

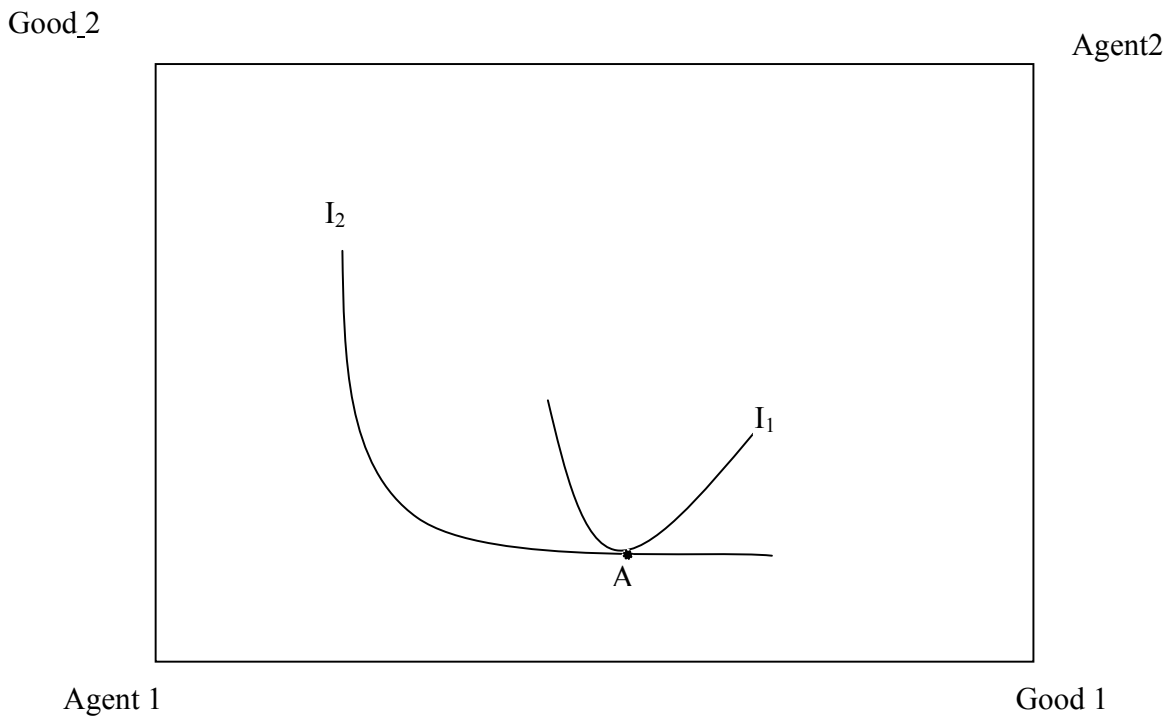
Intermediate Financial Theory
Danthine and Donaldson

Additional Exercises

Chapter 1

- 1.6. Assume a 2 goods-2 agents economy and well-behaved utility functions. Explain why the competitive equilibrium should be on the contract curve.
- 1.7. What is unusual in Figure 1 below? Is there a PO allocation? Can it be obtained as a competitive equilibrium? What is the corresponding assumption of the 2nd theorem of welfare, and why is it important?

Figure 1.2 : The Edgeworth-Bowley Box: An Unusual Configuration



Chapter 4

4.9. If you are exposed to a 50/50 probability of gaining or losing CHF 1'000 and an insurance that removes the risk costs CHF 500, at what level of wealth will you be indifferent between taking the gamble or paying the insurance? That is, what is your certainty equivalent wealth for this gamble? Assume that your utility function is $U(Y) = -1/Y$.
What would the solution be if the utility function were logarithmic?

4.10. Assume that you have a logarithmic utility function on wealth $U(Y)=\ln Y$ and that you are faced with a 50/50 probability of winning or losing CHF 1'000. How much will you pay to avoid this risk if your current level of wealth is CHF 10'000? How much would you pay if your level of wealth is CHF 1'000'000? Did you expect that the premium you were willing to pay would increase/decrease? Why?

4.11. Assume that security returns are normally distributed. Compare portfolios A and B, using both first and second-order stochastic dominance :

Case 1	Case 2	Case 3
$\sigma_a > \sigma_b$	$\sigma_a = \sigma_b$	$\sigma_a < \sigma_b$
$E_a = E_b$	$E_a > E_b$	$E_a < E_b$

4.12. An agent faces a risky situation in which he can lose some amount of money with probabilities given in the following table:

Loss	Probability
1000	10%
2000	20%
3000	35%
5000	20%
6000	15%

This agent has a utility function of wealth of the form

$$U(Y) = \frac{Y^{1-\gamma}}{1-\gamma} + 2$$

His initial wealth level is 10000 and his γ is equal to 1.2.

a. Calculate the certainty equivalent of this prospect for this agent. Calculate the risk premium. What would be the certainty equivalent of this agent if he would be risk neutral?

b. Describe the risk premium of an agent whose utility function of wealth has the form implied by the following properties: $U'(Y) > 0$ and $U''(Y) < 0$

4.13. An agent with a logarithmic utility function of wealth tries to maximize his expected utility. He faces a situation in which he will incur a loss of L with probability π . He has the possibility to insure against this loss. The insurance premium depends on the extent of the coverage. The amount covered is denoted by h and the price of the insurance per unit of coverage is p (hence the amount he has to spend on the insurance will be hp).

- a. Calculate the amount of coverage h demanded by agent as a function of his wealth level Y , the loss L , the probability π and the price of the insurance p .
- b. What is the expected gain of an insurance company offering such a contract ?
- c. If there is perfect competition in the insurance market (zero profit), what price p will the insurance company set?
- d. What amount of insurance will the agent buy at the price calculated under c. What is the influence of the form of the utility function ?

4.14. Given the following probability distributions for risky payoffs \tilde{x} and \tilde{z} :

x	Probability (x)	z	Probability (z)
-10	.1	2	.2
5	.4	3	.5
10	.3	4	.2
12	.2	30	.1

- a. If the only available choice is 100 % of your wealth in \tilde{x} or 100 % in \tilde{z} and you choose on the basis of mean and variance, which asset is preferred ?
 - b. According to the second-order stochastic dominance criterion, how would you compare them ?
- 4.15. There is an individual with a well-behaved utility function, and initial wealth Y . Let a lottery offer a payoff of G with probability π and a payoff of B with probability $1-\pi$.
- a. If the individual already owns this lottery denote the minimum price he would sell it for by P_s . Write down the expression P_s has to satisfy.
 - b. If he does not own it, write down the expression P_b (the maximum price he would be willing to pay for it) has to satisfy.
 - c. Assume now that $\pi=1/2$, $Y=10$, $G=6$, $B=26$, and the utility function is $U(Y)=Y^{1/2}$. Find buying and selling prices. Are they equal? Explain why not. Generally, can they ever be equal?

4.16. Consider the following investments:

Investment 1		Investment 2	
Payoff	Prob.	Payoff	Prob.
1	0.25	3	0.33
7	0.50	5	0.33
12	0.25	8	0.34

Check that neither investment dominates the other on the basis of

- The Mean-Variance criterion
- First Order Stochastic Dominance
- Second Order Stochastic Dominance

How could you rank these investments ?

Chapter 5

5.6. An individual with a utility function $U(c) = -\exp(-Ac)$ (where $A=(1/30)\ln 4$) and an initial wealth of \$50 must choose a portfolio of two assets. Each asset has a price of \$50. The first asset is riskless and pays off \$50 next period in each of the two possible states. The risky asset pays off z_s in state $s=1,2$. Suppose also that the individual cares only about next period consumption (denoted by c_1 or c_2 depending on the state). The probability of state 1 is denoted by π .

a. If the individual splits his wealth equally between the two assets, fill in the following table assuming that each of three scenarios is considered.

Scenario	(z_1, z_2)	π	(c_1, c_2)	$E(c)$	$\text{Var}(c)$
1	(20, 80)	1/5			
2	(38, 98)	1/2			
3	(30, 90)	1/3			

b. How would this individual rank these three scenarios? Explain and give reasons to support your argument.

c. Show that under each scenario the individual's optimal decision is to invest an equal amount on each of the two assets.

5.7. Consider an agent with a well-behaved utility function who must balance his portfolio between a riskless asset and a risky asset. The first asset, with price p_1 has the certain payoff z_1 while the second asset, with price p_2 pays off \tilde{z}_2 which is a random variable. The agent has an initial wealth Y_0 and he only cares about next period. Assuming that he holds x_1 units of the riskless asset and x_2 units of the risky asset;

a. Write down his expected utility function in terms of Y_0, x_1, x_2, z_1 , and \tilde{z}_2 . Write down his budget constraint as well. Find the equation (the FOC) that the optimal demand for the risky asset has to satisfy.

b. Assume now that his utility function is of the form: $U(Y)=a-be^{-AY}$ with $a, b>0$. Show that the demand for the risky asset is independent of the initial wealth. Explain intuitively why this is so.

5.8. Consider the savings problem of Section 4.4 :

$$\max_{s \geq 0} EU\{(y_0 - s) + \delta U(s\tilde{x})\}$$

$$\text{Assume } U(c) = Ec - \frac{1}{2}\chi\sigma_c^2$$

Show that if \tilde{x}_A SSD \tilde{x}_B ($w/E\tilde{x}_A = E\tilde{x}_B$), then $s_A > s_B$.

Chapter 7

- 7.7. Show that maximizing the Sharpe ratio, $\frac{E(r_p) - r_f}{\sigma_p}$ yields the same tangency portfolio that was obtained in the text.

Hint: Formulate the Lagrangian and solve the problem.

- 7.8. Think of a typical investor selecting his preferred portfolio along the Capital Market Line. Imagine:
1. A 1% increase in both the risk free rate and the expected rate of return on the market, so that the CML shifts in a parallel fashion
 2. An increase in the expected rate of return on the market without any change in the risk free rate, so that the CML tilts upward.

In these two situations, describe how the optimal portfolio of the typical investor is modified.

- 7.9. Questions about the Markowitz model and the CAPM.
- a. Explain why the efficient frontier must be concave.
 - b. Suppose that there are N risky assets in an economy, each being the single claim to a different firm (hence, there are N firms). Then suppose that some firms go bankrupt, i.e. their single stock disappears; how is the efficient frontier altered?
 - c. How is the efficient frontier altered if the borrowing (risk-free) rate is higher than the lending rate? Draw a picture.
 - d. Suppose you believe that the CAPM holds and you notice that an asset (call it asset A) is above the Security Market Line. How can you take advantage of this situation? What will happen to stock A in the long run?
- 7.10. Consider the case without a riskless asset. Take any portfolio p . Show that the covariance vector of individual asset returns with portfolio p is linear in the vector of mean returns if and only if p is a frontier portfolio.
- Hint: To show the "if" part is straightforward. To show the converse begin by assuming that $Vw = ae + b\mathbf{1}$ where V is the variance-covariance matrix of returns, e is the vector of mean returns, and $\mathbf{1}$ is the vector of ones.
- 7.11. Show that the covariance of the return on the minimum variance portfolio and that on any portfolio (not only those on the frontier) is always equal to the variance of the rate of return on the MVP.
- Hint: consider a 2-assets portfolio made of an arbitrary portfolio p and the MVP, with weights a and $1-a$. Show that $a=0$ satisfies the variance minimizing program; the conclusion follows.
- 7.12. Find the frontier portfolio that has an identical variance as that of its zero-covariance portfolio. (That is, determine its weights.)
- 7.13. Let there be two risky securities, a and b . Security a has expected return of 13% and volatility of 30%. Security b has expected return of 26% and volatility of 60%. The two securities are uncorrelated.
- a. Compute the portfolio on the efficient frontier that is tangent to a line from zero, the zero beta portfolio associated with that portfolio, and the minimum-variance portfolio.

- b. Assume a risk-free rate of 5%. Compute the portfolio of risky assets that investors hold. Does this portfolio differ from the tangency portfolio computed under a) ? If yes, why?
- 7.14 a. Given risk-free borrowing and lending, efficient portfolios have no unsystematic risk. True or false?
b. If the agents in the economy have different utility functions the market portfolio is not efficient. True or false?
c. The CAPM makes no provision for investor preference for skewness. True or false?

Chapter 8

8.9. Consider an exchange economy with two states. There are two agents with the same utility function $U(c) = \ln(c)$. State 1 has a probability of π . The agents are endowed with the units of the consumption good at each state. Their endowments across themselves and across states are not necessarily equal. Total endowment of this consumption good is e_1 in state 1 and e_2 in state 2. Arrow-Debreu state prices are denoted by q_1 and q_2 .

a. Write down agents' optimization problems and show that

$$\frac{q_1}{q_2} = \frac{\pi}{1-\pi} \left(\frac{y_2}{y_1} \right)$$

Assuming that $q_1 + q_2 = 1$ solve for the state prices. Hint: Recall the simple algebraic fact that $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$.

b. Suppose there are two types of asset in the economy. A riskless asset (asset 1) pays off 1 (unit of the consumption good) in each state and has market price of $P_1=1$. The risky asset (asset 2) pays off 0.5 in state 1 and 2 in state 2. Aggregate supplies of the two assets are Q_1 and Q_2 . If the two states are equally likely, show that the price of the risky asset is

$$P_2 = \frac{5Q_1 + 4Q_2}{4Q_1 + 5Q_2}$$

Hint: Note that in this case state-contingent consumption of the agents are assured, in equilibrium, through their holdings of the two assets. To solve the problem you will need to use the results of section a). There is no need to set up another optimization problem.

Chapter 10

10.5. A-D pricing

Consider two 5-year coupon bonds with different coupon rates which are simultaneously traded.

	<u>Price</u>	<u>Coupon</u>	<u>Maturity Value</u>
Bond 1	1300	8 %	1000
Bond 2	1200	6.5 %	1000

For simplicity, assume that interest payments are made once per year. What is the price of a 5-year A-D security when we assume the only state is the date.

10.6. You anticipate receiving the following cash flow which you would like to invest risk free.

t = 0	1	2	3
	\$1m	\$1,25m	

The period denotes one year. Risk free discount bonds of various maturities are actively traded, and the following price data is reported:

t = 0	1	2	3
-950	1000		
-880		1000	
-780			1000

a. Compute the term structure implied by these bond prices.

b. How much money can you create, risk free, at $t = 3$ from the above cash flow using the three mentioned instruments?

c. Show the transactions whereby you guarantee (lock in) its creation at $t = 3$.

10.7. Consider a world with two states of nature. You have the following term structure of interest rates over two periods:

$$r_1^1 = 11.1111, r_2^1 = 25.0000, r_1^2 = 13.2277, r_2^2 = 21.2678$$

where the subscript denotes the state at the beginning of period 1, and the superscript denotes the period.

For instance $\frac{1}{(1+r_j^2)^2}$ is the price at state j at the beginning of period 1 of a riskless asset paying 1 two

periods later. Construct the stationary (same every period) Arrow-Debreu state price matrix.

Chapter 13

13.6. Assume that the following two-factor model describes returns

$$r_i = a_i + b_{i1}F_1 + b_{i2}F_2 + e_i$$

Assume that the following three portfolios are observed.

Portfolio	Expected returns	b_{i1}	b_{i2}
A	12.0	1	0.5
B	13.4	3	0.2
D	12.0	3	-0.5

a. Find the equation of the plane that must describe equilibrium returns.

b. If $\tilde{r}_M - r_f = 4$, find the values for the following variables that would make the expected returns consistent with equilibrium determined by the CAPM.

i) r_f

ii) β_{pi} , the market beta of the pure portfolio associated with factor i

13.7. Based on a single factor APT model, the risk premium on a portfolio with unit sensitivity is 8% ($\lambda_1 = 8\%$). The risk free rate is 4%. You have uncovered three well-diversified portfolios with the following characteristics:

<u>Portfolio</u>	<u>Factor Sensitivity</u>	<u>Expected Return</u>
A	.80	10.4%
B	1.00	10.0%
C	1.20	13.6%

Which of these three portfolios is not in line with the APT?

13.8 A main lesson of the CAPM is that “diversifiable risk is not priced”. Is this important result supported by the various asset pricing theories reviewed in this book? Discuss.

As a provision of supplementary material we describe below how the APT could be used to construct an arbitrage portfolio to profit of security mispricing. The context is that of equity portfolios. The usefulness of such an approach will, of course, depend upon “getting the right set of factors” so that the attendant regressions have high R^2 .

13.9 An APT Exercise in Practice

- a. Step 1: select the factors; suppose there are J of them.
- b. Step 2: For a large number of firms N (big enough so that when combined in an approximately equally weighted portfolio of them the unique risks diversify away to approx. zero) undertake the following time series regressions on historical data:

$$\begin{array}{lcl}
 \text{Firm 1: IBM} & \tilde{r}_{\text{IBM}} = \hat{\alpha}_{\text{IBM}} + \hat{b}_{\text{IBM},1}\tilde{F}_1 + \dots + \hat{b}_{\text{IBM},J}\tilde{F}_J + \tilde{\epsilon}_{\text{IBM}} \\
 \text{Firm 2: BP} & \tilde{r}_{\text{BP}} = \hat{\alpha}_{\text{BP}} + \hat{b}_{\text{BP},1}\tilde{F}_1 + \dots + \hat{b}_{\text{BP},J}\tilde{F}_J + \tilde{\epsilon}_{\text{BP}} \\
 \vdots & \vdots & \vdots \\
 \text{Firm N: GE} & \tilde{r}_{\text{GE}} = \hat{\alpha}_{\text{GE}} + \hat{b}_{\text{GE},1}\tilde{F}_1 + \dots + \hat{b}_{\text{GE},J}\tilde{F}_J + \tilde{\epsilon}_{\text{GE}}
 \end{array}$$

The return to each stock are regressed on the same J factors; what differs is the factor sensitivities $\{\hat{b}_{\text{IBM},1}, \dots, \hat{b}_{\text{GE},J}\}$. Remember that:

$$\hat{b}_{\text{BP},J} = \frac{\text{cov}(\tilde{r}_{\text{BP}}, \tilde{F}_J^{\text{HIST}})}{\sigma_{F_j}^{\text{HIST}}}$$

(want high R^2)

- c. Step 3: first assemble the following data set:

$$\begin{array}{lcl}
 \text{Firm 1: IBM} & \text{AR}_{\text{IBM}}, \hat{b}_{\text{IBM},1}, \hat{b}_{\text{IBM},2}, \dots, \hat{b}_{\text{IBM},J} \\
 \text{Firm 2: BP} & \text{AR}_{\text{BP}}, \hat{b}_{\text{BP},1}, \hat{b}_{\text{BP},2}, \dots, \hat{b}_{\text{BP},J} \\
 \vdots & \vdots & \vdots \\
 \text{Firm N: GE} & \text{AR}_{\text{GE}}, \hat{b}_{\text{GE},1}, \hat{b}_{\text{GE},2}, \dots, \hat{b}_{\text{GE},J}
 \end{array}$$

The $\text{AR}_{\text{IBM}}, \text{AR}_{\text{BP}}, \dots$ etc. represent the average returns on the N stocks over the historical period chosen for the regression.

Then, regress the average returns on the factor sensitivities (we have N data points corresponding to the N firms)

(these vary across the N firms)

$$A\tilde{r}_i = r_f + \hat{\lambda}_1 \tilde{b}_{i1} + \hat{\lambda}_2 \tilde{b}_{i2} \dots \hat{\lambda}_J \tilde{b}_{iJ}$$

we obtain estimates $\{\hat{\lambda}_1, \dots, \hat{\lambda}_J\}$

In the regression sense this determines the “best” linear relationship among the factor sensitivities and the past average returns for this sample of N stocks.

This is a “cross sectional” regression. (Want a high R^2)

- d. Step 4: Compare, for the N assets, their actually observed returns with what should have been observed given their factor sensitivities; compute α_j 's:

$$\alpha_{IBM} = \underbrace{AR_{IBM}}_{\text{IBM's actually observed historical return}} - \underbrace{\left[r_f + \hat{\lambda}_1 \hat{b}_{IBM,1} + \dots + \hat{\lambda}_J \hat{b}_{IBM,J} \right]}_{\text{predicted return given its factors intensities } \hat{b}_{IBM,1}, \dots, \hat{b}_{IBM,J} \text{ according to the regression in step 3}}$$

$$\vdots$$

$$\alpha_{GE} = AR_{GE} - \left[r_f + \hat{\lambda}_1 \hat{b}_{GE,1} + \dots + \hat{\lambda}_J \hat{b}_{GE,J} \right]$$

Note that

$\alpha_j > 0$ implies the average returns exceeded what would be justified by the factor intensities => undervalued;

$\alpha_j < 0$ implies the average returns fell short of what would be justified by the factor intensities => overvalued.

e. Step 5: Form an arbitrage portfolio of the N stocks:

if $\alpha_j > 0$ - assume a long position

if $\alpha_j < 0$ - assume a short position

since N is large $e_p \equiv 0$, so ignore “unique” risks.

Remarks:

1.: In step 4 we could substitute independent (otherwise obtained) estimates of AR_i 's, and not use the historical averages.

2.: Notice that nowhere do we have to forecast future values of the factors.

3.: In forming the arbitrage portfolio we are implicitly assuming that the over and under pricing we believe exists will be eliminated in the future – to our advantage!

Chapter 15

- 15.6 Consider two agents in the context of a pure exchange economy in which there are two dates ($t = 0, 1$) and two states at $t = 1$. The endowments of the two agents are different ($e_1 \neq e_2$). Both agents have the same utility function :

$$U(c_0, c_1(\theta)) = \ln c_0 + E \ln c_1(\theta),$$

but they differ in their beliefs. In particular, agent 1 assigns probability $\frac{3}{4}$ to state 1, while agent 2 assigns state 1 a probability $\frac{1}{4}$. The agents trade Arrow-Debreu claims and the supply of each claim is 1. Neither agent receives any endowment at $t=1$.

- Derive the equilibrium state claim prices. How are they related to the relative endowments of the agents ? How are the relative demands of each security related to the agents' subjective beliefs ?
- Suppose rather than trading state claims, each agent is given a_i units of a riskless security paying one unit in each future state. Their $t=0$ endowments are otherwise unaffected. Will there be trade ? Can you think of circumstances where no trade will occur ?
- Now suppose that a risky asset is also introduced into this economy. What will be the effects ?
- Rather than introducing a risky asset, suppose an entrepreneur invents a technology that is able to convert x units of the riskless asset into x units each of $(1,0)$ and $(0,1)$. How is x and the value of these newly created securities related ? Could the entrepreneur extract a payment for the technology ? What considerations would influence the magnitude of this payment ?