Session 3: Investment

Jean Imbs

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Investment

- Dynamics mattered for Consumption
- Clearly they will equally matter for Investment. Firms invest (i.e. increase their productive capacities) in anticipation of future economic conditions - if only because it takes time for investment to become productive.
- Firms do not only look at current market conditions when they purchase new machines - they also form forecasts as to future conditions.
- In addition, it is likely increasing the stock of capital (i.e. equipment) is costly. Firms buy machines progressively - rarely all at once. There are "adjustment costs" to investment - unlike what we had for Consumption.
- Now put all these ingredients in models of Investment.

Plan

- The Standard Model (Jorgensen, 1960)
- The Dynamic Model I: Lagrangean Approach (discrete time)
- The Dynamic Model II: Bellman equation (discrete time)
- The Dynamic Model III: Hamiltonian (continuous time)
- Some Policy Analysis

The Standard Model

- Define r_t^K the "rental rate of capital" - i.e. the instantaneous costs of renting capital, or the "user cost" of capital.
- A representative firm's profits are given by

$$\pi_t = \pi(K_t) - r_t^K K_t$$

with $\pi'(K_t) > 0$ and $\pi''(K_t) < 0$. Note we take other inputs in production (labor, skills) as given, and focus on the choice of the stock of capital - namely investment.

- Optimal (static) choice of capital verifies:

$$\pi'(K_t) = r_t^K$$

Rent capital up to the point where the marginal product of capital $\pi'(K_t)$ equals its (constant, exogenous) rental / user cost.

Comparative Statics

- Comparative Statics fundamentally differ from dynamics. Here we ask how the optimal choice of capital changes in response to changes in its cost. (VERY different from asking how the dynamics of investment change as the cost of investment varies).
- Still take a total differentiation of the first order condition:

$$\pi''(K_t)\ \partial K_t = \partial r_t^K$$

$$\text{i.e.} \quad : \quad \frac{\partial K_t}{\partial r_t^K} = \frac{1}{\pi''(K_t)} < 0$$

- Desired capital stock falls as the rental cost of capital increases. Firms invest less. But that's only in response to CURRENT rental price.
- Surely missing some realism

The User Cost of Capital

- Suppose there is a market for capital, and its price is p_t^K
- Keeping one unit of capital has three costs for the firm:
  - An opportunity cost, i.e. the return it could earn by investing the amount immobilized in investment, $r_t p_t^K$
  - A depreciation cost, i.e. the amount that must be put aside to cover depreciated capital, δp_t^K
  - Capital gains or losses, i.e. a change in the price of capital. User cost will decrease if $\dot{p}_t^K > 0$
- NB: $\dot{X}_t = \partial X_t / \partial t$. And $\frac{\dot{X}_t}{X_t}$ is the % change in X, measured over an infinitesimal period.
- So user cost is given by

$$r_t^K = \left(r_t + \delta - \frac{\dot{p}_t^K}{p_t^K} \right) p_t^K$$

Limitations

- First shortcoming is this is static. Firm only consider CURRENT interest rate, and more generally current economic conditions when making an investment decision.
- Second, this implies investment that is very volatile. Suppose for instance monetary policy lowers r_t. Ceteris paribus, this means r_t^K falls, and according to comparative statics, K_t must increase instantaneously. This, in turn, means investment is positive and large. In other words, investment moves one for one with monetary policy - and presumably also with anything that moves interest rates.
- Now investment IS volatile - but not to such an extent. In particular, investment moves a lot when it does - but it does not move that often. Consistent with adjustment costs.
- Now turn to a DYNAMIC model with adjustment costs.

Dynamics Part 1

- Introduce adjustment costs so that each unit of investment imposes an additional resource of $\frac{1}{2}\phi\frac{i_t}{k_t}$, with $\phi > 0$. More costly to invest a large amount *relative to the size of existing capital stock.*
- Output is either consumed or invested - i.e. savings that we talked about last session now have a purpose. The resource constraint for the economy is

$$y_t = F(k_t) = c_t + (1 + \frac{\phi}{2}\frac{i_t}{k_t})i_t$$

- Note this assumes the price of capital $p_t^K = 1$.

Dynamics Part 1

- Plug this into a dynamic problem:

$$
\begin{aligned}
\mathcal{L}_t &= \sum_{s=0}^{\infty}\{\beta^s U(c_{t+s}) + \lambda_{t+s}[F(k_{t+s}) - c_{t+s} - i_{t+s} - \frac{\phi}{2}\frac{i_{t+s}^2}{k_{t+s}}] \\
&\quad + \mu_{t+s}[i_{t+s} - k_{t+s+1} + (1-\delta)k_{t+s}]\}
\end{aligned}
$$

- λ_{t+s} Lagrange multiplier associated with the resource constraint. μ_{t+s} Lagrange multiplier associated with the capital accumulation rule.

Dynamics Part 1

- First Order Conditions are

$$
\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial c_{t+s}} &= \beta^s U'(c_{t+s}) - \lambda_{t+s} = 0 \\
\frac{\partial \mathcal{L}_t}{\partial i_{t+s}} &= -\lambda_{t+s}(1 + \phi \frac{i_{t+s}}{k_{t+s}}) + \mu_{t+s} = 0 \\
\frac{\partial \mathcal{L}_t}{\partial k_{t+s}} &= \lambda_{t+s}\left[F'(k_{t+s}) + \frac{\phi}{2}\left(\frac{i_{t+s}}{k_{t+s}}\right)^2 \right] \\
& \quad -\mu_{t+s-1} + (1-\delta)\mu_{t+s} \\
&= 0
\end{aligned}
$$

Tobin's q, Part 1

- Simply rearranging the FOC for investment gives

$$i_{t+s} = \frac{1}{\phi}(q_{t+s} - 1)k_{t+s}$$

  where we defined $q_{t+s} = \frac{\mu_{t+s}}{\lambda_{t+s}}$. That's usually called Tobin's q.
- Notice first that positive investment will take place in period $t + s$ provided $q_{t+s} > 1$.
- When $q_{t+s} < 1$ firm wants to let its capital stock fall - "disinvest".

Tobin's q, Part 1

- What is q?
- μ_{t+s} is shadow price of relaxing capital accumulation. Denotes the marginal utility that comes from an extra unit of investment (that boosts capital stock, and so output and so utility)
- λ_{t+s} is shadow price of relaxing resource constraint. Denotes the marginal disutility at time $t+s$ that comes from installing another unit of invesment
- So q captures benefit from investment per unit of benefit of capital (i.e. current consumption). Both are expressed in terms of utility (and in marginal terms). Makes sense that you should invest if (marginal) benefit of investment exceeds (marginal) benefit of capital.
- Expressing utility in terms of output, q can also be interpreted as the ratio of the market value of one unit of investment relative to its cost.

Tobin's q, Part 1

- Can complete the equilibrium using the other FOC, to get:

$$F'(k_{t+s}) + \frac{\phi}{2}\left(\frac{1}{\phi}(q_{t+s}-1)\right)^2 = \frac{\mu_{t+s-1}}{\lambda_{t+s-1}}\frac{\lambda_{t+s-1}}{\lambda_{t+s}} - (1-\delta)\frac{\mu_{t+s}}{\lambda_{t+s}}$$

- ie

$$F'(k_{t+1}) = q_t\frac{U'(c_t)}{\beta U'(c_{t+1})} - (1-\delta)q_{t+1} - \frac{1}{2\phi}(q_{t+1}-1)^2$$

- Equilibrium for c_t, k_t, i_t and q_t then pinned down by this previous equation, along with:

$$\begin{aligned} k_{t+1} &= i_t + (1-\delta)k_t \\ F(k_t) &= c_t + (1+\frac{\phi}{2}\frac{i_t}{k_t})i_t \\ i_t &= \frac{1}{\phi}(q_t-1)k_t \end{aligned}$$

Steady State

- Focus on the long-run solution, i.e. parameter values for which all variables are constant.
- Clearly,

$$\frac{i}{k} = \frac{1}{\phi}(q - 1) \equiv \gamma$$

- So the long run value for q is:

$$q = \gamma\phi + 1$$

- With no installation costs, the steady state value of q is 1.

Tobin's q - Part 2

- Consider now the Bellman equation approach to solving the same problem. Define the value function

$$\begin{aligned} V\left(K_t\right) &= Max_{\{I_t\}_0^\infty} \sum_{t=0}^{\infty} \frac{1}{(1+r)^t}\left[\pi\left(K_t\right) - I_t - c(I_t)\right] \\ \text{subject to } K_{t+1} &= (1-\delta)K_t + I_t \end{aligned}$$

  where $\pi\left(K_t\right)$ is a profit function, $\pi'\left(K_t\right) > 0$, $\pi''\left(K_t\right) < 0$
- $c(I_t)$ is an adjustment cost, akin to what we had before. Assume $c(0) = 0$, $c'(I_t) > 0$ and $c''(I_t) > 0$.

Tobin's q - Part 2

- The Bellman equation writes

$$V\left(K_t\right) = Max_{(I_t)}\left\{\pi\left(K_t\right) - I_t - c(I_t) + \frac{1}{1+r}V\left[(1-\delta)K_t + I_t\right]\right\}$$

- The first-order condition writes

$$-1 - c'(I_t) + \frac{1}{1+r}V'\left[K_{t+1}\right] = 0$$

- The envelope theorem tells us that

$$\frac{\partial V\left(K_t\right)}{\partial K_t} = V'\left(K_t\right) = \pi'\left(K_t\right) + \frac{1-\delta}{1+r}V'\left[K_{t+1}\right]$$

Tobin's q - Part 2

- Now what is $V'\left[K_{t+1}\right]$? Use the FOC to get that:

$$\begin{aligned} \frac{1}{1+r} V'\left[K_{t+1}\right] &= 1 + c'(I_t) \\ \text{i.e. } \frac{1-\delta}{1+r} V'\left[K_{t+1}\right] &= (1-\delta)\left[1 + c'(I_t)\right] \end{aligned}$$

- Substitute back in the expression implied by the Envelope Theorem

$$V'\left(K_t\right) = \pi'\left(K_t\right) + (1-\delta)\left[1 + c'(I_t)\right]$$

- Update by one period:

$$V'\left(K_{t+1}\right) = \pi'\left(K_{t+1}\right) + (1-\delta)\left[1 + c'(I_{t+1})\right]$$

- And finally substitute back in the FOC:

$$\frac{1}{1+r}\pi'\left(K_{t+1}\right) + \frac{1-\delta}{1+r}\left[1 + c'(I_{t+1})\right] = 1 + c'(I_t)$$

- This characterizes the dynamics of investment (and capital) in this model.

Tobin's q - Part 2

- Now we had defined Tobin's q as the utility value of increasing investment today, relative to the foregone unit of capital. What is the equivalent here? Define

$$q_t = \frac{1}{1+r} V' \left[K_{t+1} \right]$$

- Recall the envelope theorem tells us that

$$V' \left(K_t \right) = \pi' \left(K_t \right) + \frac{1-\delta}{1+r} V' \left[K_{t+1} \right]$$

- Recursively, this implies

$$V' \left(K_t \right) = \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s \pi' \left(K_{t+s} \right)$$

Tobin's q - Part 2

- So in particular,

$$q_t = \frac{1}{1+r} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r} \right)^s \pi' \left(K_{t+s+1} \right)$$

- This reflects the marginal increase in future utility (profits, here) that comes from increasing the stock of capital by one unit today, i.e. from investing one unit today. This is measured relative to the foregone costs of doing so today, i.e. losing interest r. In other words, it is exactly the same definition.

Tobin's q - Part 2

- What is q here? The First Order Condition implies:

$$q_t = 1 + c'(I_t)$$

- Since $c''(.) > 0$, there exists an inverse function for $c'(.)$, defined as $c'^{-1}(.)$, which is monotonic and increasing. And we have

$$I_t = c'^{-1}(q_t - 1)$$

As in the Lagrangean case, investment increases monotonically in q_t, and it will be positive for $q_t > 1$. The intuition is the same: invest more whenever the utility (marginal) gain from investment exceeds its capital (marginal) cost.

Tobin's q - Part 2

- What can we do with the First Order Condition in closed form we obtained? Rewrite

$$\begin{aligned} \pi'(K_{t+1}) &= (1+r)\, q_t - (1-\delta)\, q_{t+1} \\ &= r q_t + \delta q_{t+1} - (q_{t+1} - q_t) \end{aligned}$$

- Corresponds to the three user costs of capital:
  - rq_t is the opportunity cost of investing q_t
  - δq_{t+1} is the depreciation cost that will be incurred next period
  - $(q_{t+1} - q_t)$ are capital gains
- Stands to reason that marginal benefits of investment should be equal to its marginal costs.

Part 3: Dynamic Programming in Continuous Time

- Introduce the Hamiltonian. Consider the standard problem

$$
\begin{aligned}
& Max_{\{u_t, x_t\}_0^T} \int_o^T J(x_t, u_t)\, dt \\
\text{subject to } \dot{x}_t &= A(x_t, u_t)
\end{aligned}
$$

- Write the Lagrangean:

$$\mathcal{L}_t = \int_o^T \{J(x_t, u_t) + \lambda_t [A(x_t, u_t) - \dot{x}_t]\}\, dt$$

- The usual procedure involves computing $\frac{\partial \mathcal{L}_t}{\partial u_t}$ and $\frac{\partial \mathcal{L}_t}{\partial x_t}$. The problem is that we have a term in $\dot{x}_t$, which is difficult to differentiate with respect to x_t.

Part 3: Dynamic Programming in Continuous Time

- Trick is to integrate by parts. In particular:

$$-\int_o^T \lambda_t \dot{x}_t dt = -\left[\lambda_T x_T - \lambda_0 x_0 - \int_o^T \dot{\lambda}_t x_t dt\right]$$

- Thus

$$\mathcal{L}_t = \int_o^T \left\{J\left(x_t, u_t\right) + \lambda_t A\left(x_t, u_t\right) + \dot{\lambda}_t x_t\right\} dt + \lambda_0 x_0 - \lambda_T x_T$$

- Now this we can differentiate:

$$\begin{aligned} \frac{\partial \mathcal{L}_t}{\partial u_t} &= J_u\left(x_t, u_t\right) + \lambda_t A_u\left(x_t, u_t\right) = 0 \\ \frac{\partial \mathcal{L}_t}{\partial x_t} &= J_x\left(x_t, u_t\right) + \lambda_t A_x\left(x_t, u_t\right) + \dot{\lambda}_t = 0 \end{aligned}$$

Part 3: Dynamic Programming in Continuous Time

- Now define the Hamiltonian:

$$H\left(x_t, u_t\right) = J\left(x_t, u_t\right) + \lambda_t A\left(x_t, u_t\right)$$

- The optimum must verify:

$$\begin{aligned} H_u\left(x_t, u_t\right) &= 0 \\ H_x\left(x_t, u_t\right) &= -\dot{\lambda}_t \\ H_\lambda\left(x_t, u_t\right) &= \dot{x}_t \end{aligned}$$

- Note the third condition is simply the law of motion.

Part 3: Dynamic Programming in Continuous Time

- Now use Hamiltonian in a continuous time version of the investment problem:

$$Max_{\{I_t\}_0^\infty} \int_0^\infty e^{-rt} \left[\pi\left(K_t\right) - I_t - c(I_t)\right] dt$$

$$\text{subject to } \dot{K}_t = I_t - \delta K_t$$

- Set up the Hamiltonian:

$$H\left(K_t, I_t\right) = e^{-rt}\left[\pi\left(K_t\right) - I_t - c(I_t)\right] + \lambda_t\left(I_t - \delta K_t\right)$$

- Then we must have at the optimum:

$$\begin{aligned} H_I\left(K_t, I_t\right) &= e^{-rt}\left[-1 - c'(I_t)\right] + \lambda_t = 0 \\ H_K\left(K_t, I_t\right) &= e^{-rt}\pi'\left(K_t\right) - \delta\lambda_t = -\dot{\lambda}_t \\ \dot{K}_t &= I_t - \delta K_t \end{aligned}$$

Part 3: Dynamic Programming in Continuous Time

- As usual we want to solve prices out, i.e. λ_t
- Once again, introduce Tobin's q:

$$q_t = \lambda_t e^{rt}$$

  q is defined as the shadow price of an additional unit of capital
- The first equation rewrites

$$q_t = 1 + c'(I_t)$$

  which by now should look familiar.
- The second equation becomes

$$\begin{aligned} \pi'(K_t) - \delta q_t &= -\dot{\lambda}_t e^{rt} \\ \pi'(K_t) &= r q_t + \delta q_t - \dot{q}_t \end{aligned}$$

  which by now should also look familiar

Part 3: Dynamic Programming in Continuous Time

- We can also derive an intuition for q. This last equation is a diffential equation in q, which rewrites

$$\dot{q}_t - (r+\delta)\, q_t + \pi'(K_t) = 0$$

- A solution to this equation writes:

$$q_t = \int_t^{\infty} e^{-(r+\delta)(s-t)} \pi'(Ks)\, ds$$

- Compare this with

$$q_t = \frac{1}{1+r} \sum_{s=0}^{\infty} \left(\frac{1-\delta}{1+r}\right)^s \pi'(K_{t+s+1})$$

Taking Stock

- Can we get an empirical counterpart to Tobin's q?
- One unit increase in K_t increases the present value of firm's profits by q_t. So it should increase the firm (equity) value by exactly q.
- Marginal q is therefore the ratio of the marginal increase in the (equity) value of a firm relative to the (marginal) cost of capital.
- These marginal increases are difficult to measure. So instead, focus on the average, and compute an average Q, as

$$Q = \frac{\text{Total Market Value of the Firm}}{\text{Replacement Cost of its Capital}}$$

Taking Stock

- Normally, investment should only change when Q does
- In particular, investment should not depend on CURRENT conditions, for instance firms' cash-flow holdings. More precisely, controlling for Q should soak up any other influence on investment.
- A enormous literature has tested whether observed investment responds more to cash flow than to Q. Seems to be the case - has rejected the model.
- More recent literature more favourable. Many measurement issues involved in estimating Q especially concerning tax features etc. When allowed for evidence more favourable.
- However investment is very lumpy and very cyclical - whilst there is a role for q investment dynamics probably much richer - Especially with irreversibility.