Review of Basic Options Concepts and Terminology

March 24, 2005

1 Introduction

The purchase of an options contract gives the buyer the right to buy (call options contract) or sell (put options contract) some other asset under pre-specified terms and circumstances. This *underlying* asset, as it is called, can in principle be anything with a well-defined price. For example, options on individual stocks, portfolios of stocks (i.e., indices such as the $S\&P500$), futures contracts, bonds, and currencies are actively traded. Note that options contracts do not represent an obligation to buy or sell and, as such, must have a positive, or at worst zero, price.

"American" style options allow the right to buy or sell (the so-called "right of exercise") at any time on or before a pre-specified future date (the "expiration" date). "European" options allow the right of exercise only at the pre-specified expiration date. Most of our discussion will be in the context of European call options. If the underlying asset does not provide any cash payments during the time to expiration (no dividends in the case of options on individual stocks), however, it can be shown that it is never wealth maximizing to exercise an American call option prior to expiration (its market price will at least equal and likely exceed its value if exercised). In this case, American and European call options are essentially the same, and are priced identically. The same statement is not true for puts.

In all applied options work, it is presumed that the introduction of options trading does not influence the price process of the underlying asset on which they are written. For a full general equilibrium in the presence of incomplete markets, however, this will not generally be the case.

2 Call and Put Options on Individual Stocks

1. European call options

   (a) Definition: A European call options contract gives the owner the right to buy a pre-specified number of shares of a pre-specified stock (the underlying asset) at a pre-specified price (the "strike" or "exercise" price) on a pre-specified future date (the expiration date). American options allow exercise "on or before" the expiration date. A contract typically represents 100 options with the cumulative right to buy 100 shares.

(b) Payoff diagram: It is customary to describe the payoff to an individual call option by its value at expiration as in Figure 1.

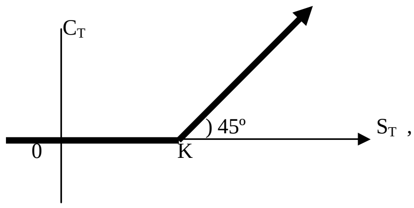

Figure 1: Payoff Diagram: European Call Option

In Figure 1, S_T denotes the possible values of the underlying stock at expiration date T, K the exercise price, and C_T the corresponding call value at expiration. Algebraically, we would write

$$C_T = \max\{0, S_T - K\}.$$

Figure 1 assumes the perspective of the buyer; the payoff to the seller (the so-called "writer" of the option) is exactly opposite to that of the buyer. See Figure 2.

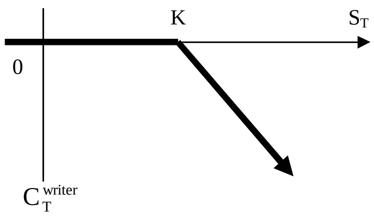

Figure 2: Payoff Diagram: European Call-Writer's Perspective

Note that options give rise to exactly offsetting wealth transfers between the buyer and the seller. The options related wealth positions of buyers and sellers must thus always sum to zero. As such we say that options are in zero net supply, and thus are not elements of "M'', the market portfolio of the classic CAPM.

(c) Remarks: The purchaser of a call option is essentially buying the expected price appreciation of the underlying asset in excess of the exercise price. As we will make explicit in a later chapter, a call option can be thought of as very highly leveraged position in the underlying stock - a property that makes it an ideal vehicle for speculation: for relatively little money (as the call option price will typically be much less than the underlying share's price) the buyer can acquire the upward potential.

There will, of course, be no options market without substantial diversity of expectations regarding the future price behavior of the underlying stock.

2. European Put Options

   (a) Definition: A European put options contract gives the buyer the right to sell a pre-specified number of shares of the underlying stock at a pre-specified price (the "exercise" or "strike" price) on a pre-specified future date (the expiration date). American puts allow for the sale on or before the expiration date. A typical contract represents 100 options with the cumulative right to sell 100 shares.

   (b) Payoff diagram: In the case of a put, the payoff at expiration to an individual option is represented in Figure 3.

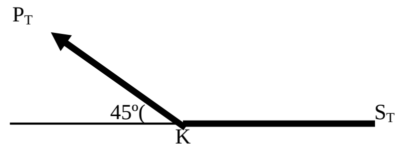

Figure 3: Payoff Diagram for a European Put

In Figure 3, P_T denotes the put's value at expiration; otherwise, the notation is the same as for calls. The algebraic equivalent to the payoff diagram is

$$P_T = \max\{0, K - S_T\}$$

The same comments about wealth transfers apply equally to the put as to the call; puts are thus also not included in the market portfolio M.

   (c) Remarks: Puts pay off when the underlying asset's price falls below the exercise price at expiration. This makes puts ideal financial instruments for "insuring" against price declines. Let us consider the payoff to the simplest "fundamental hedge" portfolio:

$$\left\{ \begin{array}{ll} \text{1 share of} & \text{1 put written on the} \\ \text{stock} & \text{stock with exercise price } K \end{array} \right\}.$$

Table 1: Payoff Table for Fundamental Hedge

Events	$S_T \leq K$	$S_T > K$
Stock	S_T	S_T
Put	$K - S_T$	0
Hedge Portfolio	K	S_T

To see how these two securities interact with one another, let us consider their net total value at expiration:

The diagrammatic equivalent is in Figure 4.

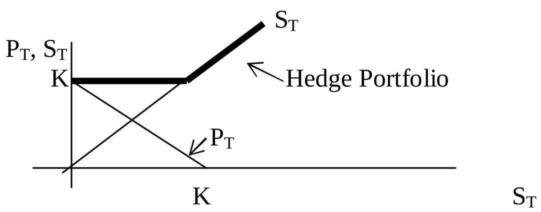

Figure 4: Payoff Diagram: Fundamental Hedge

The introduction of the put effectively bounds the share price to fall no lower than K. Such insurance costs money, of course, and its price is the price of the put.

Puts and calls are fundamentally different securities: calls pay off when the underlying asset's price at expiration exceeds K; puts pay off when its price falls short of K. Although the payoff patterns of puts and calls are individually simple, virtually any payoff pattern can be replicated by a properly constructed portfolio of these instruments.

3 The Black-Scholes Formula for a European Call Option.

1. What it presumes: The probability distribution on the possible payoffs to call ownership will depend upon the underlying stock's price process. The Black-Scholes formula gives the price of a European call under the following assumptions:

   (a) the underlying stock pays no dividends over the time to expiration;

   (b) the risk free rate of interest is constant over the time to expiration;

   (c) the continuously compounded rate of return on the underlying stock is governed by a geometric Brownian motion with constant mean and variance over the time to expiration.

This model of rate of return evolution essentially presumes that the rate of return on the underlying stock – its rate of price appreciation since there are no dividends – over any small interval of time $\Delta t \in [0, T]$ is given by

$$r_{t,\,t+\Delta t} = \frac{\Delta S_{t,\,t+\Delta t}}{S_t} = \hat{\mu}\Delta t + \hat{\sigma}\tilde{\varepsilon}\sqrt{\Delta t}\,, \tag{1}$$

where $\tilde{\varepsilon}$ denotes the standard normal distribution and $\hat{\mu}$, $\hat{\sigma}$ are, respectively, the annualized continuously compounded mean return and the standard deviation of the continuously compounded return on the stock. Under this abstraction the rate of return over any small interval of time Δt is distributed $N\left(\hat{\mu}\Delta t,\, \hat{\sigma}\sqrt{\Delta t}\right)$; furthermore, these returns are independently distributed through time. Recall (Chapter 3) that these are the two most basic statistical properties of stock returns. More precisely, Equation (1) describes the discrete time approximation to geometric Brownian motion.

True Geometric Brownian motion presumes continuous trading, and its attendant continuous compounding of returns. Of course continuous trading presumes an uncountably large number of "trades" in any finite interval of time, which is impossible. It should be thought of as a very useful mathematical abstraction.

Under continuous trading the expression analogous to Equation (1) is

$$\frac{dS}{S} = \hat{\mu}dt + \hat{\sigma}\tilde{\varepsilon}\sqrt{dt}\,. \tag{2}$$

Much more will be said about this price process in the web-complement entitled "*An Intuitive Overview of Continuous Time Finance*"

2. The formula: The Black-Scholes formula is given by

$$C_T(S, K) = SN(d_1) - e^{\hat{r}_f T} KN(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S}{K}\right) + \left(\hat{r}_f - \frac{1}{2}\hat{\sigma}^2\right)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \hat{\sigma}\sqrt{T}\,. \end{aligned}$$

In this formula:

S = the price of the stock "today" (at the time the call valuation is being undertaken);

K = the exercise price;

T = the time to expiration, measured in years;

$\hat{r}_f$ = the estimated continuously compounded annual risk free rate;

$\hat{\sigma}$ = the estimated standard deviation of the continuously compounded rate of return on the underlying asset annualized; and

$N(\ )$ is the standard normal distribution.

In any practical problem, of course, $\hat{\sigma}$ must be estimated. The risk free rate is usually unambiguous as normally there is a T-bill coming due on approximately the same date as the options contracts expire (U.S. markets).

3. An example

Suppose

S = \$68

K = \$60

T = 88 days = $\frac{88}{365}$ = .241 years

r_f = 6% (not continuously compounded)

$\hat{\sigma} = .40$

The r_f inserted into the formula is that rate which, when continuously compounded, is equivalent to the actual 6% annual rate; this must satisfy

$$e^{\hat{r}_f} = 1.06, \text{ or } \hat{r}_f = \ln(1.06) = .058.$$

Thus,

$$\begin{aligned}
d_1 &= \frac{\ln\frac{68}{60} + \left(.058 + \frac{1}{2}(.4)^2(.241)\right)}{(.40)\sqrt{.241}} = .806 \\
d_2 &= .806 - (.40)\sqrt{.241} = .610 \\
N(d_1) &= N(.806) \approx .79 \\
N(d_2) &= N(.610) \approx .729 \\
C &= \$.68(.79) - e^{-(.058)(.241)}(\$60)(.729) \\
&= \$10.60
\end{aligned}$$

4. Estimating σ.

We gain intuition about the Black-Scholes model if we understand how its inputs are obtained, and the only input with any real ambiguity is σ. Here we present a straightforward approach to its estimation based on the security's historical price series.

Since volatility is an unstable attribute of a stock, by convention it is viewed as unreliable to go more than 180 days into the past for the choice of historical period. Furthermore, since we are trying to estimate the

μ, σ of a continuously compounded return process, the interval of measurement should be, in principle, as small as possible. For most practical applications, daily data is the best we can obtain.

The procedure is as follows:

i) Select the number of chosen historical observations and index them $i = 0, 1, 2, ..., n$ with observation 0 most distant into the past and observation n most recent. This gives us $n+1$ price observations. From this we will obtain n daily return observations.

ii) Compute the equivalent continuously compounded ROR on the underlying asset over the time intervals implied by the selection of the data (i.e., if we choose to use daily data we compute the continuously compounded daily rate):

$$r_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

This is the equivalent continuously compounded ROR from the end of period $i-1$ to the end of period i.

Why is this the correct calculation? Suppose $S_i = 110$, $S_{i-1} = 100$; we want that continuously compounded return x to be such that

$$\begin{aligned} S_{i-1}e^x &= S_i, \text{ or } 100e^x = 110 \\ e^x &= (110/100) \\ x &= \ln(\frac{110}{100}) = .0953. \end{aligned}$$

This is the continuously compounded rate that will increase the price from \$100 to \$110.

iii) Compute the sample mean

$$\hat{\mu} = \frac{1}{n}\left(\sum_{i=1}^{n} r_i\right)$$

Remark: If the time intervals are all adjacent; i.e., if we have not omitted any observations, then

$$\begin{aligned} &\hat{\mu} = \tfrac{1}{n}\sum_{i=1}^{n} r_i = \tfrac{1}{n}\left[\ln\left(\tfrac{S_1}{S_0}\right) + \ln\left(\tfrac{S_2}{S_1}\right) + ... + \ln\left(\tfrac{S_n}{S_{n-1}}\right)\right] \\ &= \tfrac{1}{n}\left[\ln\left(\tfrac{S_1}{S_0}\cdot\tfrac{S_2}{S_1}\cdots\tfrac{S_n}{S_{n-1}}\right)\right] \\ &= \tfrac{1}{n}\ln\left(\tfrac{S_n}{S_0}\right) \end{aligned}$$

Note that if we omit some calendar observations – perhaps due to, say, merger rumors at the time which are no longer relevant – this shortcut fails.

iv) Estimate σ

$$\hat{\sigma} = \sqrt{\frac{1}{n-1}\left(\sum_{i=1}^{n}(r_i-\hat{\mu})^2\right)}\ ,\ or$$

$$\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} r_i^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n} r_i\right)^2}$$

Example: Consider the (daily) data in Table 2.

Table 2

Period	Closing price		$\ln\left(\frac{S_i}{S_{i-1}}\right)$
i=0	\$26		$r_1 = \ln\left(\frac{26.50}{26}\right) = .0190482$
i=1	\$26.50		$r_2 = \ln\left(\frac{26.25}{26.50}\right) = -.009479$
i=2	\$26.25		$r_3 = \ln\left(\frac{26.25}{26.25}\right) = 0$
i=3	\$26.25		$r_4 = \ln\left(\frac{26.50}{26.25}\right) = .0094787$
i=4	\$26.50		

In this case $\hat{\mu} = \frac{1}{4}\ln\left(\frac{26.50}{26}\right) = 1/4; \ln(1.0192308) = .004762$
Using the above formula,

$$\begin{aligned}
\sum_{i=1}^{4} r_i^2 &= (.0190482)^2 + (-.009479)^2 + (.0094787)^2 = .0005472 \\
\left(\sum_{i=1}^{4} r_i\right)^2 &= (.0190482 - .009479 + .0094787)^2 = (.01905)^2 = .0003628 \\
\hat{\sigma} &= \sqrt{\frac{1}{3}(.005472) - \frac{1}{12}(.003628)} \\
&= \sqrt{.0001809 - .0000302} = \sqrt{.0001507} = .0123 \\
\hat{\sigma}^2 = (.0123)^2 &= .0001507
\end{aligned}$$

v) Annualize the estimate of σ

We will assume 250 trading days per year. Our estimate for the continuously compounded annual return is thus:

$$\hat{\sigma}^2_{annual} = 250\underbrace{(.0001507)}_{\sigma^2_{daily}} = .0377$$

Remark: Why do we do this? Why can we multiply our estimate by 250 to scale things up? We can do this because of our geometric Brownian motion assumption that returns are independently distributed. This is detailed as follows:

Our objective: an estimate for var $\left(\ln\left(\frac{S_{T=1yr}}{S_0}\right)\right)$ given 250 trading days,

$$\begin{aligned}
\text{var}\left(\ln\left(\frac{S_{T=1yr}}{S_0}\right)\right) &= \text{var}\left(\ln\left(\frac{S_{day250}}{S_0}\right)\right) \\
&= \text{var}\left(\ln\left(\frac{S_{day1}}{S_0}\cdot\frac{S_{day2}}{S_{day1}}\cdots\frac{S_{day250}}{S_{day249}}\right)\right) \\
&= \text{var}\left(\ln\left(\frac{S_{day1}}{S_0}\right)+\ln\left(\frac{S_{day2}}{S_{day1}}\right)+\ldots+\ln\left(\frac{S_{day250}}{S_{day249}}\right)\right) \\
&= \text{var}\left(\ln\left(\frac{S_{day1}}{S_0}\right)\right)+\text{var}\left(\ln\left(\frac{S_{day2}}{S_{day1}}\right)\right)+\ldots+\text{var}\left(\ln\left(\frac{S_{day250}}{S_{day249}}\right)\right)
\end{aligned}$$

The latter equivalence is true because returns are uncorrelated from day to day under the geometric Brownian motion assumption.

Furthermore, the daily return distribution is presumed to be the same for every day, and thus the daily variance is the same for every day under geometric Brownian motion.

Thus, var $\left(\ln\left(\frac{S_{T=1yr}}{S_0}\right)\right) = 250\,\sigma^2_{daily}$.

We have obtained an estimate for σ^2_{daily} which we will write as $\hat{\sigma}^2_{daily}$. To convert this to an annual variance, we must thus multiply by 250.

Hence $\hat{\sigma}^2_{annual} = 250\cdot\sigma^2_{daily} = .0377$, as noted.

If a weekly $\hat{\sigma}^2$were obtained, it would be multiplied by 52.

4 The Black-Scholes Formula for an Index.

Recall that the Black-Scholes formula assumed that the underlying stock did not pay any dividends, and if this is not the case an adjustment must be made. A natural way to consider adapting the Black-Scholes formula to the dividend situation is to replace the underlying stock's price S in the formula by $S - PV(EDIVs)$, where $PV(EDIVs)$ is the present value, relative to $t = 0$, the date at which the calculation is being undertaken, of all dividends expected to be paid over the time to the option's expiration. In cases where the dividend is highly uncertain this calculation could be problematic. We want to make such an adjustment because the payment of a dividend reduces the underlying stock's value by the amount of the dividend and thus reduces the value (ceteris paribus) of the option written on it. Options are not "dividend protected," as it is said.

For an index, such as the S&P$_{500}$, the dividend yield on the index portfolio can be viewed as continuous, and the steady payment of this dividend will have a continuous tendency to reduce the index value. Let d denote the dividend yield on the index. In a manner exactly analogous to the single stock dividend treatment noted above, the corresponding Black-Scholes formula is

$$\begin{aligned} C &= Se^{-dT}N(d_1) - Ke^{-r_f T}N(d_2) \\ \text{where } d_1 &= \frac{\ln {}^{S}\!/_{K} + \left(r_f - d + {}^{\sigma^2}\!/_{2}\right)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T}\ . \end{aligned}$$