Chapter 4 : Measuring Risk and Risk Aversion

4.1 Introduction

We argued in Chapter 1 that the desire of investors to avoid risk, that is, to smooth their consumption across states of nature and for that reason avoid variations in the value of their portfolio holdings, is one of the primary motivations for financial contracting. But we have not thus far imposed restrictions on the VNM expected utility representation of investor preferences, which necessarily guarantee such behavior. For that to be the case, our representation must be further specialized.

Since the probabilities of the various state payoffs are objectively given, independently of agent preferences, further restrictions must be placed on the utility-of-money function $U()$ if the VNM (von Neumann-Morgenstern - expected utility) representation is to capture this notion of risk aversion. We will now define risk aversion and discuss its implications for $U()$.

4.2 Measuring Risk Aversion

What does the term risk aversion imply about an agent’s utility function? Consider a financial contract where the potential investor either receives an amount $h$ with probability $\frac{1}{2}$, or must pay an amount $h$ with probability $\frac{1}{2}$. Our most basic sense of risk aversion must imply that for any level of personal wealth $Y$, a risk-averse investor would not wish to own such a security. In utility terms this must mean

$$U(Y) > \left(\frac{1}{2}U(Y + h) + \frac{1}{2}U(Y - h)\right) = EU,$$

where the expression on the right-hand side of the inequality sign is the VNM expected utility associated with the random wealth levels:

- $y + h$, probability = $\frac{1}{2}$
- $y - h$, probability = $\frac{1}{2}$.

This inequality can only be satisfied for all wealth levels $Y$ if the agent’s utility function has the form suggested in Figure 4.1. When this is the case we say the utility function is strictly concave.

The important characteristics implied by this and similarly shaped utility functions is that the slope of the graph of the function decreases as the agent becomes wealthier (as $Y$ increases); that is, the marginal utility ($MU$), represented by the derivative $\frac{dU(Y)}{dY} \equiv U'(Y)$, decreases with greater $Y$. Equivalently, for twice differentiable utility functions, $\frac{d^2(U(Y))}{dY^2} \equiv U''(Y) < 0$. For this class of functions, the latter is indeed a necessary and sufficient condition for risk aversion.

Insert Figure 4.1 about here
As the discussion indicates, both consumption smoothing and risk aversion are directly related to the notion of decreasing $MU$. Whether they are envisaged across time or states, decreasing $MU$ basically implies that income (or consumption) deviations from a fixed average level diminish rather than increase utility. Essentially, the positive deviations do not help as much as the negative ones hurt.

Risk aversion can also be represented in terms of indifference curves. Figure 4.2 illustrates the case of a simple situation with two states of nature. If consuming $c_1$ in state 1 and $c_2$ in state 2 represents a certain level of expected utility $EU$, then the convex-to-the-origin indifference curve that is the appropriate translation of a strictly concave utility function indeed implies that the utility level generated by the average consumption $\frac{c_1 + c_2}{2}$ in both states (in this case a certain consumption level) is larger than $EU$.

We would like to be able to measure the degree of an investor’s aversion to risk. This will allow us to compare whether one investor is more risk averse than another and to understand how an investor’s risk aversion affects his investment behavior (for example, the composition of his portfolio).

As a first attempt toward this goal, and since $U''(Y) < 0$ implies risk aversion, why not simply say that investor $A$ is more risk averse than investor $B$, if and only if $|U_A''(Y)| \geq |U_B''(Y)|$, for all income levels $Y$? Unfortunately, this approach leads to the following inconsistency. Recall that the preference ordering described by a utility function is invariant to linear transformations. In other words, suppose $U_A(\cdot)$ and $\hat{U}_A(\cdot)$ are such that $\hat{U}_A(\cdot) = a + bU_A(\cdot)$ with $b > 0$. These utility functions describe the identical ordering, and thus must display identical risk aversion. Yet, if we use the above measure we have

$$|U_A''(Y)| > |U_A''(Y)|,$$

if, say, $b > 1$.

This implies that investor $A$ is more risk averse than he is himself, which must be a contradiction.

We therefore need a measure of risk aversion that is invariant to linear transformations. Two widely used measures of this sort have been proposed by, respectively, Pratt (1964) and Arrow (1971):

(i) absolute risk aversion $= -\frac{U''(Y)}{U'(Y)} \equiv R_A(Y)$

(ii) relative risk aversion $= -\frac{YU''(Y)}{U'(Y)} \equiv R_R(Y)$.

Both of these measures have simple behavioral interpretations. Note that instead of speaking of risk aversion, we could use the inverse of the measures proposed above and speak of risk tolerance. This terminology may be preferable on various occasions.
4.3 Interpreting the Measures of Risk Aversion

4.3.1 Absolute Risk Aversion and the Odds of a Bet

Consider an investor with wealth level \( Y \) who is offered — at no charge — an investment involving winning or losing an amount \( h \), with probabilities \( \pi \) and \( 1-\pi \), respectively. Note that any investor will accept such a bet if \( \pi \) is high enough (especially if \( \pi = 1 \)) and reject it if \( \pi \) is small enough (surely if \( \pi = 0 \)). Presumably, the willingness to accept this opportunity will also be related to his level of current wealth, \( Y \). Let \( \pi(\pi, h) \) be that probability at which the agent is indifferent between accepting or rejecting the investment. It is shown that

\[
\pi(Y, h) \approx \frac{1}{2} + \frac{1}{4} h R_A(Y),
\]

(4.1)

where \( \approx \) denotes “is approximately equal to.”

The higher his measure of absolute risk aversion, the more favorable odds he will demand in order to be willing to accept the investment. If \( R_A(1) \geq R_A(2) \), for agents 1 and 2 respectively, then investor 1 will always demand more favorable odds than investor 2, and in this sense investor 1 is more risk averse.

It is useful to examine the magnitude of this probability. Consider, for example, the family of VNM utility-of-money functions with the form:

\[
U(Y) = -\frac{1}{\nu} e^{-\nu Y},
\]

where \( \nu \) is a parameter.

For this case,

\[
\pi(Y, h) \approx \frac{1}{2} + \frac{1}{4} h \nu,
\]

in other words, the odds requested are independent of the level of initial wealth (\( Y \)); on the other hand, the more wealth at risk (\( h \)), the greater the odds of a favorable outcome demanded. This expression advances the parameter \( \nu \) as the appropriate measure of the degree of absolute risk aversion for these preferences.

Let us now derive Equation (4.1). By definition, \( \pi(Y, h) \) must satisfy

\[
\underbrace{U(Y)} = \pi(Y, h) \underbrace{U(Y + h)} + [1 - \pi(Y, h)] \underbrace{U(Y - h)}
\]

(4.2)

utility if he foregoes the bet  
expected utility if the investment is accepted

By an approximation (Taylor’s Theorem) we know that:

\[
U(Y + h) = U(Y) + hU'(Y) + \frac{h^2}{2} U''(Y) + H_1,
\]

\[
U(Y - h) = U(Y) - hU'(Y) + \frac{h^2}{2} U''(Y) + H_2,
\]

where \( H_1, H_2 \) are remainder terms of order higher than \( h^2 \). Substituting these quantities into Equation (4.2) gives
\[U(Y) = \pi(Y, h)[U(Y)+hU'(Y)+\frac{h^2}{2}U''(Y)+H_1]+(1-\pi(Y, h))[U(Y)-hU'(Y)+\frac{h^2}{2}U''(Y)+H_2]\]

Collecting terms gives
\[U(Y) = U(Y)+(2\pi(Y, h)-1)\left[hU'(Y)+\frac{h^2}{2}U''(Y)\right]+\pi(Y, h)H_1+(1-\pi(Y, h))H_2\]

Solving for \(\pi(Y, h)\) yields
\[\pi(Y, h) = \frac{1}{2} + \frac{h}{4} \left[-\frac{-U''(Y)}{U'(Y)}\right] - \frac{H}{2hU'(Y)},\] (4.4)
which is the promised expression, since the last remainder term is small - it is a weighted average of terms of order higher than \(h^2\) and is, thus, itself of order higher than \(h^2\) - and it can be ignored in the approximation.

4.3.2 Relative Risk Aversion in Relation to the Odds of a Bet

Consider now an investment opportunity similar to the one just discussed except that the amount at risk is a proportion of the investor’s wealth, in other words, \(h = \theta Y\), where \(\theta\) is the fraction of wealth at risk. By a derivation almost identical to the one above, it can be shown that
\[\pi(Y, \theta) \sim \frac{1}{2} + \frac{1}{4}\theta R_R(Y).\] (4.5)

If \(R_R^1(Y) \geq R_R^2(Y)\), for investors 1 and 2, then investor 1 will always demand more favorable odds, for any level of wealth, when the fraction \(\theta\) of his wealth is at risk.

It is also useful to illustrate this measure by an example. A popular family of VNM utility-of-money functions (for reasons to be detailed in the next chapter) has the form:
\[U(Y) = \begin{cases} Y^{1-\gamma} & \text{for } 0 > \gamma \neq 1 \\ \ln Y, & \text{if } \gamma = 1. \end{cases}\]

In the latter case, the probability expression becomes
\[\pi(Y, \theta) \approx \frac{1}{2} + \frac{1}{4}\theta.\]

In this case, the requested odds of winning are not a function of initial wealth \((Y)\) but depend upon \(\theta\), the fraction of wealth that is at risk: The lower the fraction \(\theta\), the more investors are willing to consider entering into bet that is close to being fair (a risky opportunity where the probabilities of success or
failure are both \( \frac{1}{2} \). In the former, more general, case the analogous expression is
\[
\pi(Y, \theta) \approx \frac{1}{2} + \frac{1}{4} \theta \gamma.
\]
Since \( \gamma > 0 \), these investors demand a higher probability of success. Furthermore, if \( \gamma_2 > \gamma_1 \), the investor characterized by \( \gamma = \gamma_2 \) will always demand a higher probability of success than will an agent with \( \gamma = \gamma_1 \), for the same fraction of wealth at risk. In this sense a higher \( \gamma \) denotes a greater degree of relative risk aversion for this investor class.

### 4.3.3 Risk Neutral Investors

One class of investors deserves special mention at this point. They are significant, as we shall later see, for the influence they have on the financial equilibria in which they participate. This is the class of investors who are risk neutral and who are identified with utility functions of a linear form
\[
U(Y) = cY + d,
\]
where \( c \) and \( d \) are constants and \( c > 0 \).

Both of our measures of the degree of risk aversion, when applied to this utility function give the same result:
\[
RA(Y) \equiv 0 \text{ and } RR(Y) \equiv 0.
\]

Whether measured as a proportion of wealth or as an absolute amount of money at risk, such investors do not demand better than even odds when considering risky investments of the type under discussion. They are indifferent to risk, and are concerned only with an asset’s expected payoff.

### 4.4 Risk Premium and Certainty Equivalence

The context of our discussion thus far has been somewhat artificial because we were seeking especially convenient probabilistic interpretations for our measures of risk aversion. More generally, a risk-averse agent \( (U''( ) < 0) \) will always value an investment at something less than the expected value of its payoffs. Consider an investor, with current wealth \( Y \), evaluating an uncertain risky payoff \( \tilde{Z} \). For any distribution function \( F_\tilde{Z} \),
\[
U(Y + E\tilde{Z}) \geq E[U(Y + \tilde{Z})]
\]
provided that \( U''( ) < 0 \). This is a direct consequence of a standard mathematical result known as Jensen’s inequality.

**Theorem 4.1 (Jensen’s Inequality):**
Let \( g(\cdot) \) be a concave function on the interval \((a, b)\), and \( \tilde{x} \) be a random variable such that \( \text{Prob}\{\tilde{x} \in (a, b)\} = 1 \). Suppose the expectations \( E(\tilde{x}) \) and \( Eg(\tilde{x}) \) exist; then
\[
E[g(\tilde{x})] \leq g[E(\tilde{x})].
\]
Furthermore, if \( g(\cdot) \) is strictly concave and \( \text{Prob}\{\tilde{x} = E(\tilde{x})\} \neq 1 \), then the inequality is strict.

This theorem applies whether the interval \((a, b)\) on which \( g(\cdot) \) is defined is finite or infinite and, if \( a \) and \( b \) are finite, the interval can be open or closed at either endpoint. If \( g(\cdot) \) is convex, the inequality is reversed. See De Groot (1970).

To put it differently, if an uncertain payoff is available for sale, a risk-averse agent will only be willing to buy it at a price less than its expected payoff. This statement leads to a pair of useful definitions. The (maximal) certain sum of money a person is willing to pay to acquire an uncertain opportunity defines his certainty equivalent (\( CE \)) for that risky prospect; the difference between the \( CE \) and the expected value of the prospect is a measure of the uncertain payoff’s risk premium. It represents the maximum amount the agent would be willing to pay to avoid the investment or gamble.

Let us make this notion more precise. The context of the discussion is as follows. Consider an agent with current wealth \( Y \) and utility function \( U(\cdot) \) who has the opportunity to acquire an uncertain investment \( \tilde{Z} \) with expected value \( E\tilde{Z} \). The certainty equivalent (to the risky investment \( \tilde{Z} \), \( CE(Y, \tilde{Z}) \), and the corresponding risk or insurance premium, \( \Pi(Y, \tilde{Z}) \), are the solutions to the following equations:
\[
EU(Y + \tilde{Z}) = U(Y + CE(Y, \tilde{Z})) \quad (4.6a)
\]
\[
= U(Y + E\tilde{Z} - \Pi(Y, \tilde{Z})) \quad (4.6b)
\]
which, implies
\[
CE(\tilde{Z}, Y) = E\tilde{Z} - \Pi(Y, \tilde{Z}) \quad \text{or} \quad \Pi(Y, \tilde{Z}) = E\tilde{Z} - CE(\tilde{Z}, Y)
\]
These concepts are illustrated in Figure 4.3.

Insert Figure 4.3 about here

It is intuitively clear that there is a direct relationship between the size of the risk premium and the degree of risk aversion of a particular individual. The link can be made quite easily in the spirit of the derivations of the previous section. For simplicity, the derivation that follows applies to the case of an actuarially fair prospect \( \tilde{Z} \), one for which \( E\tilde{Z} = 0 \). Using Taylor series approximations we can develop the left-hand side (LHS) and right-hand side (RHS) of the definitional Equations (4.6a) and (4.6b).
LHS:

\[ EU(Y + \tilde{Z}) = EU(Y) + E\left[\tilde{Z}U'(Y)\right] + E\left[\frac{1}{2}\tilde{Z}^2U''(Y)\right] + EH(\tilde{Z}^3) \]

\[ = U(Y) + \frac{1}{2}\sigma_\tilde{Z}^2U''(Y) + EH(\tilde{Z}^3) \]

RHS:

\[ U(Y - \Pi(Y, \tilde{Z})) = U(Y) - \Pi(Y, \tilde{Z})U'(Y) + H(\Pi^2) \]

or, ignoring the terms of order \(Z^3\) or \(\Pi^2\) or higher \((EH(\tilde{Z}^3)\) and \(H(\Pi^2)\)),

\[ \Pi(Y, \tilde{Z}) \approx \frac{1}{2}\sigma_\tilde{Z}^2 \left(\frac{-U''(Y)}{U'(Y)}\right) = \frac{1}{2}\sigma_\tilde{Z}^2 R_A(Y). \]

To illustrate, consider our earlier example in which \(U(Y) = \frac{Y^{1-\gamma}}{1-\gamma}\), and suppose \(\gamma = 3\), \(Y = $500,000\), and

\[ \tilde{Z} = \begin{cases} 
$100,000 \text{ with probability } & \frac{1}{2} \\ 
-\$100,000 \text{ with probability } & \frac{1}{2} \end{cases} \]

For this case the approximation specializes to

\[ \Pi(Y, \tilde{Z}) \approx \frac{1}{2}\sigma_\tilde{Z}^2 \left(\frac{3}{500,000}\right) = $30,000. \]

To confirm that this approximation is a good one, we must show that:

\[ U(Y - \Pi(Y, \tilde{Z})) = U(500,000 - 30,000) = \frac{1}{2}U(600,000) + \frac{1}{2}U(400,000) = EU(Y + \tilde{Z}), \]

or

\[ (4.7)^{-2} = \frac{1}{2}(6)^{-2} + \frac{1}{2}(4)^{-2}, \]

or

\[ .0452694 \approx .04513; \text{ confirmed}. \]

Note also that for this preference class, the insurance premium is directly proportional to the parameter \(\gamma\).

Can we convert these ideas into statements about rates of return? Let the equivalent risk-free return be defined by

\[ U(Y(1 + r_f)) = U(Y + CE(\tilde{Z}, Y)). \]

The random payoff \(\tilde{Z}\) can also be converted into a rate of return distribution via \(\tilde{Z} = \tilde{r}Y\), or, \(\tilde{r} = \tilde{Z}/Y\). Therefore, \(r_f\) is defined by the equation

\[ U(Y(1 + r_f)) \equiv EU(Y(1 + \tilde{r})). \]
By risk aversion, $E\tilde{r} > r_f$. We thus define the rate of return risk premium $\Pi^r$ as $\Pi^r = E\tilde{r} - r_f$, or $E\tilde{r} = r_f + \Pi^r$, where $\Pi^r$ depends on the degree of risk aversion of the agent in question. Let us conclude this section by computing the rate of return premium in a particular case. Suppose $U(Y) = \ln Y$, and that the random payoff $\tilde{Z}$ satisfies

$$\tilde{Z} = \begin{cases} 
$100,000 \text{ with probability } &= \frac{1}{2} \\
-50,000 \text{ with probability } &= \frac{1}{2} 
\end{cases}$$

from a base of $Y = 500,000$. The risky rate of return implied by these numbers is clearly

$$\tilde{r} = \begin{cases} 
20\% \text{ with probability } &= \frac{1}{2} \\
-10\% \text{ with probability } &= \frac{1}{2} 
\end{cases}$$

with an expected return of 5%. The certainty equivalent $CE(Y, \tilde{Z})$ must satisfy

$$\ln(500,000 + CE(Y, \tilde{Z})) = \frac{1}{2} \ln(600,000) + \frac{1}{2} \ln(450,000),$$

or

$$CE(Y, \tilde{Z}) = 19,618,$$

so that

$$(1 + r_f) = \frac{519,618}{500,000} = 1.0392.$$ 

The rate of return risk premium is thus 5% - 3.92% = 1.08%. Let us be clear: This rate of return risk premium does not represent a market or equilibrium premium. Rather it reflects personal preference characteristics and corresponds to the premium over the risk-free rate necessary to compensate, utility-wise, a specific individual, with the postulated preferences and initial wealth, for engaging in the risky investment.

### 4.5 Assessing the Level of Relative Risk Aversion

Suppose that agents’ utility functions are of the form $U(Y) = Y^{1-\gamma} (1-\gamma)$ class. As noted earlier, a quick calculation informs us that $R_R(Y) \equiv \gamma$, and we say that $U(\cdot)$ is of the constant relative risk aversion class. To get a feeling as to what this measure means, consider the following uncertain payoff:

$$\tilde{Z} = \begin{cases} 
$50,000 \text{ with probability } &= \pi = 0.5 \\
$100,000 \text{ with probability } &= \pi = 0.5 
\end{cases}$$

Assuming your utility function is of the type just noted, what would you be willing to pay for such an opportunity (i.e., what is the certainty equivalent for this uncertain prospect) if your current wealth were $Y$? The interest in asking such a question resides in the fact that, given the amount you are willing to pay,
it is possible to infer your coefficient of relative risk aversion $R_R(Y) = \gamma$, provided your preferences are adequately represented by the postulated functional form. This is achieved with the following calculation.

The $CE$, the maximum amount you are willing to pay for this prospect, is defined by the equation

$$\frac{(Y + CE)^{1-\gamma}}{1-\gamma} = \frac{1}{2}(Y + 50,000)^{1-\gamma} + \frac{1}{2}(Y + 100,000)^{1-\gamma}$$

Assuming zero initial wealth ($Y = 0$), we obtain the following sample results (clearly, $CE > 50,000$):

- $\gamma = 0$, $CE = 75,000$ (risk neutrality)
- $\gamma = 1$, $CE = 70,711$
- $\gamma = 2$, $CE = 66,667$
- $\gamma = 5$, $CE = 58,566$
- $\gamma = 10$, $CE = 53,991$
- $\gamma = 20$, $CE = 51,858$
- $\gamma = 30$, $CE = 51,209$

Alternatively, if we suppose a current wealth of $Y = $100,000 and a degree of risk aversion of $\gamma = 5$, the equation results in a $CE = $ 66,532.

### 4.6 The Concept of Stochastic Dominance

In response to dissatisfaction with the standard ranking of risky prospects based on mean and variance, a theory of choice under uncertainty with general applicability has been developed. In this section we show that the postulates of expected utility lead to a definition of two weaker alternative concepts of dominance with wider applicability than the concept of state-by-state dominance. These are of interest because they circumscribe the situations in which rankings among risky prospects are preference free, or, can be defined independently of the specific trade-offs (among return, risk, and other characteristics of probability distributions) represented by an agent’s utility function.

We start with an illustration. Consider two investment alternatives, $Z_1$ and $Z_2$, with the characteristics outlined in Table 4.1:

<table>
<thead>
<tr>
<th>Payoffs</th>
<th>10</th>
<th>100</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prob $Z_1$</td>
<td>.4</td>
<td>.6</td>
<td>0</td>
</tr>
<tr>
<td>Prob $Z_2$</td>
<td>.4</td>
<td>.4</td>
<td>.2</td>
</tr>
</tbody>
</table>

$EZ_1 = 64, \sigma_{z_1} = 44$

$EZ_2 = 444, \sigma_{z_2} = 779$
First observe that under standard mean-variance analysis, these two investments cannot be ranked: Although investment \( Z_2 \) has the greater mean, it also has the greater variance. Yet, all of us would clearly prefer to own investment 2. It at least matches investment 1 and has a positive probability of exceeding it.

To formalize this intuition, let us examine the cumulative probability distributions associated with each investment, \( F_1(Z) \) and \( F_2(Z) \) where \( F_i(Z) = \text{Prob}(Z_i \leq Z) \).

In Figure 4.4 we see that \( F_1(\cdot) \) always lies above \( F_2(\cdot) \). This observation leads to Definition 4.1.

**Definition 4.1:**
Let \( F_A(\tilde{x}) \) and \( F_B(\tilde{x}) \), respectively, represent the cumulative distribution functions of two random variables (cash payoffs) that, without loss of generality assume values in the interval \([a, b]\). We say that \( F_A(\tilde{x}) \) first order stochastically dominates \( F_B(\tilde{x}) \) if and only if \( F_A(x) \leq F_B(x) \) for all \( x \in [a, b] \).

Distribution \( A \) in effect assigns more probability to higher values of \( x \); in other words, higher payoffs are more likely. That is, the distribution functions of \( A \) and \( B \) generally conform to the following pattern: if \( F_A \) \( \text{FSD} \) \( F_B \), then \( F_A \) is everywhere below and to the right of \( F_B \) as represented in Figure 4.5. By this criterion, investment 2 in Figure 4.5 stochastically dominates investment 1. It should, intuitively, be preferred. Theorem 4.2 summarizes our intuition in this latter regard.

Theorem 4.2:
Let \( F_A(\tilde{x}) \), \( F_B(\tilde{x}) \), be two cumulative probability distributions for random payoffs \( \tilde{x} \in [a, b] \). Then \( F_A(\tilde{x}) \) \( \text{FSD} \) \( F_B(\tilde{x}) \) if and only if \( E_AU(\tilde{x}) \geq E_BU(\tilde{x}) \) for all non-decreasing utility functions \( U(\ ) \).

**Proof:**
See Appendix.

Although it is not equivalent to state-by-state dominance (see Exercise 4.8), \( \text{FSD} \) is an extremely strong condition. As is the case with the former, it is so strong a concept that it induces only a very incomplete ranking among uncertain prospects. Can we find a broader measure of comparison, for instance, which would make use of the hypothesis of risk aversion as well?

Consider the two independent investments in Table 4.2.\(^1\)

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\(^1\)In this example, contrary to the previous one, the two investments considered are statistically independent.
Table 4.2: Two Independent Investments

<table>
<thead>
<tr>
<th></th>
<th>Investment 3</th>
<th>Investment 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Prob.</td>
<td>0.25</td>
<td>0.33</td>
</tr>
<tr>
<td>Payoff</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>Prob.</td>
<td>0.50</td>
<td>0.33</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>Prob.</td>
<td>0.25</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Which of these investments is better? Clearly, neither investment (first order) stochastically dominates the other as Figure 4.6 confirms. The probability distribution function corresponding to investment 3 is not everywhere below the distribution function of investment 4. Yet, we would probably prefer investment 3. Can we formalize this intuition (without resorting to the mean/variance criterion, which in this case accords with intuition: \( ER_3 = 5, ER_4 = 5.75; \sigma_3 = 2.9, \) and \( \sigma_4 = 1.9 \))? This question leads to a weaker notion of stochastic dominance that explicitly compares distribution functions.

**Definition 4.2: Second Order Stochastic Dominance (SSD).**
Let \( F_A(\tilde{x}) \), \( F_B(\tilde{x}) \), be two cumulative probability distributions for random payoffs in \([a, b]\). We say that \( F_A(\tilde{x}) \) second order stochastically dominates (SSD) \( F_B(\tilde{x}) \) if and only if for any \( x \):

\[
\int_{-\infty}^{x} \left[ F_B(t) - F_A(t) \right] dt \geq 0.
\]

(with strict inequality for some meaningful interval of values of \( t \)).

The calculations in Table 4.3 reveal that, in fact, investment 3 second order stochastically dominates investment 4 (let \( f_i(x) \), \( i = 3, 4 \), denote the density functions corresponding to the cumulative distribution function \( F_i(x) \)). In geometric terms (Figure 4.6), this would be the case as long as area \( B \) is smaller than area \( A \).

Insert Figure 4.6 about here

As Theorem 4.3 shows, this notion makes sense, especially for risk-averse agents:

**Theorem 4.3:**
Let \( F_A(\tilde{x}) \), \( F_B(\tilde{x}) \), be two cumulative probability distributions for random payoffs \( \tilde{x} \) defined on \([a, b]\). Then, \( F_A(\tilde{x}) \) SSD \( F_B(\tilde{x}) \) if and only if \( E_A U(\tilde{x}) \geq E_B U(\tilde{x}) \) for all nondecreasing and concave \( U \).

**Proof:**
See Laffont (1989), Chapter 2, Section 2.5
Table 4.3: Investment 3 Second Order Stochastically Dominates Investment 4

<table>
<thead>
<tr>
<th>Values of x</th>
<th>$\int_0^x f_3(t)dt$</th>
<th>$\int_0^x F_3(t)dt$</th>
<th>$\int_0^x f_4(t)dt$</th>
<th>$\int_0^x F_4(t)dt$</th>
<th>$\int_0^x [F_4(t) - F_3(t)]dt$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1/3</td>
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<tr>
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<td>.25</td>
<td>1/3</td>
<td>4/3</td>
<td>13/12</td>
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<tr>
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<td>.75</td>
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<tr>
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<td>.75</td>
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</tr>
<tr>
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<tr>
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<td>1</td>
<td>6.5</td>
<td>1</td>
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<tr>
<td>13</td>
<td>1</td>
<td>7.5</td>
<td>1</td>
<td>9</td>
<td>3/2</td>
</tr>
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</table>

That is, all risk-averse agents will prefer the second-order stochastically dominant asset. Of course, FSD implies SSD: If for two investments $Z_1$ and $Z_2$, $Z_1 \text{ FSD } Z_2$, then it is also true that $Z_1 \text{ SSD } Z_2$. But the converse is not true.

4.7 Mean Preserving Spreads

Theorems 4.2 and 4.3 attempt to characterize the notion of “better/worse” relevant for probability distributions or random variables (representing investments). But there are two aspects to such a comparison: the notion of “more or less risky” and the trade-off between risk and return. Let us now attempt to isolate the former effect by comparing only those probability distributions with identical means. We will then review Theorem 4.3 in the context of this latter requirement.

The concept of more or less risky is captured by the notion of a mean preserving spread. In our context, this notion can be informally stated as follows: Let $f_A(x)$ and $f_B(x)$ describe, respectively, the probability density functions on payoffs to assets $A$ and $B$. If $f_B(x)$ can be obtained from $f_A(x)$ by removing some of the probability weight from the center of $f_A(x)$ and distributing it to the tails in such a way as to leave the mean unchanged, we say that $f_B(x)$ is related to $f_A(x)$ via a mean preserving spread. Figure 4.7 suggests what this notion would mean in the case of normal-type distributions with identical mean, yet different variances.

Insert Figure 4.7 about here
How can this notion be made both more intuitive and more precise? Consider a set of possible payoffs $\tilde{x}_A$ that are distributed according to $F_A(\cdot)$. We further randomize these payoffs to obtain a new random variable $\tilde{x}_B$ according to

$$\tilde{x}_B = \tilde{x}_A + \tilde{z} \quad (4.7)$$

where, for any $x_A$ value, $E(\tilde{z}) = \int \tilde{z} dH_{\tilde{x}_A}(\tilde{z}) = 0$; in other words, we add some pure randomness to $\tilde{x}_A$. Let $F_B(\cdot)$ be the distribution function associated with $\tilde{x}_B$. We say that $F_B(\cdot)$ is a mean preserving spread of $F_A(\cdot)$.

A simple example of this is as follows. Let

$$\tilde{x}_A = \begin{cases} 5 \text{ with prob } 1/2 \\ 2 \text{ with prob } 1/2 \end{cases}$$

and suppose

$$\tilde{z} = \begin{cases} +1 \text{ with prob } 1/2 \\ -1 \text{ with prob } 1/2 \end{cases}$$

Then,

$$\tilde{x}_B = \begin{cases} 6 \text{ with prob } 1/4 \\ 4 \text{ with prob } 1/4 \\ 3 \text{ with prob } 1/4 \\ 1 \text{ with prob } 1/4 \end{cases}$$

Clearly, $E\tilde{x}_A = E\tilde{x}_B = 3.5$; we would also all agree that $F_B(\cdot)$ is intuitively riskier.

Our final theorem (Theorem 4.4) relates the sense of a mean preserving spread, as captured by Equation (4.7), to our earlier results.

**Theorem 4.4:**

Let $F_A(\cdot)$ and $F_B(\cdot)$ be two distribution functions defined on the same state space with identical means. If this is true, the following statements are equivalent:

(i) $F_A(\tilde{x}) SSD F_B(\tilde{x})$
(ii) $F_B(\tilde{x})$ is a mean preserving spread of $F_A(\tilde{x})$ in the sense of Equation (4.7).

**Proof:**


But what about distributions that are not stochastically dominant under either definition and for which the mean-variance criterion does not give a relative ranking? For example, consider (independent) investments 5 and 6 in Table 4.4.

In this case we are left to compare distributions by computing their respective expected utilities. That is to say, the ranking between these two investments is preference dependent. Some risk-averse individuals will prefer investment 5 while other risk-averse individuals will prefer investment 6. This is not bad.
Table 4.4: Two Investments; No Dominance

<table>
<thead>
<tr>
<th>Investment 5</th>
<th>Investment 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff</td>
<td>Prob.</td>
</tr>
<tr>
<td>1</td>
<td>0.25</td>
</tr>
<tr>
<td>7</td>
<td>0.5</td>
</tr>
<tr>
<td>12</td>
<td>0.25</td>
</tr>
</tbody>
</table>

There remains a systematic basis of comparison. The task of the investment advisor is made more complex, however, as she will have to elicit more information on the preferences of her client if she wants to be in position to provide adequate advice.

4.8 Conclusions

The main topic of this chapter was the VNM expected utility representation specialized to admit risk aversion. Two measures of the degree of risk aversion were presented. Both are functions of an investor’s current level of wealth and, as such, we would expect them to change as wealth changes. Is there any systematic relationship between $R_A(Y)$, $R_R(Y)$, and $Y$ which it is reasonable to assume?

In order to answer that question we must move away from the somewhat artificial setting of this chapter. As we will see in Chapter 5, systematic relationships between wealth and the measures of absolute and relative risk aversion are closely related to investors’ portfolio behavior.

References


Appendix: Proof of Theorem 4.2

⇒ There is no loss in generality in assuming $U(\ )$ is differentiable, with $U'(\ ) > 0$. 
Suppose $F_A(x)$ FSD $F_B(x)$, and let $U(\cdot)$ be a utility function defined on $[a, b]$ for which $U'(\cdot) > 0$. We need to show that

$$E_A U(\tilde{x}) = \int_a^b U(\tilde{x})dF_A(\tilde{x}) > \int_a^b U(\tilde{x})dF_B(\tilde{x}) = E_B U(\tilde{x}).$$

This result follows from integration by parts (recall the relationship $\int_a^b udv = uv|_a^b - \int_a^b vdu$).

$$\int_a^b U(\tilde{x})dF_A(\tilde{x}) - \int_a^b U(\tilde{x})dF_B(\tilde{x}) = U(b)F_A(b) - U(a)F_A(a) - \int_a^b F_A(\tilde{x})U'(\tilde{x})d\tilde{x} - \int_a^b F_B(\tilde{x})U'(\tilde{x})d\tilde{x} = - \int_a^b F_A(\tilde{x})U'(\tilde{x})d\tilde{x} + \int_a^b F_B(\tilde{x})U'(\tilde{x})d\tilde{x},$$

(since $F_A(b) = F_B(b) = 1$, and $F_A(a) = F_B(a) = 0$)

$$\int_a^b F_B(\tilde{x}) - F_A(\tilde{x})U'(\tilde{x})d\tilde{x} \geq 0.$$

The desired inequality follows since, by the definition of FSD and the assumption that the marginal utility is always positive, both terms within the integral are positive. If there is some subset $(c, a) \subset [a, b]$ on which $F_A(x) > F_B(x)$, the final inequality is strict.

$\Leftarrow$ Proof by contradiction. If $F_A(\tilde{x}) \leq F_B(\tilde{x})$ is false, then there must exist an $\bar{x} \in [a, b]$ for which $F_A(\bar{x}) > F_B(\bar{x})$. Define the following nondecreasing function $\hat{U}(x)$ by

$$\hat{U}(x) = \begin{cases} 1 & \text{for } b \geq x > \bar{x} \\ 0 & \text{for } a \leq x < \bar{x} \end{cases}.$$

We’ll use integration by parts again to obtain the required contradiction.

$$\int_a^b \hat{U}(\tilde{x})dF_A(\tilde{x}) - \int_a^b \hat{U}(\tilde{x})dF_B(\tilde{x}) = \int_a^b \hat{U}(\tilde{x}) [dF_A(\tilde{x}) - dF_B(\tilde{x})] = \int_a^b 1 [dF_A(\tilde{x}) - dF_B(\tilde{x})] = F_A(b) - F_B(b) - [F_A(\bar{x}) - F_B(\bar{x})] - \int_\bar{x}^b [F_A(\tilde{x}) - F_B(\tilde{x})](0)d\tilde{x} = F_B(\bar{x}) - F_A(\bar{x}) < 0.$$
Thus we have exhibited an increasing function $\bar{U}(x)$ for which
\[ \int_a^b \bar{U}(\tilde{x})dF_A(\tilde{x}) < \int_a^b U(\tilde{x})dF_B(\tilde{x}), \]
a contradiction. $\Box$