t investment in R&D
- R&D sector uses this investment to produce new inputs, as

$$\dot{N}_t = \eta Z_t$$

- Crucial (but also complicating considerably the model) is the assumption that R&D is NOT competitive. In particular, they hold a perpetual patent on the variety they invent.

Models with Expanding Varieties and RD

- Easy (if tedious) to show production will be linear in this model. Based on monopoly power of innovator. Final good firms purchase each input at a price that is set by the innovator, to maximize their profit

$$\max_{x_{it}} \left[\int_0^{N_t} (x_{it})^{1-\beta} \, di \right] L^\beta - \int_0^{N_t} p_{it} x_{it} di - w_t L$$

- Profit maximization means their demand function is given by

$$x_{it} = (p_{it})^{-\frac{1}{\beta}} L$$

- Then R&D firms maximize THEIR profits, faced with this demand function:

$$\max_{x_{it}, p_{it}} \int_t^\infty e^{-rs} \left[p_{is} x_{is} - \psi x_{is} \right] ds$$

where ψ denotes the cost of producing an innovation. Note that profits are discounted over the infinite horizon from t onwards. Once an innovation is discovered, the innovator enjoys a perpetual patent.

Models with Expanding Varieties and RD

- This can be shown to imply

$$p_{it} = \frac{\psi}{1-\beta}$$

- All R&D firms charge the same price - i.e. all innovation cost the same. That price increases with the cost of production, and with the market power of the innovator (captured by how substitutable diffent varieties are - i.e. by β).
- Normalizing (arbitrarily - but for the sake of clarity) $\psi = 1 - \beta$, one obtains

$$x_{it} = L$$

- All firms produce the same quantity of their input.

Models with Expanding Varieties and RD

- Substituted in the production function of the final good, this gives

$$Y_t = \frac{1}{1-\beta} L N_t$$

- Production is LINEAR in the accumulated factor - which here is the number of varieties. This is how long run growth is possible.
- Just as in the previous case, the "decentralized equilibrium" will not be optimal.
- Why? Monopoly power. R&D firms do not produce enough once they found a new variety, to maintain their profits at a high level. So the usage of varieties is sub-optimal, and so is growth.
- There is an externality here as well, in that R&D firms do not internalize the effect their innovations have on the economy at large. They just maximize their own profits. A "pecuniary" externality.

Models with Expanding Varieties and RD

- Policy implications:
  - Subsidy to capital inputs: can alleviate the pecuniary externality.
  - Subsidy to research. Useful for instance if innovation has spillover effects - "standing on the shoulder of giants". In terms of the model, this would mean

$$P\dot{N}_t = \eta Z_t L_{RD}$$

    i.e. introduce another externality via knowledge spillovers.

- Note both can be (and are often) financed via a distortionary tax and may defeat the initial purpose. The manner in which these subsidies are financed of course not part of the model

Models with Expanding Varieties and RD

- Just as in the previous case, there will be a scale effect. Here, countries with "more scientists" (more labor allocated to the R&D sector) will grow permanently faster. Empirically invalidated - number of scientists has soared in OECD - not growth rates.
- Why are increases in the level of R&D not having growth effects?
- Many recent efforts in that direction. Main idea is to introduce diminishing returns to research

$$\dot{N}_t = \eta Z_t L_{RD}^{\phi} \text{ with } \phi < 1$$

Models with Schumpeterian Creative Destruction

- The same setup, but now new varieties can be of higher quality, and thus displace earlier innovations.
- Modify production function into

$$Y_t = \frac{1}{1-\beta}\left[\int_0^1 q_{it}\left(x_{it}\right)^{1-\beta} di\right] L^{\beta}$$

  where q_{it} is the quality of machine (input) i at time t. Note the integral is now between 0 and 1.
- Following the same reasoning, one can show that demand by final good firms is given by

$$x_{it} = \left(\frac{p_{it}}{q_{it}}\right)^{-\frac{1}{\beta}} L$$

Models with Schumpeterian Creative Destruction

- Profit maximizing prices set by innovators are given by

$$p_{it} = \frac{\psi}{1-\beta} q_{it}$$

  As before, prices increase with costs and with market power - and they also increase with the quality of the variety.Finally, the same intuition gives production

$$Y_t = \frac{1}{1-\beta} L Q_t$$

  with $Q_t = \int_0^1 q_{it} di$ a measure of the overall quality in the economy.
- Same predictions on long run growth, and on scale effects.
- Same property that decentralized equilibrium is not optimal. As before, innovators fail to capture the social benefits of their innovations because innovation is cumulative. Higher overall quality can be used by subsequent innovators.

Models with Schumpeterian Creative Destruction

- But in addition, an interesting political implication. In this model, only entrants innovate - incumbents have no incentive to do so, since the profits arising from their previous innovation go to zero when it becomes displaced by a better one. (Arrow's Replacement Effect)
- So the model predicts that incumbents will actually support a tax on R&D, so that their innovation get displaced to a lesser extent.
- This happens despite the fact that a tax on R&D generally lowers growth rate. There may be a constituency supporting a tax on R&D.