An Intuitive Overview of Continuous Time Finance

June 1, 2005

1 Introduction

If we think of stock prices as arising from the equilibration of traders demands and supplies, then the binomial model is implicitly one in which security trading occurs at discrete time intervals, however short, and this is factually what actually happens. It will be mathematically convenient, however, to abstract from this intuitive setting and hypothesize that trading takes place "continuously." This is consistent with the notion of continuous compounding. But it is not fully realistic: it implies that an uncountable number of individual transactions may transpire in any interval of time, however small, which is physically impossible.

Continuous time finance is principally concerned with techniques for the pricing of derivative securities under the fiction of continuous trading. These techniques frequently allow closed form solutions to be obtained – at the price of working in a context that is less intuitive than discrete time. In this Appendix we hope to convey some idea as to how this is done.

We will need first to develop a continuous time model of a stock's price evolution through time. Such a model must respect the basic statistical regularities which are known to characterize, empirically, equity returns:

(i) stock prices are lognormally distributed, which means that returns (continuously compounded) are normally distributed;

(ii) for short time horizons stock returns are independently and identically distributed over non-overlapping time intervals.

After we have faithfully represented these equity regularities in a continuous time setting, we will move on to a consideration of derivatives pricing. In doing so we aim to give some idea how the principles of risk neutral valuation carry over to this specialized setting. The discussion aims at intuition; no attempt is made to be mathematically complete.

In all cases this intuition has its origins in the discrete time context. This leads to a discussion of random walks.

2 Random Walks and Brownian Motion

Consider a time horizon composed of N adjacent time intervals each of duration Δt, and indexed by $t_0, t_1, t_2, ...t_N$; that is,

$$t_i - t_{i-1} = \Delta t, \;\; i = 1,\, 2,\, ...,\, N\, .$$

We define a discrete time stochastic process on this succession of time indices by

$$\begin{aligned} &x(t_0) = 0 \\ &x(t_{j+1}) = x(t_j) + \tilde{\varepsilon}(t_j)\sqrt{\Delta t}, \;\; j = 0,\, 1,\, 2,\, ...,\, N-1\,, \end{aligned}$$

where, for all j, $\tilde{\varepsilon}(t_j) \sim N(0,\, 1)$. It is further assumed that the random factors $\tilde{\varepsilon}(t_j)$are independent of one another; i.e.,

$$E\left(\tilde{\varepsilon}(t_j)\tilde{\varepsilon}(t_i)\right) = 0, \;\; i \neq j\,.$$

This is a specific example of a random walk, specific in the sense that the uncertain disturbance term follows a normal distribution [1].

We are interested to understand the behavior of a random walk over extended time periods. More precisely, we want to characterize the statistical properties of the difference

$x(t_k) - x(t_j)$ for any $j < k$.

Clearly,

$$\tilde{x}(t_k) - x(t_j) = \sum_{i=j}^{k-1} \tilde{\varepsilon}(t_i)\sqrt{\Delta t}\,.$$

Since the random disturbances $\tilde{\varepsilon}(t_i)$ all have mean zero,

$$E\left(\tilde{x}(t_k) - x(t_j)\right) = 0.$$

Furthermore,

$$\begin{aligned} \operatorname{var}\left(x(t_k) - x(t_j)\right) &= E\left(\sum_{i=j}^{k-1} \tilde{\varepsilon}(t_i)\sqrt{\Delta t}\right)^2 \\ &= E\left(\sum_{i=j}^{k-1} \left[\tilde{\varepsilon}(t_i)\right]^2 \Delta t\right) \text{ (by independance)} \\ &= \sum_{i=j}^{k-1} (1)\Delta t = (k-j)\Delta t, \text{since} \\ E\left[\,\tilde{\varepsilon}(t_i)\right]^2 &= 1\,. \end{aligned}$$

[1] In particular, a very simple random walk could be of the form $x(t_{j+1}) = x(t_j) + n(t_j)$, where for all $j = 0, 1, 2...$

$$n(t_j) = \begin{array}{l} +1, \text{ if a coin is flipped and a head appears} \\ -1, \text{ if a coin is flipped and a tail appears .} \end{array}$$

At each time interval $x(t_j)$ either increases or diminishes by one depending on the outcome of the coin toss. Suppose we think of $x(t_0) \equiv 0$as representing the center of the sidewalk where an intoxicated person staggers one step to the right or to the left of the center in a manner that is consistent with independent coin flips (heads implies to the right). This example is the source of the name "random walk."

If we identify

$$x_{t_j} = \ln q^e_{t_j},$$

where $q^e_{t_j}$ is the price of the stock at time t_j, then this simple random walk model becomes a candidate for our model of stock price evolution beginning from $t = 0$: At each node t_j, the logarithm of the stock's price is distributed normally, with mean $\ln q^e_{t_0}$and variance $j\Delta t$.

Since the discrete time random walk is so respectful of the empirical realities of stock prices, it is natural to seek its counterpart for continuous time. This is referred to as a "Brownian Motion" (or a Weiner process), and it represents the limit of the discrete time random walk as we pass to continuous time; i.e., as $\Delta t \mapsto 0$. It is represented symbolically by

$$dz = \tilde{\varepsilon}(t)\sqrt{dt}\,,$$

where $\tilde{\varepsilon}(t) \sim N(0,1)$, and for any times t, t' where $t \neq t'$, and $E\left(\tilde{\varepsilon}(t')\tilde{\varepsilon}(t)\right) = 0$. We used the word "symbolically" not only because the term dz does not represent a differential in the terminology of ordinary calculus but also because we make no attempt here to describe how such a limit is taken. Following what is commonplace notation in the literature we will also not write a $\sim$ over z even though it represents a random quantity.

More formally, a stochastic process $z(t)$ defined on $[0,T]$ is a Brownian motion provided the following three properties are satisfied:

(i) for any $t_1 < t_2$, $z(t_2) - z(t_1)$ is normally distributed with mean zero and variance $t_2 - t_1$;

(ii) for any $0 \leq t_1 < t_2 \leq t_3 < t_4$, $z(t_4) - z(t_3)$ is statistically independent of $z(t_2) - z(t_1)$; and

(iii) $z(t_0) \equiv 0$ with probability one.

A Brownian motion is a very unusual stochastic process, and we can only give a hint about what is actually transpiring as it evolves. Three of its properties are considered below:

1. First, a Brownian motion is a continuous process. If we were able to trace out a sample path $z(t)$ of a Brownian motion, we would not see any jumps[2].

2. However, this sample path is not at all "smooth" and is, in fact, as "jagged as can be" which we formalize by saying that it is nowhere differential. A function must be essentially smooth if it is to be differentiable. That is, if we magnify a segment of its time path enough, it will appear approximately linear. This latter "smoothness" is totally absent with a Brownian motion.

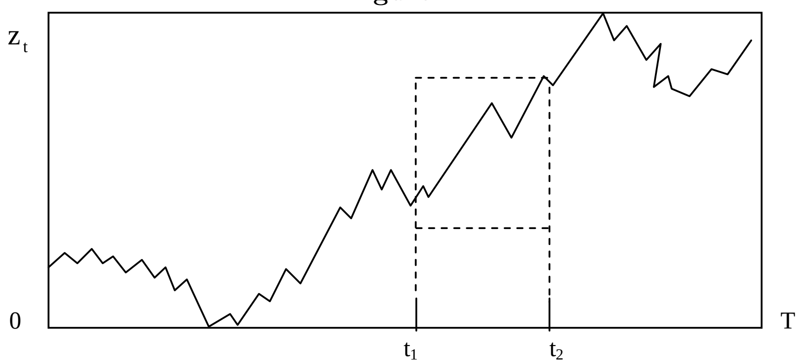
3. Lastly a Brownian motion is of "unbounded variation." This is perhaps the least intuitive of its properties. By this is intended the idea that if we could take one of those mileage wheels which are drawn along a route on a map to assess the overall distance (each revolution of the wheel corresponding to a fixed number of kilometers) and apply it to the sample path of a Brownian motion,

[2]At times such as the announcement of a take over bid, stock prices exhibit jumps. We will not consider such "jump processes," although considerable current research effort is being devoted to studying them, and to the pricing of derivatives written on them.

then no matter how small the time interval, the mileage wheel would record "an infinite distance" (if it ever got to the end of the path!)

One way of visualizing such a process is to imagine a rough sketch of a particular sample path where we connect its position at a sequence of discrete time intervals by straight lines. Figure 1 proposes one such path.

Figure 1



Suppose that we were next to enlarge the segment between time intervals t_1 and t_2. We would find something on the order of Figure 2.

Figure 2

Continue this process of taking a segment, enlarging it, taking another subsegment of that segment, enlarging it etc., etc. (in Figure 2 we would next enlarge the segment from t_3 to t_4). Under a typical differentiable function of bounded variation, we would eventually be enlarging such a small segment that it would appear as a straight line. With a Brownian motion, however, this will never happen. No matter how much we enlarge even a segment that corresponds to an arbitrarily short time interval, the same "sawtooth" pattern will appear, and there will be many, many "teeth".

A Brownian motion process represents a very special case of a continuous process with independent increments. For such processes, the standard deviation per unit of time becomes unbounded as the interval becomes smaller and smaller:

$$\lim_{\Delta t \mapsto 0} \frac{\sigma\sqrt{\Delta t}}{\Delta t} = \lim_{\Delta t \mapsto 0} \frac{\sigma}{\sqrt{\Delta t}} = \infty.$$

No matter how small the time period, proportionately, a *lot* of variation remains. This constitutes our translation of the abstraction of a discrete time random walk to a context of continuous trading[3].

3 More General Continuous Time Processes

A Brownian motion will be the principal building block of our description of the continuous time evolution of a stock's price – it will be the "engine" or "source" of the uncertainty. To it is often added a deterministic component which is intended to capture the "average" behavior through time of the process. Together we have something of the form

$$dx(t) = adt + b\tilde{\varepsilon}(t)\sqrt{dt} = adt + bdz, \tag{1}$$

where the first component is the deterministic one and a is referred to as the drift term.

This is an example of a generalized Brownian motion or, to use more common terminology, a generalized Weiner process. If there were no uncertainty x(t) would evolve deterministically; if we integrate

$dx(t) = adt$, we obtain

$x(t) = x(0) + at$

The solution to (1) is thus of the form

$$x(t) = x(0) + at + bz(t)\,, \tag{2}$$

where the properties of $z(t)$ were articulated earlier (recall properties (1), (2), and (3) of the definition). These imply that:

$$\begin{aligned}&E\left(x(t)\right) = x(0) + at,\\ &\text{var}\left(x(t)\right) = b^2 t, \text{ and}\\ &s.d\left(x(t)\right) = b\sqrt{t}.\end{aligned}$$

Equation (2) may be further generalized to allow the coefficients to depend upon the time and the current level of the process:

$$dx(t) = a\left(x(t), t\right) dt + b\left(x(t), t\right) dz. \tag{3}$$

[3] The name Brownian motion comes from a 19^{th} century physicist Brown, who studied the behavior of dust particles floating on the surface of water. Under a microscope dust particles are seen to move randomly about in a manner similar to the sawtooth pattern above except that the motion can be in any 360^o direction. The interpretation of the phenomena is that the dust particles experience the effect of random collisions by moving water molecules.

In this latter form, it is referred to as an Ito process after one of the earliest and most important developers of this field. An important issue in the literature - but one we will ignore - is to determine the conditions on $a\,(x(t),t)$ and $b\,(x(t),t)$ in order for equation (3) to have a solution. Equations (1) and (3) are generically referred to as stochastic differential equations.

Given this background, we now return to the original objective of modeling the behavior of a stock's price process.

4 A Continuous-Time Model of Stock Price Behavior

Let us now restrict our attention only to those stocks which pay no dividends, so that stock returns are exclusively determined by price changes. Our basic discrete time model formulation is:

$$\ln q^e(t+\Delta t) - \ln q^e(t) = \mu\Delta t + \sigma\tilde{\varepsilon}\sqrt{\Delta t}. \tag{4}$$

Notice that the stochastic process is imposed on differences in the logarithm of the stock's price. Equation (4) thus asserts that the continuously compounded return to the ownership of the stock over the time period t to $t+\Delta t$ is distributed normally with mean $\mu\Delta t$ and variance $\sigma^2\Delta t$.

This is clearly a lognormal model:

$$\ln\left(q^e(t+\Delta t)\right) \sim N\left(\ln q^e(t) + \mu\Delta t, \sigma\tilde{\varepsilon}\sqrt{\Delta t}\right).$$

It is a more general formulation than a pure random walk as it admits the possibility that the mean increase in the logarithm of the price is positive. The continuous time analogue of (4) is

$$d\ln q^e(t) = \mu dt + \sigma dz. \tag{5}$$

Following (2), it has the solution

$$\ln q^e(t) = \ln q^e(0) + \mu t + \sigma z(t)\,, \tag{6}$$

where

$$\begin{aligned} &E\ln q^e(t) = \ln q^e(0) + \mu t\,, \text{ and} \\ &\operatorname{var} q^e(t) = \sigma^2 t \end{aligned}$$

Since $\ln q^e(t)$ on average grows linearly with t (so that, on average, $q^e(t)$ will grow exponentially), Equations (5) and (6) are, together, referred to as a geometric Brownian motion(GBM). It is clearly a lognormal process: $\ln q^e(t) \sim N\left(\ln q^e(0) + \mu t, \sigma\sqrt{t}\right)$. The webcomplement entitled "Review of Basic Options Concepts and Terminology" illustrates how the parameters μ and σ can be estimated under the maintained assumption that time is measured in years.

While Equation (6) is a complete description of the evolution of the logarithm of a stock's price, we are rather interested in the evolution of the price itself. Passing from a continuous time process on $\ln q^e(t)$ to one on $q^e(t)$is not a trivial matter, however, and we need some additional background to make the conversion correctly. This is considered in the next few paragraphs.

The essence of lognormality is the idea that if a random variable $\tilde{y}$ is distributed normally, then the random variable $\tilde{w} = e^{\tilde{y}}$ is distributed lognormally. Suppose, in particular, that $\tilde{y} \sim N(\mu_y, \sigma_y)$. A natural question is: how are μ_w and σ_w related to μ_y and σ_y when $\tilde{w} = e^{\tilde{y}}$? We first note that

$$\mu_w \neq e^{\mu_y}, \text{ and } \sigma_w \neq e^{\sigma_y}.$$

Rather, it can be shown that

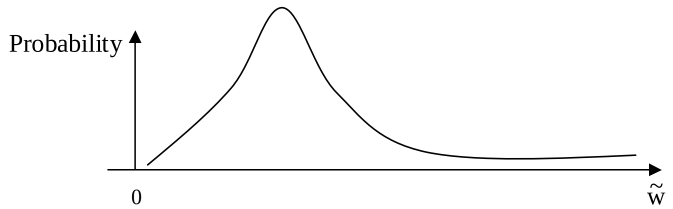
$$\mu_w = e^{\mu_y + 1/2\sigma_y^2} \tag{7}$$

and

$$\sigma_w = e^{\mu_y + 1/2\sigma_y^2} \left(e^{\sigma_y^2} - 1 \right)^{1/2}. \tag{8}$$

These formulae are not obvious, but we can at least shed some light on (7). Why the variance of $\tilde{y}$ should have an impact on the mean of $\tilde{w}$. To see why this is so, let us remind ourselves of the shape of the lognormal probability density function as found in Figure 3

Figure 3: A lognormal density function



Suppose there is an increase in variance. Since this distribution is pinched off to the left at zero, a higher variance can only imply (within the same class of distributions) that probability is principally shifted to higher values of $\tilde{w}$. But this will have the simultaneous effect of increasing the mean of $\tilde{w}$. The variance of $\tilde{y}$ and the mean of $\tilde{y}$ cannot be specified independently. The mean and standard deviation of the lognormal variable $\tilde{w}$ are thus each related to both the mean and variance of $\tilde{y}$ as per the relationships in Equations (7) and (8).

These results allow us to express the man and standard deviation of $q^e(t)$ (by analogy, $\tilde{w}$) in relation to $\ln q^e(t) + \mu t$ and $\sigma^2 t$ (by analogy, the mean and variance of $\tilde{y}$) via Equations (5) and (6):

$$
\begin{aligned}
Eq^e(t) &= e^{\ln q^e(0)+\mu t+\frac{1}{2}\sigma^2 t} = q^e(0)e^{\mu t+\frac{1}{2}\sigma^2 t} && (9)\\
\text{s.d.}q^e(t) &= e^{\ln q^e(0)+\mu t+\frac{1}{2}\sigma^2 t}\left(e^{\sigma^2 t}-1\right)^{\frac{1}{2}} && (10)\\
&= q^e(0)e^{\mu t+\frac{1}{2}\sigma^2 t}\left(e^{\sigma^2 t}-1\right)^{\frac{1}{2}}.
\end{aligned}
$$

We are now in a position, at least at an intuitive level, to pass from a stochastic differential equation describing the behavior of $\ln q^e(t)$ to one that governs the behavior of $q^e(t)$. If $\ln q^e(t)$ is governed by Equation (5), then

$$\frac{dq^e(t)}{q^e(t)} = \left(\mu + \frac{1}{2}\sigma^2\right)dt + \sigma dz(t) \tag{11}$$

where $\frac{dq^e(t)}{q^e(t)}$ can be interpreted as the instantaneous (stochastic) rate of price change. Rewriting Equation (11) slightly differently yields

$$dq^e(t) = \left(\mu + \frac{1}{2}\sigma^2\right)q^e(t)dt + \sigma q^e(t)dz(t) \tag{12}$$

which informs us that the stochastic differential equation governing the stocks price represents an Ito process since the coefficients of dt and $dz(t)$ are both time dependent.

We would also expect that if $q^e(t)$ were governed by

$$
\begin{aligned}
dq^e(t) &= \mu q^e(t)dt + \sigma q^e(t)dz(t), \text{ then} && (13)\\
d\ln q^e(t) &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dz(t). && (14)
\end{aligned}
$$

Equations (13) and (14) are fundamental to what follows.

5 Simulation and Call Pricing

5.1 Ito Processes

Ito Processes and their constituents, most especially the Brownian motion, are difficult to grasp at this abstract level and it will assist our intuition to describe how we might simulate a discrete time approximation to them.

Suppose we have estimated $\hat{\mu}$ and $\hat{\sigma}$ for a stock's price process as suggested in the webcomplement "Review of Options..". Recall that these estimates are derived from daily price data properly scaled up to reflect the fact that in this literature it is customary to measure time in years. We have two potential stochastic differential equations to guide us – Equations (13) and (14) – and each has a discrete time approximate counterpart.

(i) Discrete Time Counterpart to Equation (13)

If we approximate the stochastic differential $dq^e(t)$ by the change in the stock's price over a short interval of time Δt we have,

$$\begin{aligned} q^e(t+\Delta t) - q^e(t) &= \hat{\mu} q^e(t)\Delta t + \hat{\sigma} q^e(t)\tilde{\varepsilon}(t)\sqrt{\Delta t}, \text{ or} \\ q^e(t+\Delta t) &= q^e(t)\left[1 + \hat{\mu}\Delta t + \hat{\sigma}\tilde{\varepsilon}(t)\sqrt{\Delta t}\right] \end{aligned} \tag{15}$$

There is a problem with this representation, however, because for any $q^e(t)$, the price next "period," $q^e(t+\Delta t)$, is normally distributed (recall that $\tilde{\varepsilon}(t) \sim N(0,1)$) rather than lognormal as a correct match to the data requires. In particular, there is the unfortunate possibility that the price could go negative, although for small time intervals Δt, this is exceedingly unlikely.

(ii) Discrete Time Counterpart to Equation (14)

Approximating $d \ln q^e(t)$ by successive log values of the price over small time intervals Δt yields

$$\begin{aligned} \ln q^e(t+\Delta t) - \ln q^e(t) &= \left(\hat{\mu} - \tfrac{1}{2}\hat{\sigma}^2\right)\Delta t + \hat{\sigma}\tilde{\varepsilon}(t)\sqrt{\Delta t}, \text{ or} \\ \ln q^e(t+\Delta t) &= \ln q^e(t) + \left(\hat{\mu} - \tfrac{1}{2}\hat{\sigma}^2\right)\Delta t + \hat{\sigma}\tilde{\varepsilon}\sqrt{\Delta t}\,. \end{aligned} \tag{16}$$

Here it is the logarithm of the price in period $t+\Delta t$ that is normally distributed, as required, and for this reason we'll limit ourselves to (16) and its successors. For simulation purposes, it is convenient to express equation (16) as

$$q^e(t+\Delta t) = q^e(t)e^{\left(\hat{\mu}-\frac{1}{2}\hat{\sigma}^2\right)\Delta t + \hat{\sigma}\tilde{\varepsilon}(t)\sqrt{\Delta t}}. \tag{17}$$

It is easy to generate a possible sample path of price realizations for (17). First select an interval of time Δt, and the number of successive time periods of interest (this will be the length of the sample path), say N. Using a random number generator, next generate N successive draws from the standard normal distribution. By construction, these draws are independent and thus successive rates of return $\left(\frac{q^e(t+\Delta t)}{q^e(t)} - 1\right)$ will be statistically independent of one another. Let this series of N draws be represented by $\{\varepsilon_j\}_{j=1}^{N}$. The corresponding sample path (or "time series") of prices is thus created as per Equation (18)

$$q^e(t_{j+1}) = q^e(t_j)e^{\left(\hat{\mu}-\frac{1}{2}\hat{\sigma}^2\right)\Delta t + \hat{\sigma}\tilde{\varepsilon}_j\sqrt{\Delta t}}, \tag{18}$$

where $t_{j+1} = t_j + \Delta t$.

This is not the price path that would be used for derivatives pricing, however.

5.2 The binomial model

Under the binomial model, call valuation is undertaken in a context where the probabilities have been changed in such a way so that all assets, including the underlying stock, earn the risk free rate. The simulation-based counterpart to this transformation is to replace $\hat{\mu}$ by $\ln(1+r_f)$ in Equations (17) and (18):

$$q^e(t+\Delta t) = q^e(t)e^{\left(\ln(1+r_f)-\frac{1}{2}\hat{\sigma}^2\right)\Delta t + \sigma\tilde{\varepsilon}(t)\sqrt{\Delta t}}, \tag{19}$$

where r_f is the one year risk free rate (not continuously compounded) and $\ln(1+r_f)$ is its continuously compounded counterpart.

How would we proceed to price a call in this simulation context? Since the value of the call at expiration is exclusively determined by the value of the underlying asset at that time, we first need a representative number of possible "risk neutral prices" for the underlying asset at expiration. The entire risk neutral sample path - as per equation (18) - is not required. By "representative" we mean enough prices so that their collective distribution is approximately lognormal. Suppose it was resolved to create J sample prices (to be even reasonably accurate$J \geq 1000$) at expiration, T years from now. Given random draws $\{\varepsilon_k\}_{k=1}^{J}$ from $N(0,1)$, the corresponding underlying stock price realizations are $\{q_k^e(T)\}_{k=1}^{J}$ as given by

$$q_k^e(T) = q^e(0) e^{\left(\ln(1+r_f) - \frac{1}{2}\hat{\sigma}^2\right)T + \sigma\tilde{\varepsilon}_k\sqrt{\Delta T}} \tag{20}$$

For each of these prices, the corresponding call value at expiration is

$$C_k^T = \max\{0, q_k^e(T) - E\}, \quad k = 1, 2, ..., J.$$

The average expected payoff across all these possibilities is

$$C_{Avg}^T = \frac{1}{J}\sum_{k=1}^{J} C_k^T.$$

Since under risk neutral valuation the expected payoff of any derivative asset in the span of the underlying stock and a risk free bond is discounted back at the risk free rate, our estimate of the calls value today (when the stock's price is $q^e(0)$) is

$$C^0 = e^{-\ln(1+r_f)T} C_{Avg}^T. \tag{21}$$

In the case of an Asian option or some other path dependent option, a large number of sample paths must to be generated since the exercise price of the option (and thus its value at expiration) is dependent upon the entire sample path of underlying asset prices leading to it.

Monte Carlo simulation is, as the above method is called, not the only pricing technique where the underlying idea is related to the notion of risk neutral valuation. There are ways that stochastic differential equations can be solved directly.

6 Solving Stochastic Differential Equations: A First Approach

Monte Carlo simulation employs the notion of risk neutral valuation but it does not provide closed form solutions for derivatives prices, such as the Black Scholes formula in the case of calls [4]. How are such closed firm expressions obtained? In what follows we provide a non-technical outline of the first of two available methods. The context will once again be European call valuation where the underlying stock pays no dividends.

The idea is to obtain a partial differential equation whose solution, given the appropriate boundary condition is the price of the call. This approach is due to Black and Scholes (1973) and, in a more general context, Merton (1973). The latter author's arguments will guide our discussion here.

In the same spirit as the replicating portfolio approach mentioned in Section 4, Merton (1973) noticed that the payoff to a call can be represented in continuous time by a portfolio of the underlying stock and a risk free bond whose quantities are continuously adjusted. Given the stochastic differential equation which governs the stock's price (13) and another non stochastic differential equation governing the bond's price evolution, it becomes possible to construct the stochastic differential equation governing the value of the replicating portfolio. This latter transformation is accomplished via an important theorem which is referred to in the literature as Ito's Lemma. Using results from the stochastic calculus, this expression can be shown to imply that the value of the replicating portfolio must satisfy a particular partial differential equation. Together with the appropriate boundary condition (e.g., that $C(T) = \max\{q^e(T) - E, 0\}$), this partial differential equation has a known solution – the Black Scholes formula.

In what follows we begin first with a brief overview of this first approach; this is accomplished in three steps.

6.1 The Behavior of Stochastic Differentials.

In order to motivate what follows, we need to get a better idea of what the object dz(t) means. It is clearly a random variable of some sort. We first explore its moments. Formally, $dz(t)$ is

$$\lim_{\Delta t \mapsto 0} z(t+\Delta t) - z(t), \tag{22}$$

where we will not attempt to be precise as to how the limit is taken. We are reminded, however, that

$$\begin{aligned}
&E\left[z(t+\Delta t) - z(t)\right] = 0, \text{ and} \\
&\operatorname{var}\left[z(t+\Delta t) - z(t)\right] = \left(\sqrt{\Delta t}\right)^2 = \Delta t\,, \text{ for all } \Delta t.
\end{aligned}$$

It is not entirely surprising, therefore that

$$E\left(dz(t)\right) \equiv \lim_{\Delta t \mapsto 0} E\left[z(t+\Delta t) - z(t)\right] = 0, \text{ and} \tag{23}$$

$$var\left(dz(t)\right) = \lim_{\Delta t \mapsto 0} E\left[\left(z(t+\Delta t) - z(t)\right)^2\right] = dt. \tag{24}$$

[4] The estimate obtained using Monte Carlo simulation will coincide with the Black Scholes value to a high degree of precision, however, if the number of simulated underlying stock prices is large (≥10,000) and the parameters r_f, E, σ, T are identical.

The object $dz(t)$ may thus be viewed as denoting an infinitesimal random variable with zero mean and variance dt (very small, but we are in a world of infinitesimals).

There are several other useful relationships:

$$
\begin{aligned}
E\left(dz(t)\,dz(t)\right) &\equiv var\left(dz(t)\right) = dt && (25)\\
var\left(dz(t)\,dz(t)\right) &= \lim_{\Delta t\mapsto 0} E\left[\left(z(t+\Delta t)-z(t)\right)^4 - (\Delta t)^2\right] && (26)\\
&\approx 0\\
E\left(dz(t)\,dt\right) &= \lim_{\Delta t\mapsto 0} E\left[\left(z(t+\Delta t)-z(t)\right)\Delta t\right] = 0 && (27)\\
var\left(dz(t)\,dt\right) &= \lim_{\Delta t\mapsto 0} E\left[\left(z(t+\Delta t)-z(t)\right)^2 (\Delta t)^2\right] \approx 0 && (28)
\end{aligned}
$$

Equation (28) and (26) imply, respectively, that (25) and (**??**) are not only satisfied in expectation but with equality. Expression (25) is, in particular, quite surprising, as it argues that the square of a Brownian motion random process is effectively deterministic.

These results are frequently summarized as in Table 2 where $(dt)^2$ is negligible in the sense that it is very much smaller than dt and we may treat it as zero.

Table 2: The Product of Stochastic Differentials

	dz	dt
dz	dt	0
dt	0	0

The power of these results is apparent if we explores their implications for the computation of a quantity such as $\left(dq^e(t)\right)^2$:

$$
\begin{aligned}
\left(dq^e(t)\right)^2 &= \left(\mu dt + \sigma dz(t)\right)^2\\
&= \mu^2 (dt)^2 + 2\mu\sigma dt dz(t) + \sigma^2 \left(dz(t)\right)^2\\
&= \sigma^2 dt,
\end{aligned}
$$

since, by the results in Table 2, $(dt)(dt) = 0$ and $dtdz(t) = 0$.

The object $dq^e(t)$ thus behaves in the manner of a random walk in that its variance is proportional to the length of the time interval.

We will use these results in the context of Ito's lemma.

6.2 Ito's Lemma

A statement of this fundamental result is outlined below.

<u>Theorem (Ito's Lemma)</u>.

Consider an Ito process $dx(t)$ of form $dx(t) = a(x(t),t)\,dt + b(x(t),t)\,dz(t)$, where $dz(t)$ is a Brownian motion, and consider a process $y(t) = F(x(t),t)$. Under quite general conditions $y(t)$ satisfies the stochastic differential equation

$$dy(t) = \frac{\partial F}{\partial x} dx(t) + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} (dx(t))^2 . \tag{29}$$

The presence of the right most term (which would be absent in a standard differential equation) is due to the unique properties of a stochastic differential equation. Taking advantage of results (in Table 2) let us specialize Equation (29) to the standard Ito process, where for notational simplicity, we suppress the dependence of coefficients $a(\ )$ and $b(\ )$ on $x(t)$ and t:

$$\begin{aligned} dy(t) &= \tfrac{\partial F}{\partial x}(adt + bdz(t)) + \tfrac{\partial F}{\partial t} dt + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial x^2}(adt + bdz(t))^2 \\ &= \tfrac{\partial F}{\partial x} adt + \tfrac{\partial F}{\partial x} bdz(t) + \tfrac{\partial F}{\partial t} dt + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial x^2}\left(a^2(dt)^2 + abdtdz(t) + b^2 (dz(t))^2\right) \end{aligned}$$

Note that $(dt)^2 = 0$, $dtdz(t) = 0$, and $(dz(t))^2 = dt$.

Making these substitutions and collecting terms gives

$$dy(t) = \left(\frac{\partial F}{\partial x} a + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} b^2\right) dt + \frac{\partial F}{\partial x} bdz(t). \tag{30}$$

As a simple application, let us take as given

$$dq^e(t) = \mu q^e(t)dt + \sigma q^e(t)dz(t),$$

and attempt to derive the relationship for $d\ln q^e(t)$.

Here we have $a(q^e(t),t) \equiv \mu q^e(t)$, $b(q^e(t),t) \equiv \sigma q^e(t)$, and

$$\frac{\partial F}{\partial q^e(t)} = \frac{1}{q^e(t)}, \text{ and } \frac{\partial^2 F}{\partial q^e(t)^2} = -\frac{1}{q^e(t)^2}.$$

Lastly $\frac{\partial F(\ )}{\partial t} = 0$.

Substituting these results into Equation (30) yields.

$$\begin{aligned} d\ln q^e(t) &= \left[\frac{1}{q^e(t)}\mu q^e(t) + 0 + \frac{1}{2}(-1)\left(\frac{1}{q^e(t)}\right)^2 (\sigma q^e(t))^2\right] dt + \frac{1}{q^e(t)}\sigma q^e(t) dz(t) \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dz(t), \end{aligned}$$

as was observed earlier.

This is the background.

6.3 The Black Scholes Formula

Merton (1973) requires four assumptions:

1. There are no market imperfections (perfect competition), transactions costs, taxes, short sales constraints or any other impediment to the continuous trading of securities.

2. There is unlimited riskless borrowing and lending at the constant risk free rate. If $q^b(t)$ is the period t price of a discount bond, then $q^b(t)$ is governed by the differential equation

$$\begin{aligned} dq^b(t) &= r_f q^b(t)dt, \text{ or} \\ B(t) &= B(0)e^{r_f t}; \end{aligned}$$

3. The underlying stock's price dynamics are given by a geometric Brownian motion of the form

$$\begin{aligned} &dq^e(t) = \mu q^e(t)dt + \sigma q^e(t)dz(t), \\ &q^e(0) > 0; \end{aligned}$$

4. There are no arbitrage opportunities across the financial markets in which the call, the underlying stock and the discount bond are traded.

Attention is restricted to call pricing formulae which are functions only of the stock's price currently and the time (so, e.g., the possibility of past stock price dependence is ignored); that is,

$$C = C\left(q^0(t), t\right).$$

By a straightforward application of Ito's lemma the call's price dynamics must be given by

$$dC = \left[\mu q^e(t)\frac{\partial C}{\partial q^e(t)} + \frac{\partial C}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 C}{\partial q^e(t)^2}\right]dt + \sigma q^e(t)\frac{\partial C}{\partial q^e(t)}dz(t),$$

which is of limited help since the form of $C\left(q^e(t), t\right)$ is precisely what is not known. The partials with respect to $q^e(t)$ and t of $C\left(q^e(t), t\right)$ must be somehow circumvented.

Following the replicating portfolio approach, Merton (1973) defines the value of the call in terms of the self financing continuously adjustable portfolio P composed of $\Delta\left(q^e(t), t\right)$ shares and $N\left(q^e(t), t\right)$ risk free discount bonds:

$$V\left(q^e(t), t\right) = \Delta\left(q^e(t),\, t\right)\, q^e(t) + N\left(q^e(t), t\right) q^b(t). \tag{31}$$

By a straightforward application of Ito's lemma, the value of the portfolio must evolve according to (suppressing functional dependence in order to reduce the burdensome notation):

$$dV = \Delta dq^e + N dq^b + d\Delta q^e + dN q^b + (d\Delta)dq^e \tag{32}$$

Since $V(\ )$ is assumed to be self-financing, any change in its value can only be due to changes in the values of the constituent assets and not in the numbers of them. Thus it must be that

$$dV = \Delta dq^e + N dq^b, \tag{33}$$

which implies that the remaining terms in Equation (33) are identically zero:

$$d\Delta q^e + dN q^b + (d\Delta) dq^e \equiv 0. \tag{34}$$

But both $\Delta(\ )$ and $N(\ )$ are functions of $q^e(t)$, and t and thus Ito's lemma can be applied to represent their evolution in terms of $dz(t)$ and dt. Using the relationships of Table 2, and collecting terms, both those preceding $dz(t)$ and those preceding dt must individually be zero.

Together these relationships imply that the value of the portfolio must satisfy the following partial differential equation:

$$\frac{1}{2}\sigma^2 q^e V_{q^e q^e} + r_f q_e V_{q^e} + V_t = r_f V, \tag{35}$$

which has as its solution the Black Scholes formula when coupled with the terminal condition V$(q^e\,(T)\,, T) = \max[0, q^e\,(T) - E]$.

7 A Second Approach: Martingale Methods

This method originated in the work of Harrison and Kreps (1979). It is popular as a methodology because it frequently allows for simpler computations than in the PDE approach. The underlying mathematics, however, is very complex and beyond the scope of this book. In order to convey a sense of what is going on, we present a brief heuristic argument that relys on the binomial abstraction.

Recall that in the binomial model, we undertook our pricing in a tree context where the underlying asset's price process had been modified. In particular, the true probabilities of the "up" and "down" state were replaced by the corresponding risk neutral probabilities. All assets (including the underlying stock) displayed an expected return equal to the risk free rate in the transformed setting.

Under geometric Brownian motion, the underlying price process is represented by an Ito stochastic differential equation of the form

$$dq^e(t) = \mu q^e(t) dt + \sigma q^e(t) dz(t). \tag{36}$$

In order to transform this price process into a risk neutral setting, two changes must be made.

1. The expression μ defines the mean and it must be replaced by r_f; Only with this substitution will the mean return on the underlying stock become r_f. Note that r_f denotes the corresponding continuously compounded risk free rate.

2. The standard Brownian motion process must be modified. In particular, we replace dz by dz^*, where the two processes are related via the transformation:

$$dz^*(t) = dz(t) + \left(\frac{\mu - r_f}{\sigma}\right) dt$$

The transformed price process is thus

$$dq^e(t) = r_f q^e(t)dt + \sigma q^e(t)dz^*(t). \tag{37}$$

By Equation (14) the corresponding process on $\ln q^e(t)$ is

$$d\ln q^e(t) = \left(r_f - \frac{1}{2}\sigma^2\right) dt + \sigma dz^*(t) \tag{38}$$

Let T denote the expiration date of a simple European call option. In the same spirit as the binomial model the price of a call must be the present value of its expected payoff at expiration under the transformed process.

Equation (38) informs us that in the transformed economy,

$$\ln\left(\frac{q^e(t)}{q^e(0)}\right) \sim N\left(\left(r_f - \left(\frac{1}{2}\right)\sigma^2\right)T, \sigma^2 T\right). \tag{39}$$

Since, in the transformed economy,

$$\text{prob}\left(q^e(t) \geq E\right) = \text{prob}\left(\ln q^e(t) \geq \ln E\right),$$

we can compute the call's value using the probability density implied by Equation (39):

$$C = e^{-r_f T} \int_{\ln E}^{\infty} (e^s - E) f(s) ds,$$

where $f(s)$ is the probability density on the ln of the stock's price.

Making the appropriate substitutions yields

$$C = e^{-r_f T}\left(\frac{1}{\sqrt{2\pi\sigma^2 T}}\right) \int_{\ln K}^{\infty} (e^s - K) e^{\frac{-[s - \ln q^e(t) - r_f T + \frac{\sigma^2 T}{2}]}{2\sigma^2 T}} ds \tag{40}$$

which, when the integration is performed yields the Black Scholes Formula.

8 Applications

We make reference to a number that have been considered earlier in the text.

8.1 The Consumption-Savings Problem.

This is a classic economic problem and we considered it fairly thoroughly in Chapter 5. Without the requisite math background there is not a lot we can say about the continuous time analogue than to set up the problem, but even that first step will be helpful.

Suppose the risky portfolio ("M") is governed by the following price process
$dq^M(t) = q^M(t)[\mu_M dt + \sigma_M dz(t)]$,
$q^M(0)$ given, and the risk free asset by
$dq^b(t) = rq^b(t)dt, q^b(0)$ given.

If an investor has initial wealth $Y(0)$, and chooses to invest the proportion $w(t)$ (possibly continuously varying) in the risky portfolio, then his wealth $Y(t)$ will evolve according to

$$dY(t) = Y(t)[w(t)(\mu_M - r_t) + r_f]dt + Y(t)[\pi(t)\sigma dz(t)] - c(t)dt, \tag{41}$$

where $c(t)$ is his consumption path. With objective function

$$\max_{c(t),w(t)} E \int_0^T e^{-\gamma t} U(c(t))dt, \tag{42}$$

the investors 'problem' is one of maximizing Equation (42) subject to Equation (41) and initial conditions on wealth, and the constraint that $Y\,(t) \geq 0$ for all t.

A classic result allows us to transform this problem to one that it turns out can be solved much more easily:

$$\max_{c(t),w(t)} E \int_0^T e^{-\gamma t} u(c(t))dt \tag{43}$$

$$\text{s.t. } PV_0(c(t)) = E^* \int_0^T e^{-r_f t} c(t)dt \leq Y\,(0)$$

where E^* is the transformed risk neutral measure under which the investor's wealth follows a Brownian motion process.

In what we have presented above, all the notation is directly analogous to that of Chapter 5: $U(\ )$ is the investors utility of (instantaneous) consumption, γ his (instantaneous) subjective discount rate, and T his time horizon,

8.2 An Application to Portfolio Analysis.

Here we hope to give a hint of how to extend the portfolio analysis of Chapters 5 or 6 to a setting where trading is (hypothetically) continuous and individual security returns follow geometric Brownian motions.

Let there be $i = 1, 2, \ldots, N$ equity securities, each of whose return is governed by the process

$$\frac{dq_i^e(t)}{q_i^e(t)} = \mu_i d_t + \sigma_i dz(t) \tag{44}$$

where $\sigma_i > 0$. These processes may also be correlated with one another in a manner that we can represent precisely. Conducting a portfolio analysis in this setting has been found to have two principal advantages. First, it provides new insights concerning the implications of diversification for long run portfolio returns, and, second, it allows for an easier solution to certain classes of problems. We will note these advantages with the implicit understanding that the derived portfolio rules must be viewed as guides for practical applications. Literally interpreted, they will imply, for example, continuous portfolio rebalancing – at an unbounded total expense, if the cost of doing each rebalancing is positive - which is absurd. In practice one would rather employ them weekly or perhaps daily.

The stated objective will be to maximize the expected rate of appreciation of a portfolio's value; equivalently, to maximize its expected terminal value which is the terminal wealth of the investor who owns it. Most portfolio managers would be familiar with this goal.

To get an idea of what this simplest criterion implies, and to make it more plausible in our setting, we first consider the discrete time equivalent (and, by implication) the discrete time approximation to GBM.

8.3 Digression to Discrete Time

Suppose a CRRA investor has initial wealth $Y(0)$ at time $t = 0$ and is considering investing in any or all of a set of non-dividend paying stocks whose returns are iid. Since the rate of expected appreciation of the portfolio is its expected rate of return, and since the return distributions of the available assets are iid, the investors' optional portfolio proportions will be invariant to the level of his wealth, and the distribution of his portfolio's returns will itself be iid. At the conclusion of his planning horizon, T periods from the present, the investors' wealth will be

$$Y_T = Y_0 \prod_{s=1}^{T} \tilde{R}_s^P, \tag{45}$$

where $\tilde{R}_s^P$denotes the (iid,) gross portfolio return in period s. It follows that

$$\begin{aligned} \ln\left(\frac{Y_T}{Y_0}\right) &= \sum_{s=1}^{T} \ln\tilde{R}_s^P, \text{and} \\ \ln\left(\frac{Y_T}{Y_0}\right)^{\frac{1}{T}} &= \left(\frac{1}{T}\right)\sum_{s=1}^{T} \ln\tilde{R}_s^P. \end{aligned} \tag{46}$$

Note that whenever we introduce the ln we effectively assume continuous compounding within the time period. As the number of periods in the time horizon grows without bound, $T \longmapsto \infty$, by the Law of Large Numbers,

$$\left(\frac{Y_T}{Y_0}\right)^{\frac{1}{T}} \longmapsto e^{E \ln \tilde{R}^P}, \text{ or} \tag{47}$$

$$Y_T \longmapsto Y_0 e^{TE \ln \tilde{R}_P} \tag{48}$$

Consider an investor with a <u>many</u> period time horizon who wishes to maximize her expected terminal wealth under continuous compounding. The relationship in Equation (48) informs her that

(1) it is sufficient, under the aforementioned assumptions, for her to choose portfolio proportions which maximize $E \ln \tilde{R}^P$, the expected logarithm of the one period return, and

(2) by doing so the average growth in rate of her wealth will approach a deterministic limit.

Before returning to the continuous time setting let us also entertain a brief classic example, one in which an investor must decide what fractions of his wealth to assign to a highly risky stock and to a risk free asset (actually, the risk free asset is equivalent to keeping money in a shoebox under the bed). For an amount $Y(0)$ invested in either asset, the respective returns are found in Figure 4.

Figure 4: Two alternative Investment Returns

Let w represent the proportion in the stock, and notice that the expected gross return to either asset under continuous compounding is <u>zero</u>:

Stock : $ER^e = \frac{1}{2}\ln(2) + \frac{1}{2}\ln(\frac{1}{2}) = 0$

Shoebox : $ER^{sb} = \frac{1}{2}\ln(1) + \frac{1}{2}\ln(1) = 0$

With each asset paying the same expected return, and the stock being wildly risky, at first appearance the shoebox would seem to be the investment of choice. But according to Equation (48) the investor ought to allocate his wealth among the two assets so as to maximize the expected ln of the portfolio's one-period gross return:

$\max \mathrm{E} \ln \tilde{R}^P = \max_w \left\{\frac{1}{2}\ln(2w + (1-w)) + \frac{1}{2}\ln\left(\frac{1}{2}w + (1-w)\right)\right\}$

A straight forward application of the calculus yields $w = 3/4$, with the consequent portfolio returns in each state as show in Figure 5

Figure 5 Optimal Portfolio Returns in Each State

$\ln(1 + w) = \ln(1.75) = 0.5596$, prob. $= \frac{1}{2}$

$\ln(1-\frac{1}{2}\ln w) = \ln(0.625) = -0.47$, prob. $= \frac{1}{2}$

As a result, $E\ln R^{\tilde{P}} = 0.0448$ with an effective risk free period return (for a very long time horizon) of 4.5% ($e^{0.448} = 1.045$) .

This result is surprising and the intuition is not obvious. Briefly, the optimal proportions of $w = 3/4$ and $1 - w = 1/4$ reflect the fact that by always keeping a fixed fraction of wealth in the risk free asset, the worst trajectories can be avoided. By "frequent" trading, although each asset has an expected return of zero, a combination will yield an expected return that is strictly positive, and over a long time horizon effectively riskless. Frequent trading expands market opportunities.

8.4 Return to Continuous Time

The previous setup applies directly to a continuous time setting as all of the fundamental assumptions are satisfied. In particular, there are a very large number of periods (an uncountable number, in fact) and the returns to the various securities are iid through time. Let us make the added generalization that the individual asset returns are correlated through their Brownian motion components. By an application of Ito's lemma, we may write

$cov(dz_i, dz_j) = E(dz_i(t)dz_j(t)) = \sigma_{ij}dt,$

where σ_j denotes the (i,j) entry of the (instantaneous) variance – covariance matrix.

As has been our custom, denote the portfolio's proportions for the N assets by $w_1, \ldots, w_N$ and let the superscript P denote the portfolio itself. As in earlier chapters, the process on the portfolio's instantaneous rate of return, $\frac{dY^P(t)}{Y^P(t)}$ is the weighted average of the instantaneous constituent asset returns (as given in Equation (44)):

$$\begin{aligned}
\frac{dY^P(t)}{Y^P(t)} &= \sum_{i=1}^{N} w_i \frac{dq_i^e(t)}{q_i^e(t)} = \sum_{i=1}^{N} w_i(\mu_i dt + dz_i(t)) \qquad (49)\\
&= \left(\sum_{i=1}^{N} w_i\mu_i\right) dt + \sum_{i=1}^{N} w_i dz_i(t),
\end{aligned}$$

where the variance of the stochastic term is given by

$$\begin{aligned}
E\left(\sum_{i=1}^{N} w_i dz_i(t)\right)^2 &= E\left\{\left(\sum_{i=1}^{N} w_i dz_i(t)\right)\left(\sum_{j=1}^{N} w_j dz_j(t)\right)\right\}\\
&= \left(\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}\right) dt
\end{aligned}$$

Equation (49) describes the process on the portfolio's rate of return and we see that it implies that the portfolio's value, at any future time horizon T will be lognormally distributed; furthermore, an uncountable infinity of periods will have passed. By analogy (and formally) our discrete time reflections suggest that an investor should in this context also choose portfolio proportion so as to maximize the mean growth rate, V_P, of the portfolio as given by

$$E\left\{\ln \frac{Y^P(t)}{Y(0)}\right\} = T_{vp}$$

Since the portfolio's value itself follows a Brownian motion (with drift $\sum_{i=1}^{N} w_i\mu_i$ and disturbance $\sum_{i=1}^{N} w_i dz_i$),

$$\begin{aligned} E\left[\ln \frac{Y^P(t)}{Y(0)}\right] &= \left(\sum_{i=1}^{N} w_i\mu_i\right)T - \frac{1}{2}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_i \sigma_{ij}\right)T, \text{ thus} && (50) \\ \nu_P &= \left(\frac{1}{T}\right)E\left[\ln \frac{Y^P(t)}{Y(0)}\right] = \sum_{i=1}^{N} w_i\mu_i - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}. && (51) \end{aligned}$$

The investor should choose portfolio proportions to maximize this latter quantity.

Without belaboring this development much further, it behooves us to recognize the message implicit in (51). This can be accomplished most straightforwardly in a context of an equally weighted portfolio where each of the N assets is independently distributed of one another ($\sigma_{ij} = 0$ for $i \neq j$), and all have the same mean and variance $((\mu_i\sigma_i) = (\mu,\sigma),\ i = 1, 2, \ldots, N)$.

In this case (51) reduces to

$$\nu_P = \mu - \left(\frac{1}{2N}\right)\sigma^2, \tag{52}$$

with the direct implication that the more identical stocks the investor adds to the portfolio the greater the mean instantaneous return. In this sense it is useful to search for many similarly volatile stocks whose returns are independent of one another: by combining them in a portfolio where we continually (frequently) rebalance to maintain equal proportions, not only will portfolio variance decline ($\frac{1}{2N}\sigma^2$), as in the discrete time case, but the mean return will rise (which is *not* the case in discrete time!).

8.5 The Consumption CAPM in Continuous Time

Our final application concerns the consumption CAPM of Chapter 9, and the question we address is this: What is the equilibrium asset price behavior in a Mehra-Prescott asset pricing context when the growth rate in consumption follows a GBM? Specializing preferences to be of the customary forms $U(c) = (c^{1-\gamma}/1-\gamma)$, pricing relationship (9.4) reduces to:

$$\begin{aligned}
P_t &= E_t\left\{Y_t \sum_{j=1}^{\infty} \beta^j x_{t+j}^{1-\gamma}\right\} \\
&= Y_t \sum_{j=1}^{\infty} \beta^j E_t\left\{x_{t+j}^{1-\gamma}\right\}
\end{aligned}$$

where x_{t+j} is the growth rate in output (equivalently, consumption in the Mehra-Prescott economy) from period j to period $j+1$.

We hypothesize that the growth rate x follows a GBM of the form

$$dx = \mu x dt + \sigma x dz,$$

where we interpret x_{t+j} as the discrete time realisation of $x\,(t)$ at time $t+j$.

One result from statistics is needed. Suppose w is lognormally distributed which we write $\tilde{w} \sim L\,(\xi, \eta)$ where $\xi = E \ln \tilde{w}$ and $\eta^2 = var \ln \tilde{w}$. Then for any real number q,

$$E\left\{w^q\right\} = e^{q\xi + \frac{1}{2}q^2\eta^2}$$

By the process on the growth rate just assumed, $x\,(t) \sim L(\left(\mu - \frac{1}{2}\sigma^2\right) t, \sigma\sqrt{t})$ so that at time $t+j, x_{t+j} \sim L(\left(\mu - \frac{1}{2}\sigma^2\right) j, \sigma\sqrt{j})$. By this result,

$$\begin{aligned}
E\left\{x_{t+j}^{1-\gamma}\right\} &= e^{(1-\gamma)\left(\mu - \frac{1}{2}\sigma^2\right)j + \frac{1}{2}(1-\gamma)^2\sigma^2 j} \\
&= e^{(1-\gamma)\left(\mu - \frac{1}{2}\gamma\sigma^2\right)j},
\end{aligned}$$

and thus,

$$\begin{aligned}
P_t &= Y_t \sum_{j=1}^{\infty} \beta^j e^{(1-\gamma)\left(\mu - \frac{1}{2}\gamma\sigma^2\right)j} \\
&= Y_t \sum_{j=1}^{\infty} \left(\beta e^{(1-\gamma)\left(\mu - \frac{1}{2}\gamma\sigma^2\right)}\right)^j
\end{aligned}$$

which is well defined (the sum has a finite value) if $\beta e^{(1-\gamma)\left(\mu - \frac{1}{2}\gamma\sigma^2\right)} < 1$, wich we will assume to be the case. Then

$$P_t = Y_t \frac{\beta e^{(1-\gamma)\left(\mu - \frac{1}{2}\gamma\sigma^2\right)}}{1 + \beta e^{(1-\gamma)\left(\frac{1}{2}\gamma\sigma^2\right)}}$$

This is an illustration of the fact that working in continuous time often allows convenient closed form solutions. Our remark are taken from Mehra and Sah (2001).

9 Final Comments

There is much more to be said. There are many more extensions of the CCAPM style models to a continuous time setting. Another issue is the sense in which a continuous time price process (e.g. Equation (13)) can be viewed as an equilibrium price process in the sense of that concept as presented in this book. This remains a focus of research.

Continuous time is clearly different from discrete time, but does it use (as a derivatives pricing tool) enrich our economic understanding of the large financial and macroeconomics reality? That is less clear.