

Session 2: Consumption

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- The second revolution in modern macroeconomics comes from the introduction of dynamic decision making on the part of (rational) agents.
- Decisions today are influenced by (expectations about) the future.
- As a result, (expectations about) the future may have an effect on the current state of the economy. A flavor of Keynes' "animal spirits"
- We first introduce dynamics in the context of consumption-saving choice over time. Eminently dynamic, since obviously less consumption today means more saving today, which has to have future consumption as an objective.
- Will take us through classics in macroeconomics: the Permanent Income Hypothesis, the Random Walk hypothesis, the Consumption-based Capital Asset Pricing Model and the Equity Premium puzzle.

The Consumer Maximization Problem

- Introduce a "Value Function" V_t , defined as the VALUE of optimal behavior from period t onwards.
- By definition, maximizing agents choose consumption to solve the following problem:

$$\text{Max } V_t = \sum_{s=0}^{\infty} \beta^s U(c_{t+s})$$

$$\text{subject to : } a_{t+1} = (1 + r_t)a_t + y_t - c_t$$

- As we will see, there are (at least) three manners in which the problem can be solved. First is as a simple constrained maximization problem, i.e. using a Lagrangean. (The second will be using so-called "Bellman equation" and the third the so-called "Hamiltonian".)

Constrained Optimization

- Set up a Lagrangean:

$$\mathcal{L} = \sum_{s=0}^{\infty} \{ \beta^s U(c_{t+s}) + \lambda_{t+s} [y_{t+s} + (1 + r_{t+s})a_{t+s} - c_{t+s} - a_{t+s+1}] \}$$

- Notice the timing of the constraint.
- Standard constrained maximization problem. Take first-order conditions:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_{t+s}} &= \beta^s U'(c_{t+s}) - \lambda_{t+s} = 0 \\ \frac{\partial \mathcal{L}}{\partial a_{t+s}} &= \lambda_{t+s}(1 + r_{t+s}) - \lambda_{t+s-1} = 0 \end{aligned}$$

The Euler equation

- Gives

$$\frac{\beta U'(c_{t+1})}{U'(c_t)}(1 + r_{t+1}) = 1$$

- The so called Euler equation. Take a first order Taylor series expansion of $U'(c_{t+1})$ around c_t :

$$\begin{aligned}\frac{U'(c_{t+1})}{U'(c_t)} &\simeq 1 + \frac{U''}{U'} \Delta c_{t+1} \\ &\simeq 1 - \sigma \frac{\Delta c_{t+1}}{c_t}\end{aligned}$$

where $\sigma = -\frac{U'' c_t}{U'}$ is coefficient of relative risk aversion. Therefore, with $\theta = 1/\beta - 1 = \frac{1-\beta}{\beta}$,

$$\frac{\Delta c_{t+1}}{c_t} \simeq \frac{r_{t+1} - \theta}{\sigma(1 + r_{t+1})}$$

The Euler equation

- $\frac{\Delta c_{t+1}}{c_t} \simeq \frac{r_{t+1} - \theta}{\sigma(1+r_{t+1})}$
- If $r_{t+1} = \theta$ then consumption growth is zero
- If $r_{t+1} > \theta$ then consumption profile upward sloping (agent saves today)
- If $r_{t+1} < \theta$ then consumption profile downward sloping (agent dissaves today)
- If increase r_{t+1} then increase $\frac{\Delta c_{t+1}}{c_t}$ so that (current) c_t falls i.e. consumption falls when interest rates rise
- $1/\sigma$ is the Intertemporal elasticity of substitution and determines how interest sensitive consumption/savings are

The Intertemporal Budget Constraint

- Intertemporal Budget Constraint

$$a_{t+1} = (1 + r_t)a_t + y_t - c_t$$

$$a_{t+2} = (1 + r_{t+1})a_{t+1} + y_{t+1} - c_{t+1}$$

$$a_{t+n} = \prod_{j=1}^{N-1} (1 + r_{t+j})(1 + r_t)a_t + \sum_{s=0}^{N-1} \prod_{j=s+1}^{N-1} (1 + r_{t+j})(y_{t+s} - c_{t+s})$$

- Therefore

$$\frac{a_{t+n}}{\prod_{j=1}^{N-1} (1 + r_{t+j})} = (1 + r_t)a_t + \sum_{s=0}^{N-1} \frac{(y_{t+s} - c_{t+s})}{\prod_{j=1}^s (1 + r_{t+j})}$$

The Intertemporal Budget Constraint

- Assume transversality condition

$$\lim_{n \rightarrow \infty} \frac{a_{t+n}}{\prod_{j=1}^{N-1} (1 + r_{t+j})} = 0$$

- No Ponzi Game condition - then

$$\sum_{s=0}^{\infty} \frac{c_{t+s}}{\prod_{j=1}^s (1 + r_{t+j})} = (1 + r_t) a_t + \sum_{s=0}^{\infty} \frac{y_{t+s}}{\prod_{j=1}^s (1 + r_{t+j})}$$

- Consumption in present value must equate initial assets plus Income in present value.

The Consumption Function

- Assume $r_{t+j} = r, \forall j$. and let $r = \theta$. This implies $c_{t+s} = c_t \forall s$ so that our budget constraint is

$$\sum_{s=0}^{\infty} \frac{c_{t+s}}{(1+r)^s} = \frac{1+r}{r} c_t = (1+r_t) a_t + \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s}$$

$$c_t = \frac{r}{1+r} \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s} + r a_t$$

- Famous Friedman result that consumption equals permanent income
- Marginal propensity to consume out of current income is $\frac{r}{1+r}$
- But if $y_{t+s} = y_t \forall s$ (so that any change in current income is perceived as being permanent) then

$$c_t = y_t + r a_t$$

- A more "Keynesian" style consumption function - consume just current income. Therefore $mpc=1$

The Permanent Income Hypothesis

- PIH can help reconcile some empirical evidence based on Keynesian consumption function.
- In particular, estimating

$$C_t = a + bY_t + e_t$$

gives different results in cross-section (across households), in time series (in a country over time), or in different groups (Black vs. White)

- Can rationalise this if differentiate $Y_t = Y_t^T + Y_t^P$, a transitory (T) and a permanent (P) component. Do it as an exercise (either with Martin or in Romer). A direct application of the "signal extraction" problem we saw.

- Savings are defined as total income less consumption

$$s_t = y_t + ra_t - c_t$$

- Using our consumption function gives

$$\begin{aligned} s_t &= y_t - \frac{r}{1+r} \sum_{s=0}^{\infty} \frac{y_{t+s}}{(1+r)^s} \\ &= -\frac{r}{1+r} \sum_{s=1}^{\infty} \frac{y_{t+s} - y_t}{(1+r)^s} \\ &= -\sum_{s=1}^{\infty} \frac{\Delta y_{t+s}}{(1+r)^s} \end{aligned}$$

- Campbell's rainy day savings hypothesis - savings predicts future declines in labour income
- If savings are positive and consumption equals permanent income then current income must be above future income
- If savings are negative and consumption equals permanent income then current income must be below future income
- Suggests that savings should be procyclical over the business cycle (which they aren't..)

Introduce Uncertainty

- So far we have assumed perfect foresight but in reality agents have to make expectations of future variables
- Need to revise our problem to

$$\begin{aligned} \text{Max } E_t V_t &= \text{Max } E_t \sum_{s=0}^{\infty} \beta^s U(c_{t+s}) \\ \text{s.t. } a_{t+1} &= (1 + \tilde{r}_t) a_t + y_t - c_t \end{aligned}$$

where E_t is the expectations operator conditional on information at time t and \tilde{r}_t is the (stochastic) interest rate at time t .

- This creates a problem when using the Lagrangean approach.
Consider

$$\mathcal{L} = E_t \sum_{s=0}^{\infty} \{ \beta^s U(c_{t+s}) + \lambda_{t+s} [y_{t+s} + (1 + \tilde{r}_{t+s}) a_{t+s} - c_{t+s} - a_{t+s+1}] \}$$

Introduce Uncertainty

- The first-order conditions write

$$\frac{\partial \mathcal{L}}{\partial c_{t+s}} = E_t \{ \beta^s U'(c_{t+s}) - \lambda_{t+s} \} = 0$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+s}} = E_t \{ \lambda_{t+s}(1 + \tilde{r}_{t+s}) - \lambda_{t+s-1} \} = 0$$

- The second equation is NOT equivalent to

$$\frac{\partial \mathcal{L}}{\partial a_{t+s}} = E_t(\lambda_{t+s})E_t(1 + \tilde{r}_{t+s}) = \lambda_{t+s-1}$$

There is indeed potentially a covariance term. So we cannot solve using this approach

The Bellman equation

- Recognize instead that the definition of V_t implies the following two-period problem

$$\begin{aligned} \text{Max}_{c_t} E_t V(a_t) &= \text{Max}_{c_t} \{U(c_t) + \beta E_t V(a_{t+1})\} \\ \text{subject to} \quad &: a_{t+1} = (1 + \tilde{r}_t)a_t + y_t - c_t \end{aligned}$$

- The first-order condition writes

$$\begin{aligned} \frac{\partial E_t V(a_t)}{\partial c_t} &= U'(c_t) + \beta E_t \frac{\partial V [(1 + \tilde{r}_t)a_t + y_t - c_t]}{\partial C_t} = 0 \\ &= U'(c_t) + \beta E_t \left[\frac{\partial V(a_{t+1})}{\partial c_{t+1}} \frac{\partial c_{t+1}}{\partial a_{t+1}} \frac{\partial a_{t+1}}{\partial c_t} \right] = 0 \end{aligned}$$

- Now,

$$c_{t+1} = (1 + \tilde{r}_{t+1})a_{t+1} + y_{t+1} - a_{t+2}$$

so that

$$\frac{\partial c_{t+1}}{\partial a_{t+1}} = 1 + \tilde{r}_{t+1}$$

The Bellman equation

- And

$$\frac{\partial a_{t+1}}{\partial c_t} = -1$$

Now

$$V(a_{t+1}) = \text{Max}_{c_{t+1}} \{U(c_{t+1}) + \beta E_t V(a_{t+2})\}$$

so that

$$\frac{\partial V(a_{t+1})}{\partial c_{t+1}} = U'(c_{t+1}) + 0$$

- Note this made use of the envelope theorem, which tells us the derivative of a maximum is equal to the derivative of the objective function, evaluated at the maximum. That is why the second term is zero: evaluated at maximum consumption c_{t+1}^* , $V(a_{t+2}) = 0$ since it is already maximized with respect to c_{t+1} .
- The first order condition becomes the Euler equation

$$\beta E_t \left[\frac{U'(c_{t+1})}{U'(c_t)} (1 + \tilde{r}_{t+1}) \right] = 1$$

Bob Hall's Random Walk Result

- Introduction of uncertainty prevents previous derivations of expressions for consumption growth and consumption function
- In general,

$$E_t [U'(c_{t+1})] \neq U'(E_t c_{t+1})$$

Cannot "take the expectation operator out". EXCEPT if assume $U'(\cdot)$ is linear, i.e. if assume $U(\cdot)$ is quadratic. Hall (1978) does that, with $U(c_t) = -(c_t - b)^2$.

- Quadratic utility implies the "certainty equivalent", i.e. agents care only about mean outcomes and not volatility. Then Euler equation becomes

$$\beta E_t \left[\frac{c_{t+1} - b}{c_t - b} (1 + \tilde{r}_{t+1}) \right] = 1$$

Bob Hall's Random Walk Result

- For simplicity assume now interest rate constant $\tilde{r}_{t+1} = r$ and $\beta(1+r) = 1$, to get

$$E_t c_{t+1} = c_t$$

- Not a consumption function - just pins down the expected profile for consumption. Implies that consumption growth should be unpredictable by any t period information
- In practice, consumption does possess a unit root, but consumption growth shows positive short term persistence. Future consumption growth seems predictable by current income one period ahead.

Introducing Risk Aversion

- Quadratic utility implies certainty equivalence so that uncertainty doesn't affect the outcome.
- More plausibly we can assume that the utility function is $U(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$ (so called CRRA utility)
- The Euler equation in this case is

$$\beta E_t \left\{ \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \tilde{R}_{t+1} \right\} = 1$$

- To get further we need to assume that $\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \tilde{R}_{t+1}$ are jointly log normally distributed. If $\ln X \sim N(\mu, \sigma^2)$ then $E(X) = \exp(\mu + \frac{1}{2}\sigma^2)$.

Introducing Risk Aversion

- This utility function and stochastic assumption forms the Hansen-Singleton approach to consumption
- From this equation we will later develop implications for financial markets.
- The analytical solution to the Euler equation is now

$$E_t \Delta \ln c_{t+1} = \frac{1}{\sigma} \ln \beta + \frac{1}{\sigma} E_t \ln R_{t+1} + \alpha$$

where α represents terms in the variance and covariance of consumption growth and interest rates

- Derive this as an exercise.

Introducing Risk Aversion

- Unlike Hall consumption growth is no longer unpredictable
- Higher interest rates lead to faster future consumption growth and so lower current consumption
- Consumption growth should have the same serial correlation properties as interest rates - but it doesn't
- More uncertainty there is about the future then larger is α and so the lower is current consumption (because consumption growth is faster)
- This last role adds a precautionary motive to savings - savings will reflect future income growth and precautionary motive.
- Exercise: Precautionary motive will also become apparent with
$$U(c_t) = -\frac{1}{\alpha} \exp(-\alpha C_t).$$

An Assessment

- Powerful framework that offers unifying approach to macroeconomic issues
- Focus on fundamental primitives of the economy and expectations of the future
- Focus on variables being jointly determined - a general equilibrium framework
- BUT, empirical problems:
 - Consumption growth serially correlated and predictable
 - Consumption growth sensitive to individual shocks
 - Consumption growth not as volatile as permanent income suggests
 - Consumption growth not tied to rates of return that closely

Extension: Borrowing Constraints

- Consumer now has to maximise utility subject to a constraint $a_t \geq \Psi$.
Leads to a first order condition

$$\beta E_t \{ (1 + \tilde{r}_{t+1}) U'(c_{t+1}) + \lambda_{t+1} \} = U'(c_t) + \lambda_t$$

where λ is Lagrange multiplier attached to borrowing constraint

- If $\lambda = 0$ then borrowing constraint not binding and usual Euler Equation holds
- But if $\lambda_t > 0$ then consumption has very different dynamics and will be more closely tied to income
- If λ_t binds even for only one period affects level of consumption for all future periods even though normal Euler equation applies

Extension: Habits

- So far assumed utility depends only on current consumption e.g. no hangovers
- Habits says our level of utility depends on past consumption decisions
- Simple model of habits has a utility function

$$U(c_t, c_{t-1}) = \frac{(c_t - \gamma c_{t-1})^{1-\sigma}}{1-\sigma}$$

where γ governs level of habits ($\gamma = 0$ no habits, $\gamma < 1$)

Extension: Habits

- Marginal utility now more closely linked to growth of consumption. Smoothing marginal utility no longer same as smoothing consumption
- Consumption growth dynamics now very different and show positive serial correlation
- Consumption growth correlated with past income and consumption adjusts to changes in income in a more sluggish way

The Consumption Capital Asset Pricing Model (CAPM)

- The consumer's intertemporal first order condition applies for *any* asset. In other words

$$U'(c_t) = \beta E_t [U'(c_{t+1})R_{1t+1}]$$

$$U'(c_t) = \beta E_t [U'(c_{t+1})R_{2t+1}]$$

- Define R_{1t} as a safe asset. In other words R_{1t+1} is known at t . Therefore

$$U'(c_t) = \beta R_{1t+1} E_t U'(c_{t+1}) = \beta E_t [U'(c_{t+1})R_{2t+1}]$$

- Because $Cov(X, Y) = E(XY) - E(X)E(Y)$ we have

$$R_{1t+1} = \frac{Cov(R_{2t+1}, U'(c_{t+1})) + E_t [R_{2t+1}] E_t [U'(c_{t+1})]}{E_t U'(c_{t+1})}$$

The Consumption Capital Asset Pricing Model (CAPM)

- From this we get the key equation of the Consumption CAPM model

$$E_t R_{2t+1} = R_{1t+1} - \frac{\text{Cov}(R_{2t+1}, U'(c_{t+1}))}{E_t U'(c_{t+1})}$$

- Under risk aversion $U'' < 0$ therefore $\text{Cov}(R_{2t+1}, U'(c_{t+1}))$ has opposite sign to $\text{Cov}(R_{2t+1}, U(c_{t+1}))$
- If $\text{Cov}(R_{2t+1}, U'(c_{t+1})) < 0$ then $E_t R_{2t+1} > R_{1t+1}$ - if R_{2t+1} pays a high return when consumption is high then pays higher return than risk free rate
- If $\text{Cov}(R_{2t+1}, U'(c_{t+1})) > 0$ then $E_t R_{2t+1} < R_{1t+1}$ - if R_{2t+1} pays a high return when consumption is low then pays lower return than risk free rate - it provides insurance

Equity Prices

- Assume risk free rate of return is a constant $1 + r_1$ and assume $-\frac{\text{Cov}(R_{2t+1}, U'(c_{t+1}))}{E_t U'(c_{t+1})}$ is a constant risk premium γ_2
- Return on equity is $\frac{p_{t+1} + d_t}{p_t}$ where p_t is real price of share and d_t is real dividend payment

$$\frac{E_t(p_{t+1} + d_t)}{p_t} = 1 + r + \gamma_2$$
$$p_t = \frac{E_t d_t}{1 + r + \gamma_2} + \frac{E_t p_{t+1}}{1 + r + \gamma_2}$$

- Solving this equation recursively and imposing the condition that $\lim_{n \rightarrow \infty} \frac{p_{t+n}}{(1+r+\gamma_2)^n} = 0$ we arrive at following price equation for equity

$$p_t = \sum_{j=0}^{\infty} \frac{E_t d_{t+j}}{(1 + r + \gamma_2)^{j+1}}$$

- If we assume dividends growth at a constant rate g , that is $d_{t+j} = (1 + g)^j d_t$ then we have the Gordon dividend growth model

$$p_t = d_t \sum_{j=0}^{\infty} \frac{(1 + g)^j}{(1 + r + \gamma_2)^j}$$

$$\frac{d_t}{p_t} = \frac{r + \gamma_2 - g}{1 + r + \gamma_2}$$

- Historically $r = 0.02$ (2%), $g = 0.02$, and average risk premium = 0.054
- Therefore dividend yield in past has averaged around 5%
- When dividend yield below 5% then equity prices historically high relative to dividends and the opposite if yield above 5%

Equity Prices - A Random Walk Result

- We know that

$$p_t = \sum_{j=0}^{\infty} \frac{E_t d_{t+j}}{(1+r+\gamma_2)^j}$$

and

$$p_{t+1} = \sum_{j=0}^{\infty} \frac{E_{t+1} d_{t+1+j}}{(1+r+\gamma_2)^j}$$

- So that

$$p_{t+1} - (1+r+\gamma_2)p_t = (E_{t+1} - E_t) \sum_{j=0}^{\infty} \frac{d_{t+1+j}}{(1+r+\gamma_2)^j} - d_t$$

- d_t is known as of time t

Equity Prices - A Random Walk Result

- But the expected value of an expectational error (a "surprise") is zero! Thus the share price rises at the rate $(1 + r + \gamma_2)$ plus an unpredictable expectational error. Share prices follow a random walk with drift
- George Gibson (1889): "when shares become publicly known in an open market, the value they acquire may be regarded as the judgement of the best intelligence concerning them"
- Fama (1965): day-to-day Dow Jones close to random walk.
- Jensen (1978): "the efficient market hypothesis is the best established fact in all of social sciences"
- Cutler, Poterba and Summers (1990) found serial correlations in returns for 13 countries – some evidence that in short term changes in share prices show small positive correlation and at longer horizons negative correlation
- Shares are not very predictable – random walk model a good approximation

The Equity Premium Puzzle

- Go back to Euler equation with CRRA utility, $U(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$. We had:

$$\beta E_t \left\{ \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} (1 + \tilde{r}_{t+1}) \right\} = 1$$

- Rearrange defining $g_c = \frac{c_{t+1}-c_t}{c_t}$ and $\beta = \frac{1}{1+\rho}$. And get rid of time subscripts for simplicity:

$$E \left\{ (1 + g_c)^{-\sigma} (1 + \tilde{r}) \right\} = 1 + \rho$$

- Now take a Taylor expansion around $r = g = 0$ to get

$$(1 + g_c)^{-\sigma} (1 + \tilde{r}) \simeq 1 + \tilde{r} - \sigma g_c + \frac{g_c^2}{2} \sigma(\sigma + 1) - \sigma \tilde{r} g_c$$

The Equity Premium Puzzle

- Take expectations:

$$\begin{aligned} E \{ (1 + g_c)^{-\sigma} (1 + \tilde{r}) \} &\simeq 1 + E(\tilde{r}) - \sigma E(g_c) \\ &\quad - \sigma [E(\tilde{r})E(g_c) + \text{cov}(\tilde{r}, g_c)] \\ &\quad + \frac{1}{2}\sigma(\sigma + 1) [E(g_c)^2 + \text{var}(g_c)] \end{aligned}$$

- So from Euler equation, and omitting the negligible (product) terms:

$$E(\tilde{r}) - \sigma E(g_c) - \sigma \text{cov}(\tilde{r}, g_c) + \frac{1}{2}\sigma(\sigma + 1)\text{var}(g_c) \simeq \rho$$

- Rearrange:

$$E(\tilde{r}) \simeq \rho + \sigma E(g_c) + \sigma \text{cov}(\tilde{r}, g_c) - \frac{1}{2}\sigma(\sigma + 1)\text{var}(g_c)$$

The Equity Premium Puzzle

- Now it sounds horrible, but the exact same derivation can be performed for a riskless asset, with return r :

$$(1 + r) E \{ (1 + g_c)^{-\sigma} \} = 1 + \rho$$

- Which can be linearized into

$$(1 + g_c)^{-\sigma} (1 + r) \simeq 1 + r - \sigma g_c + \frac{g_c^2}{2} \sigma(\sigma + 1) - \sigma r g_c$$

- And:

$$r \simeq \rho + \sigma E(g_c) - \frac{1}{2} \sigma(\sigma + 1) \text{var}(g_c)$$

The Equity Premium Puzzle

- So finally, the equity PREMIUM is:

$$E(\tilde{r}) - r \simeq \sigma \text{cov}(\tilde{r}, g_c)$$

- For the US 1889-1978 (the classic Mehra-Prescott (1983) study) we have:

$$E(r_s) = 0.07 \quad E(r) = 0.01 \quad \text{Cov}(g_c, r_s) = 0.0022$$

- Uses returns on bonds to get r .
- This is the equity risk premium - why on average have equities earned 6% per annum more per decade than the risk free rate of return on bonds?
- Can only be explained if assume extremely high levels of risk aversion e.g σ of 80.
- Need to magnify limited volatility of consumption through high levels of risk aversion to justify equity risk premium

Resolving the Puzzle

- *Incomplete Markets* - If agents unable to buy insurance then they will demand an even higher return on risky assets that pay high rates of return when consumption is high
- *Borrowing Constraints* - A form of incomplete markets
- *Habits* - A variety of proposals have been made. Campbell and Cochrane most successful
- Define utility as $U_t = \frac{(c_t - \lambda c_{t-1})^{1-\sigma} - 1}{1-\sigma}$
- For more plausible (but still high) levels of σ this can account for risk premium. Trick is it defines very high levels of local risk aversion (for given c_{t-1})
- What happens is that marginal utility depends not on consumption but $c_t - \lambda c_{t-1}$ - so what matters is not volatility of consumption growth but volatility of change in $c_t - \lambda c_{t-1}$