Appendix

Proof of Proposition 3.2

Proposition 3.2 follows from the two following Lemmas.

Lemma 8.1 *There exists* $K_1 > 0$ *such that*

$$V(x_2, c) - V(x_1, c) \leq K_1 (x_2 - x_1)$$

for all $0 \leq x_1 \leq x_2$ *and* $c \leq \min\{\overline{c}, p\}$.

Proof. Take $\varepsilon > 0$ and $C \in \Pi_{x_2,c,\overline{c}}$ such that

$$J(x_2; C) \geq V(x_2, c) - \varepsilon. \tag{36}$$

Then the associated control process is given by

$$X_t^C = x_2 + \int_0^t (p - C_s) ds - \sum_{i=1}^{N_t} U_i.$$

Let τ be the ruin time of the process X_t^C. Assume first that $\overline{c} \leq p$ and define $\widetilde{C} \in \Pi_{x_1,c,\overline{c}}$ as $\widetilde{C}_t = C_t$,

where

$$X_t^{\widetilde{C}} = x_1 + \int_0^t (p - C_s) ds - \sum_{i=1}^{N_t} U_i.$$

For the ruin time $\widetilde{\tau} \leq \tau$ of the process $X_t^{\widetilde{C}}$, it holds $X_t^C - X_t^{\widetilde{C}} = x_2 - x_1$ for $t \leq \widetilde{\tau}$. Since $\overline{c} \leq p$, ruin can occur only at the arrival of a claim. Hence, using (36) we have

$$\begin{array}{rcl} V(x_2, c) - V(x_1, c) & \leq & J(x_2; C) - J(x_1; \widetilde{C}) + \varepsilon \\ & = & \mathbb{E}[\int_{\widetilde{\tau}}^{\tau} C_s e^{-qs} ds] + \varepsilon \\ & \leq & \mathbb{E}[\sum_{j=1}^{\infty} I_{\{\widetilde{\tau}=\tau_j \text{ and } \tau>\tau_j\}} \left(\int_{\tau_j}^{\tau} C_s e^{-qs} ds\right)] + \varepsilon \\ & \leq & \frac{\overline{c}}{q} \mathbb{E}[\sum_{j=1}^{\infty} e^{-q\tau_j} I_{\{\widetilde{\tau}=\tau_j \text{ and } \tau>\tau_j\}}] + \varepsilon. \end{array} \tag{37}$$

With the definitions

$$\mathcal{U}_{j-1} := \sum_{i=1}^{j-1} U_i \text{ and } A_t^C := \int_0^t (p - C_s) ds, \tag{38}$$

we have

$$\{\widetilde{\tau} = \tau_j \text{ and } \tau > \tau_j\} = \left\{x_2 + A_{\tau_j}^C - \mathcal{U}_{j-1} \geq U_j \ > x_1 + A_{\tau_j}^C - \mathcal{U}_{j-1}\right\},$$

and by the i.i.d. assumptions τ_j, U_j and $\mathcal{U}_{j-1}$ are mutually independent. This implies

$$\begin{array}{rl} & \mathbb{E}[\sum_{j=1}^{\infty} e^{-q\tau_j} I_{\{\widetilde{\tau}=\tau_j, \tau>\tau_j\}}] \\ \leq & K(x_2 - x_1)\beta \sum_{j=1}^{\infty} \left[\int_0^{\infty} e^{-qt} \left(\frac{\beta^{j-1} t^{j-1}}{(j-1)!}\right) e^{-\beta t} dt\right] \\ \leq & K \frac{\beta}{q} (x_2 - x_1), \end{array} \tag{39}$$

because $F(A_t + x_2 - \mathcal{U}_{j-1}) - F(x_1 + A_t - \mathcal{U}_{j-1}) \leq K(x_2 - x_1)$. From (37) and (39) we get the result with $K_1 = K\beta\overline{c}/q^2$.

Consider now $c \leq p < \overline{c}$. The main difference in this case is that ruin can occur not only at the arrival of a claim but also if dividends are paid with current surplus zero at a rate greater than p.

Let us prove first the result for $c = p$. Consider $C \in \Pi_{x_2,p,\overline{c}}$ as in (36) and

$$T = \min\left\{t : \int_0^t (C_s - p)\, ds = x_2 - x_1\right\}. \tag{40}$$

We put $T=\infty$ in the event

$$\int_0^{\tau}(C_s-p)\,ds<x_2-x_1.$$

Define $\overline{C}\in\Pi_{x_1,p,\overline{c}}$ as follows: $\overline{C}_t=p$ for $t\le T$ and then $\overline{C}_t=C_t$ and $\overline{\tau}\le\tau$ as the ruin time of the controlled process $X_t^{\overline{C}}$. Note that if $T\le\overline{\tau}$ we have $X_T^C=X_T^{\overline{C}}$ because

$$X_T^C-X_T^{\overline{C}}=x_2-x_1+\int_0^T(p-C_s)ds=0$$

and so $X_t^C=X_t^{\overline{C}}$ for $T\le t\le\overline{\tau}=\tau$. In the event that $T>\overline{\tau}$, we have $0<X_t^C-X_t^{\overline{C}}\le x_2-x_1$ for all $t\le\overline{\tau}$; also $\overline{\tau}$ coincides with the arrival of a claim since $\overline{C}_s=p$ for $s\le\overline{\tau}$. Therefore, from (40) and using the proof of (39) we can write

$$\begin{array}{ll} V(x_2,p)-V(x_1,p) & \\ \le & J(x_2;C)-J(x_1;\overline{C})+\varepsilon \\ = & \mathbb{E}[I_{T\le\overline{\tau}}\left(\int_0^T(C_s-p)\,e^{-qs}ds\right)] \\ & +\mathbb{E}\left[I_{T>\overline{\tau}}\int_0^{\overline{\tau}}\left(C_s-\overline{C}_s\right)e^{-qs}ds\right]+\mathbb{E}\left[I_{T>\overline{\tau}}\int_{\overline{\tau}}^{\tau}C_se^{-qs}ds\right]+\varepsilon \\ \le & 2(x_2-x_1)+\mathbb{E}[I_{\overline{\tau}\le T}\sum_{j=1}^{\infty}I_{\left\{\overline{\tau}=\tau_j,\tau>\tau_j\right\}}\left(\int_{\tau_j}^{\tau}C_se^{-qs}ds\right)]+\varepsilon \\ \le & \left(2+\overline{c}\,K\frac{\beta}{q^2}\right)(x_2-x_1)+\varepsilon \end{array} \tag{41}$$

and so we get the result with $K_1=2+\overline{c}\,K\beta/q^2$.

Let us consider now the case $c<p<\overline{c}$, $C\in\Pi_{x_2,c,\overline{c}}$ as in (36) and define

$$T_1=\min\{t:C_t\ge p\};$$

if $C_t\le p$ for all $t\le\tau$ then $T_1=\infty$.

Since $V(\cdot,p)$ is non-decreasing and continuous, we can find (as in Lemma 1.2 of [7]) an increasing sequence (y_i) with $y_1=0$ such that if $y\in[y_i,y_{i+1})$ then $0\le V(y,p)-V(y_i,p)\le\varepsilon/2$; consider admissible strategies $\widehat{C}^i\in\Pi_{y_i,p,\overline{c}}$ such that $V(y_i,p)-J(y_i,\widehat{C}^i)\le\varepsilon/2$. Let us define the dividend payment strategy $\overline{C}\in\Pi_{x_1,c,\overline{c}}$ as follows: $\overline{C}_t=C_t$ for $t<T_1$ and $\overline{C}_t=\widehat{C}^i_{t-T_1}$ for $t\ge T_1$ in the case that $X_{T_1}^C\in[y_i,y_{i+1})$; note that, with this definition, the strategy $\overline{C}$ turns out to be Borel measurable and so it is admissible. With arguments similar to the ones used before, we obtain

$$V(x_2,p)-V(x_1,p)\le\left(2+2\overline{c}\,K\frac{\beta}{q^2}\right)(x_2-x_1).\ \blacksquare$$

Lemma 8.2 *There exists $K_2>0$ such that*

$$0\le V(x,c_1)-V(x,c_2)\le K_2\,(c_2-c_1)$$

for all $x\ge0$ and $0\le c_1\le c_2\le\min\{\overline{c},p\}$.

Proof. Take $\varepsilon>0$ and $C\in\Pi_{x,c_1,\overline{c}}$ such that

$$J(x;C)\ge V(x,c_1)-\varepsilon \tag{42}$$

and define the stopping time

$$\widehat{T}=\min\{t:C_t\ge c_2\}. \tag{43}$$

Recall that τ is the ruin time of the process X_t^C. Consider first the case $\overline{c}\le p$ and define $\widetilde{C}\in\Pi_{x,c_2,\overline{c}}$ as $\widetilde{C}_t=c_2I_{t<\widehat{T}}+C_tI_{t\ge\widehat{T}}$; denote by X_t^C the associated controlled surplus process and by $\overline{\tau}\le\tau$ the corresponding ruin time. Since $\overline{c}\le p$, both X_t^C and $X_t^{\widetilde{C}}$ are non-decreasing between claim arrivals, and ruin can only occur at the arrival of a claim. We also have that $\widetilde{C}_s-C_s\le c_2-c_1$. We can write

$$\begin{array}{rcl} V(x,c_1)-V(x,c_2) & \le & J(x;C)+\varepsilon-J(x;\widetilde{C}) \\ & = & \mathbb{E}\left[\int_0^{\overline{\tau}}\left(C_s-\widetilde{C}_s\right)e^{-qs}ds\right]+\mathbb{E}\left[\int_{\overline{\tau}}^{\tau}C_se^{-qs}ds\right]+\varepsilon \\ & \le & \frac{\overline{c}}{q}\sum_{j=1}^{\infty}\mathbb{E}\left[I_{\left\{\overline{\tau}=\tau_j,\tau>\tau_j\right\}}e^{-q\tau_j}\right]+\varepsilon. \end{array} \tag{44}$$

Then,

$$\mathbb{E}\left[\left(e^{-q\overline{\tau}}-e^{-q\tau}\right)I_{\left\{\overline{\tau}=\tau_j,\ \tau>\tau_j\right\}}\right]\leq\mathbb{E}\left[e^{-q\tau_j}I_{\left\{\overline{\tau}=\tau_j,\ \tau>\tau_j\right\}}\right].$$

Using the definitions given in (38), we have

$$\begin{array}{cl} & \{\overline{\tau}=\tau_j,\ \tau>\tau_j\} \\ = & \left\{X^{C}_{\tau_j}=x+A^{C}_{\tau_j}-\mathcal{U}_{j-1}\geq 0 \text{ and } X^{\widetilde{C}}_{\tau_j}=x+A^{\widetilde{C}}_{\tau_j}-\mathcal{U}_{j-1}<0\right\} \\ = & \left\{x+A^{C}_{\tau_j}-\mathcal{U}_{j-1}\ \geq U_j\ >x+A^{\widetilde{C}}_{\tau_j}-\mathcal{U}_{j-1}\right\} \\ \subseteq & \left\{x+A^{\widetilde{C}}_{\tau_j}+(c_2-c_1)\tau_j-\mathcal{U}_{j-1}\ \geq U_j\ >x+A^{\widetilde{C}}_{\tau_j}-\mathcal{U}_{j-1}\right\}. \end{array}$$

Note that by the i.i.d. assumptions of the compound Poisson process we have that τ_j, U_j and $\mathcal{U}_{j-1}$ are mutually independent. Hence,

$$\begin{array}{cl} & \mathbb{E}[\sum_{j=1}^{\infty}e^{-q\tau_j}I_{\left\{\widetilde{\tau}=\tau_j,\tau>\tau_j\right\}}] \\ \leq & K(c_2-c_1)\beta\sum_{j=1}^{\infty}\left[\int_0^{\infty}e^{-qt}\left(\frac{\beta^{j-1}t^{j-1}}{(j-1)!}\right)te^{-\beta t}dt\right] \\ \leq & K\frac{\beta}{q^2}(c_2-c_1), \end{array} \tag{45}$$

because $F(x+A^{\widetilde{C}}_t+(c_2-c_1)t-u)-F(x+A^{\widetilde{C}}_t-u)\leq(c_2-c_1)t$. From (44) and (45) we get the result with $K_2=K\beta\overline{c}/q^3$.

Let us consider now the case $\overline{c}>p$. Take $C\in\Pi_{x,c_1,\overline{c}}$ as in (42) and $\widehat{T}$ as in (43). For

$$T_1:=\min\{t:C_t\geq p\},$$

since $c_2\leq p$, we have that $T_1\geq\widehat{T}$. Consider the increasing sequence (y_i) and the admissible strategies $\widehat{C}^i\in\Pi_{y_i,p,\overline{c}}$ introduced in the proof of Lemma 8.1, and define the dividend payment strategy $\overline{C}\in\Pi_{x,c_2,\overline{c}}$ as follows: take rate c_2 for $t\leq\widehat{T}$, C_t for $\widehat{T}\leq t<T_1$ and for $t\geq T_1$ take $\overline{C}_t=\widehat{C}^i_{t-T_1}$ in the case that $X^{C}_{T_1}\in[y_i,y_{i+1})$; as before, the strategy $\overline{C}$ turns out to be Borel measurable and so it is admissible. With arguments similar to the ones used before, we obtain,

$$V(x,c_1)-V(x,c_2)\leq\left(\frac{2}{q}+2\overline{c}\ K\frac{\beta}{q^3}\right)(c_2-c_1).\ \blacksquare$$

Proof of Proposition 3.3

Proposition 3.3 follows from the following two lemmas:

Lemma 8.3 *Assume that $\overline{c}>p$, then there exist constants $K_2>0$ and $K_3>0$ such that*

$$V(x_2,c)-V(x_1,c)\leq\left[K_2+\frac{K_3}{c-p}\right](x_2-x_1)$$

for all $0\leq x_1\leq x_2$ and $p<c\leq\overline{c}$.

Proof. Take $\varepsilon>0$ and $C\in\Pi_{x_2,c,\overline{c}}$ such that

$$J(x_2;C)\geq V(x_2,c)-\varepsilon. \tag{46}$$

Define $\widetilde{C}\in\Pi_{x_1,c,\overline{c}}$ as $\widetilde{C}_t=C_t$, and let us call $\widetilde{\tau}\leq\tau$ the ruin time of the process $X^{\widetilde{C}}_t$, then $X^{C}_t-X^{\widetilde{C}}_t=x_2-x_1$ for $t\leq\widetilde{\tau}$. Hence, using (46) and (39) we have,

$$\begin{array}{cl} & V(x_2,c)-V(x_1,c) \\ = & \mathbb{E}[\int_{\widetilde{\tau}}^{\tau}C_s e^{-qs}ds]+\varepsilon \\ \leq & \mathbb{E}[\sum_{j=1}^{\infty}\left(I_{\left\{\widetilde{\tau}=\tau_j,\ \tau>\tau_j\right\}}\int_{\tau_j}^{\tau}C_s e^{-qs}ds\right)]+\mathbb{E}[\sum_{j=1}^{\infty}\left(I_{\left\{\widetilde{\tau}\in(\tau_{j-1},\tau_j)\right\}}\int_{\widetilde{\tau}}^{\tau}e^{-qs}C_s ds\right)]+\varepsilon \\ \leq & \frac{\overline{c}}{q}\mathbb{E}[\sum_{j=1}^{\infty}e^{-q\tau_j}I_{\left\{\widetilde{\tau}=\tau_j,\tau>\tau_j\right\}}]+\frac{\overline{c}}{q}\mathbb{E}[\sum_{j=1}^{\infty}I_{\left\{\widetilde{\tau}\in(\tau_{j-1},\tau_j)\right\}}\left(e^{-q\widetilde{\tau}}-e^{-q\tau}\right)]+\varepsilon. \\ \leq & \overline{c}K\frac{\beta}{q^2}(x_2-x_1)+\frac{\overline{c}}{q}\mathbb{E}[\sum_{j=1}^{\infty}I_{\left\{\widetilde{\tau}\in(\tau_{j-1},\tau_j)\right\}}\left(e^{-q\widetilde{\tau}}-e^{-q\tau}\right)]+\varepsilon. \end{array} \tag{47}$$

We also get

$$\mathbb{E}\big[\sum_{j=1}^{\infty} I_{\left\{\widetilde{\tau}\in(\tau_{j-1},\tau_j)\right\}}\left(e^{-q\tau}-e^{-q\widetilde{\tau}}\right)\big] \leq q\,\mathbb{E}\big[\sum_{j=1}^{\infty} e^{-q\tau_{j-1}}\left(\tau-\widetilde{\tau}\right)\big]. \tag{48}$$

Assume now that $\widetilde{\tau}\in(\tau_{j-1},\tau_j)$ (and so $\widetilde{\tau}<\tau$). Then

$$0 = X_{\widetilde{\tau}}^{\widetilde{C}} = x_1 + \int_0^{\widetilde{\tau}} (p - C_s)\,ds - \sum_{k=1}^{j-1} U_k \text{ and } 0 \leq X_{\tau^-}^{C} \leq x_2 + \int_0^{\tau} (p - C_s)\,ds - \sum_{k=1}^{j-1} U_k.$$

Hence, we get

$$0 \leq X_{\tau^-}^{C} - X_{\widetilde{\tau}}^{\widetilde{C}} \leq x_2 - x_1 + \int_{\widetilde{\tau}}^{\tau} (p - C_s)\,ds \leq x_2 - x_1 + (p-c)(\tau-\widetilde{\tau})$$

and this implies

$$\tau - \widetilde{\tau} \leq \frac{x_2 - x_1}{c-p}. \tag{49}$$

We also have

$$\begin{array}{rcl} \mathbb{E}[\sum_{j=1}^{\infty} e^{-q\tau_{j-1}}] & = & 1 + \int\limits_0^{\infty} e^{-qs}\beta \sum_{k=1}^{\infty}\left(\frac{\beta^{k-1}s^{k-1}}{(k-1)!}\right) e^{-\beta s} ds \\ & \leq & 1+\beta/q. \end{array} \tag{50}$$

So, from (47), (48), (49) and (50), we get the result with $K_2 = \overline{c}K\beta/q^2$ and $K_3 = \overline{c}(1+\beta/q)$. ■

Lemma 8.4 *Assume that $\overline{c} > p$, then there exist constants $K_2 > 0$ and $K_3 > 0$ such that*

$$V(x,c_1) - V(x,c_2) \leq \left[K_2 + \frac{K_3 x}{(c_1-p)^2}\right](c_2-c_1)$$

for all $x \geq 0$ and $p < c_1 \leq c_2 \leq \overline{c}$.

Proof. If $x=0$, $V(x,c)=0$ for all $c>p$. Consider now $x>0$ and $p<c_1<c_2\leq\overline{c}$. Take $\varepsilon>0$ and $C\in\Pi_{x,c_1,\overline{c}}$ such that $J(x;C)\geq V(x,c_1)-\varepsilon$; we define the admissible strategy

$$\widehat{T} = \min\{t : C_t \geq c_2\}.$$

$\overline{C}\in\Pi_{x,c_2,\overline{c}}$ as $\overline{C}_t = c_2 I_{\left\{t<\widehat{T}\right\}} + C_t I_{\left\{t\geq\widehat{T}\right\}}$, and the ruin times τ and $\overline{\tau}$ of the processes X_t^C and $X_t^{\overline{C}}$ respectively. In this case both τ and $\overline{\tau}$ are finite with $\tau \geq \overline{\tau}$. Note that

$$\tau \leq \frac{x}{c_1-p}. \tag{51}$$

Let us define as $T_0 = \min\{t : x + \int_0^t (p-\overline{C}_s)ds = 0\}$ as the ruin time of the controlled process $X_t^{\overline{C}}$. In the event of no claims, we have $\overline{\tau}\leq T_0$. Since $\overline{c}\geq\overline{C}_s\geq c_2>p$, T_0 is finite and satisfies

$$\frac{x}{\overline{c}-p} \leq T_0 \leq \frac{x}{c_2-p}. \tag{52}$$

So we have

$$0 \leq \int_0^t \left(\overline{C}_s - C_s\right) ds \leq \left\{ \begin{array}{lll} (c_2-c_1)t & \text{if} & t\leq\widehat{T} \\ (c_2-c_1)\widehat{T} & \text{if} & t>\widehat{T} \end{array} \right. ,$$

and then

$$X_{\overline{\tau}}^{C} \leq X_{\overline{\tau}^-}^{C} \leq X_{\overline{\tau}^-}^{C} - X_{\overline{\tau}^-}^{\overline{C}} \leq (c_2-c_1)\overline{\tau} \leq (c_2-c_1)T_0 \leq (c_2-c_1)\frac{x}{c_2-p}. \tag{53}$$

We can write, using (45),

$$\begin{aligned}
&V(x,c_1)-V(x,c_2)\\
\leq\ & J(x;C)-J(x;\overline{C})+\varepsilon\\
\leq\ & \tfrac{\overline{c}}{q}\sum_{j=1}^{\infty}\mathbb{E}\left[I_{\left\{\overline{\tau}=\tau_j,\ \tau>\tau_j\right\}}e^{-q\tau_j}\right]+\tfrac{\overline{c}}{q}\sum_{j=1}^{\infty}\mathbb{E}\left[I_{\left\{\overline{\tau}\in(\tau_{j-1},\tau_j),\ \tau>\overline{\tau}\right\}}\left(e^{-q\overline{\tau}}-e^{-q\tau}\right)\right]+\varepsilon\\
\leq\ & \tfrac{\overline{c}\beta K}{q^3}(c_2-c_1)+\tfrac{\overline{c}}{q}\sum_{j=1}^{\infty}\mathbb{E}\left[I_{\left\{\overline{\tau}\in(\tau_{j-1},\tau_j),\ \tau>\overline{\tau}\right\}}\left(e^{-q\overline{\tau}}-e^{-q\tau}\right)\right]+\varepsilon.
\end{aligned} \tag{54}$$

In the case that $\overline{\tau}\in(\tau_{j-1},\tau_j)$ and $\tau>\overline{\tau}$, we have that

$$X_{\overline{\tau}}^{C}+\int_{\overline{\tau}}^{\tau}(p-c_1)ds\geq X_{\tau^-}^{C}\geq 0.$$

Then we get, from (53),

$$0\leq\tau-\overline{\tau}\leq\frac{X_{\overline{\tau}}^{C}}{c_1-p}\leq\frac{x}{(c_1-p)(c_2-p)}(c_2-c_1). \tag{55}$$

Hence, by virtue of (48), (50) and (55),

$$\begin{aligned}
&\tfrac{\overline{c}}{q}\sum_{j=1}^{\infty}\mathbb{E}\left[I_{\left\{\overline{\tau}\in(\tau_{j-1},\tau_j),\ \tau>\overline{\tau}\right\}}\left(e^{-q\overline{\tau}}-e^{-q\tau}\right)\right]\\
\leq\ & \tfrac{\overline{c}}{q}\sum_{j=1}^{\infty}\mathbb{E}\left[I_{\left\{\overline{\tau}\in(\tau_{j-1},\tau_j),\ \tau>\overline{\tau}\right\}}q\left(\tau-\overline{\tau}\right)e^{-q\tau_{j-1}}\right]\\
\leq\ & \sum_{j=1}^{\infty}\mathbb{E}\left[I_{\left\{\overline{\tau}\in(\tau_{j-1},\tau_j),\ \tau>\overline{\tau}\right\}}e^{-q\tau_{j-1}}\right]\tfrac{\overline{c}x}{(c_1-p)(c_2-p)}(c_2-c_1)\\
\leq\ & \tfrac{\overline{c}x}{(c_1-p)(c_2-p)}(c_2-c_1)\sum_{j=1}^{\infty}\mathbb{E}\left[e^{-q\tau_{j-1}}\right]\\
=\ & \tfrac{\overline{c}x}{(c_1-p)(c_2-p)}\left(1+\tfrac{\beta}{q}\right)(c_2-c_1).
\end{aligned}$$

Therefore, from (54) the result is established with $K_2=\overline{c}\beta K/q^3$ and $K_3=\overline{c}\,(1+\beta/q)$. ■

Proof of Proposition 3.4

The proof of Proposition 3.4 is quite technical. In addition to some technical lemmas below, we will use the exponential inequality

$$e^{-\frac{\gamma}{z^\eta}}\ \leq\frac{e^{-\frac{1}{\eta}}}{(\gamma\eta)^{1/\eta}}z \tag{56}$$

for $z>0$, $\gamma>0$ and $\eta>0$, as well as the following elementary remark about convolutions of independent distribution functions.

Remark 8.1 The distribution function F_j of the random variable $\mathcal{U}_j=U_1+...+U_j$ is Lipschitz with the same Lipschitz constant as F. To see this, consider $\mathcal{U}_2=U_1+U_2$. Then

$$P(a\leq U_1+U_2\leq a+h)=\int_0^{a+h}\int_{a-u}^{a+h-u}dF(v)dF(u)\leq Kh\int_0^{a+h}dF(u)\leq Kh.$$

With a recursive argument the proof extends to all $\mathcal{U}_j$ for $j\geq 1$.

Let us call J_x^c the value function of the strategy in $\Pi_{x,c,\overline{c}}$ that pays dividends at a constant rate c until ruin. We first compare J_x^p with J_x^c for $c>p$.

Lemma 8.5 *If $c>p$, there exists a positive constant $\overline{K}$, such that,*

$$-\frac{c-p}{q}\leq J_x^p-J_x^c\leq\overline{K}\left[1+\frac{1}{x}+\frac{e^{-\frac{1}{1-\alpha}}}{(xq(1-\alpha))^{1/(1-\alpha)}}+\frac{x}{(c-p)^{1-\alpha}}\right](c-p),$$

for any $0<\alpha<1$ and $x>0$.

Proof. Let us call $C \in \Pi_{x,p,\bar{c}}$ the constant strategy $C_t = p$ and $\overline{C} \in \Pi_{x,c,\bar{c}}$ the constant strategy $\overline{C}_t = c > p$ for all t. Define again τ as the ruin time of the process X_t^C and $\overline{\tau}$ the one of the process $X_t^{\overline{C}}$. We have that τ coincides with the arrival of a claim and $\overline{\tau} \leq \tau$, so we get the first inequality since

$$J_x^c \leq \int_0^\infty (c-p)\, e^{-qs} ds + \int_0^{\overline{\tau}} p e^{-qs} ds \leq \frac{c-p}{q} + J_x^p.$$

We can write, using (45),

$$\begin{array}{rcl} J_x^p - J_x^c & \leq & \frac{p}{q}\mathbb{E}\left[e^{-q\overline{\tau}} - e^{-q\tau}\right]. \\ & \leq & \frac{p}{q}\sum_{j=1}^{\infty}\mathbb{E}\left[I_{\left\{\overline{\tau}=\tau_j,\ \tau>\tau_j\right\}}e^{-q\tau_j}\right] + \frac{p}{q}\sum_{j=1}^{\infty}\mathbb{E}\left[I_{\left\{\overline{\tau}\in(\tau_{j-1},\tau_j),\ \tau>\overline{\tau}\right\}}e^{-q\overline{\tau}}\right] \\ & \leq & \frac{p\beta K}{q^3}(c-p) + \frac{p}{q}\sum_{j=1}^{\infty}\mathbb{E}\left[I_{\left\{\overline{\tau}\in(\tau_{j-1},\tau_j),\ \tau>\overline{\tau}\right\}}e^{-q\overline{\tau}}\right]. \end{array} \quad (57)$$

Note that if $\overline{\tau} \in (\tau_{j-1}, \tau_j)$, then $\tau > \overline{\tau}$.

In the event that $\overline{\tau} \in (0, \tau_1)$ we have $\overline{\tau} = x/(c-p)$. From (56), we get

$$\mathbb{E}\left[e^{-q\overline{\tau}} I_{\{\overline{\tau}\in(0,\tau_1)\}}\right] \leq e^{-q\frac{x}{c-p}}\mathbb{E}\left[I_{\{\overline{\tau}\in(0,\tau_1)\}}\right] \leq \frac{e^{-1}}{qx}(c-p)\mathbb{E}\left[I_{\{\overline{\tau}\in(0,\tau_1)\}}\right]. \quad (58)$$

In the event that $\overline{\tau} \in (\tau_1, \tau_2)$, we have $X_{\tau_1}^{\overline{C}} = x - (c-p)\tau_1 - U_1 > 0$. We consider two cases: $X_{\tau_1}^{\overline{C}} \leq x\,(c-p)^\alpha$ and $X_{\tau_1}^{\overline{C}} > x\,(c-p)^\alpha$. In the first case, using the Lipschitz condition on F, we obtain

$$\begin{array}{l} \mathbb{E}\left[e^{-q\overline{\tau}} I_{\{\overline{\tau}\in(\tau_1,\tau_2)\}} I_{\left\{0<X_{\tau_1}^{\overline{C}}\leq x(c-p)^\alpha\right\}}\right] \\ \leq \quad \mathbb{E}\left[e^{-q\overline{\tau}} I_{\{\overline{\tau}\in(\tau_1,\tau_2)\}} I_{\{x+(p-c)\tau_1 - x(c-p)^\alpha \leq U_1 < x+(p-c)\tau_1\}}\right] \\ \leq \quad \mathbb{E}\left[e^{-q\tau_1} I_{\{\overline{\tau}\in(\tau_1,\tau_2)\}} I_{\{x+(p-c)\tau_1 - x(c-p)^\alpha \leq U_1 < x+(p-c)\tau_1\}}\right] \\ \leq \quad Kx\,(c-p)^\alpha\,\mathbb{E}\left[e^{-q\tau_1} I_{\{\overline{\tau}\in(\tau_1,\tau_2)\}}\right]. \end{array}$$

In the second case, we have $(\overline{\tau} - \tau_1) = X_{\tau_1}^{\overline{C}}/(c-p) \geq x/\,(c-p)^{1-\alpha}$, and (56) yields

$$\begin{array}{rcl} \mathbb{E}\left[e^{-q\overline{\tau}} I_{\{\overline{\tau}\in(\tau_1,\tau_2)\}} I_{\left\{X_{\tau_1}^{\overline{C}} > x(c-p)^\alpha\right\}}\right] & = & \mathbb{E}\left[e^{-q(\overline{\tau}-\tau_1)} e^{-q\tau_1} I_{\{\overline{\tau}\in(\tau_1,\tau_2)\}}\right] \\ & \leq & e^{-\frac{qx}{(c-p)^{1-\alpha}}}\mathbb{E}\left[e^{-q\tau_1} I_{\{\overline{\tau}\in(\tau_1,\tau_2)\}}\right]. \end{array}$$

Hence,

$$\mathbb{E}\left[e^{-q\overline{\tau}} I_{\{\overline{\tau}\in(\tau_1,\tau_2)\}}\right] \leq \mathbb{E}\left[e^{-q\tau_1}\right]\left(Kx\,(c-p)^\alpha + \frac{e^{-\frac{1}{1-\alpha}}}{(qx(1-\alpha))^{1/(1-\alpha)}}(c-p)\right). \quad (59)$$

In a similar way, and using Remark 8.1, we obtain,

$$\mathbb{E}\left[e^{-q\overline{\tau}} I_{\left\{\overline{\tau}\in(\tau_{j-1},\tau_j),\ \tau>\overline{\tau}\right\}}\right] \leq \mathbb{E}\left[e^{-q\tau_j}\right]\left(Kx\,(c-p)^\alpha + \frac{e^{-\frac{1}{1-\alpha}}}{(xq(1-\alpha))^{1/(1-\alpha)}}(c-p)\right)$$

for any $j \geq 3$ and so from (50), (57) and (58) we get the second inequality. ■

In the next lemma, we give an alternative version of the Lipschitz condition for $x > 0$ and $c > p$. Here, for $x_2 > x_1 \geq \delta > 0$, the growth of the Lipschitz bound as $c \to p^+$, goes to infinity but slower than the bound obtained in Lemma 8.3.

Lemma 8.6 *For any $0 < \alpha < 1$ there exists a positive constant $\widetilde{K}$ such that*

$$V(x_2, c) - V(x_1, c) \leq \widetilde{K}\left[1 + \frac{1}{x_1} + \frac{e^{-\frac{1}{1-\alpha}}}{(qx_1(1-\alpha))^{1/(1-\alpha)}} + \frac{x_1}{(c-p)^{1-\alpha}}\right](x_2 - x_1),$$

where $p < c \leq \bar{c}$ and $0 < x_1 < x_2$.

Proof. Take $\varepsilon > 0$ and $C \in \Pi_{x_2,c,\overline{c}}$ such that

$$J(x_2; C) \geq V(x_2, c) - \varepsilon \tag{60}$$

and call τ the ruin time of the process X_t^C. Define $\widetilde{C} \in \Pi_{x_1,c,\overline{c}}$ as $\widetilde{C}_t = C_t$ and call $\widetilde{\tau}$ the ruin time of the process $X_t^{\widetilde{C}}$; it holds that $\widetilde{\tau} \leq \tau$ and $X_t^C - X_t^{\widetilde{C}} = x_2 - x_1$ for $t \leq \widetilde{\tau}$. In the event that $\widetilde{\tau} \in (\tau_{j-1}, \tau_j)$ (and so $\widetilde{\tau} < \tau$), $X_{\widetilde{\tau}}^{\widetilde{C}} = 0$ and so $X_{\widetilde{\tau}}^C = X_{\widetilde{\tau}}^{\widetilde{C}} + (x_2 - x_1) = x_2 - x_1$. Hence, since $C_s \geq C_{\widetilde{\tau}}$ for $s \geq \widetilde{\tau}$,

$$\tau - \widetilde{\tau} \leq \frac{1}{C_{\widetilde{\tau}} - p} \int_{\widetilde{\tau}}^{\tau} (C_s - p)\, ds \leq \frac{x_2 - x_1}{C_{\widetilde{\tau}} - p}. \tag{61}$$

From (47) and (61), we get

$$\begin{array}{ll} V(x_2, c) - V(x_1, c) & \\ \leq \ \ \overline{c} K \frac{\beta}{q^2}(x_2 - x_1) + \frac{\overline{c}}{q} \mathbb{E}[\sum_{j=1}^{\infty} I_{\left\{\widetilde{\tau} \in (\tau_{j-1}, \tau_j)\right\}} e^{-q\widetilde{\tau}} \left(1 - e^{-q(\tau - \widetilde{\tau})}\right)] + \varepsilon & \\ \leq \ \ \overline{c} K \frac{\beta}{q^2}(x_2 - x_1) + \overline{c} \mathbb{E}[\sum_{j=1}^{\infty} I_{\left\{\widetilde{\tau} \in (\tau_{j-1}, \tau_j)\right\}} e^{-q\widetilde{\tau}} \frac{1}{C_{\widetilde{\tau}} - p}] \,(x_2 - x_1) + \varepsilon & \end{array} \tag{62}$$

since $1 - e^{-ay} \leq ay$.

In the event that $\widetilde{\tau} \in (0, \tau_1)$,

$$(C_{\widetilde{\tau}} - p)\widetilde{\tau} \geq \int_0^{\widetilde{\tau}} (C_s - p)\, ds = x_1,$$

so $\widetilde{\tau} \geq x_1/(C_{\widetilde{\tau}} - p)$. By (56), we get

$$\mathbb{E}\left[\frac{e^{-q\overline{\tau}}}{C_{\widetilde{\tau}} - p} I_{\{\overline{\tau} \in (0, \tau_1)\}}\right] \leq \frac{e^{-1}}{q x_1}. \tag{63}$$

In the event that $\widetilde{\tau} \in (\tau_1, \tau_2)$, we consider two cases: $X_{\tau_1}^{\widetilde{C}} > x_1 (C_{\widetilde{\tau}} - p)^{\alpha}$ and $0 < X_{\tau_1}^{\widetilde{C}} \leq x_1 (C_{\widetilde{\tau}} - p)^{\alpha}$. Analogously to the proof of Lemma 8.5, we use the Lipschitz condition on F in the first case and (56) in the second case to obtain

$$\begin{array}{rcl} \mathbb{E}\left[I_{\left\{\widetilde{\tau} \in (\tau_1, \tau_2\right\}} \frac{e^{-q\widetilde{\tau}}}{C_{\widetilde{\tau}} - p}\right] & = & \mathbb{E}\left[\frac{e^{-q\widetilde{\tau}}}{C_{\widetilde{\tau}} - p} I_{\{\widetilde{\tau} \in (\tau_1, \tau_2)\}} I_{0 < X_{\tau_1} \leq x_1 (C_{\widetilde{\tau}} - p)^{\alpha}}\right] \\ & & + \mathbb{E}\left[\frac{e^{-q\widetilde{\tau}}}{C_{\widetilde{\tau}} - p} I_{\{\widetilde{\tau} \in (\tau_1, \tau_2)\}} I_{X_{\tau_1} > x_1 (C_{\widetilde{\tau}} - p)^{\alpha}}\right] \\ & \leq & \left(\frac{K x_1}{(c-p)^{1-\alpha}} + \frac{e^{-\frac{1}{1-\alpha}}}{(q x_1 (1-\alpha))^{1/(1-\alpha)}}\right) \mathbb{E}\left[e^{-q\tau_1}\right]. \end{array}$$

In a similar way, and using Remark 8.1, we obtain

$$\mathbb{E}\left[I_{\left\{\widetilde{\tau} \in (\tau_{j-1}, \tau_j)\right\}} e^{-q\widetilde{\tau}} \frac{1}{C_{\widetilde{\tau}} - p}\right] \leq \left(\frac{K x_1}{(c - p)^{1-\alpha}} + \frac{e^{-\frac{1}{1-\alpha}}}{(q x_1 (1 - \alpha))^{1/(1-\alpha)}}\right) \mathbb{E}\left[e^{-q\tau_{j-1}}\right]$$

for any $j \geq 3$ and so from (50), (63) and (62), we get the result. ∎

Proof of Proposition 3.4. Consider $x > 0$, we need to prove that $\lim_{c \to p^+} V(x, c) = V(x, p)$. Let us call, as before, J_y^c the value function of the strategy in $\Pi_{y,c,\overline{c}}$ that pays dividends at a constant rate c until ruin. Then, by Remark 2.1, $V(0, p) = J_0^p$. Also, we get $0 \leq J_y^p - J_0^p = J_y^p - V(0, p)$, and from Proposition 3.2 there exists a $K_1 > 0$ such that $V(y, p) - V(0, p) \leq K_1 y$. Hence,

$$V(y, p) - J_y^p \leq V(y, p) - V(0, p) + V(0, p) - J_y^p \leq K_1 y.$$

So, given $\varepsilon > 0$ small enough and taking $\delta \leq \varepsilon / K_1$, we have

$$V(y, p) - J_y^p \leq \varepsilon \tag{64}$$

for all initial surplus levels $0 \leq y \leq \delta$. We assume $\delta < \min\{1/4, x\}$, so $\delta^{3/2} < \delta/2$. Consider $C \in \Pi_{x,p,\overline{c}}$ such that $J(x; C) \geq V(x, p) - \varepsilon$ and define $T_1 := \min\left\{t \geq 0 : X_t^C \leq \delta\right\}$ and T_2 such that

$$\int_{T_2}^{\infty} e^{-qs}\overline{c}ds = \frac{\overline{c}}{q}e^{-qT_2} \leq \varepsilon.$$

Take $c \in (p, \overline{c})$ such that

$$c - p \leq \min\{\delta^{3/2}/T_2, (\varepsilon/T_2)^5, \varepsilon, \delta^{3/2}\}. \tag{65}$$

Let us define $\widehat{T} := \min\{t : C_t \geq c\}$. Since $V(\cdot, c)$ is non-decreasing and continuous, we can find (as in Lemma 8.1) an increasing sequence (y_i) with $y_1 = 0$ such that if $y \in [y_i, y_{i+1})$, then $0 \leq V(y, c) - V(y_i, c) \leq \varepsilon/2$. Consider admissible strategies $\widehat{C}^i \in \Pi_{y_i,c,\overline{c}}$ such that $V(y_i, c) - J(y_i, \widehat{C}^i) \leq \varepsilon/2$.

Let us now define the admissible strategy $\overline{C} \in \Pi_{x,c,\overline{c}}$ as follows: $\overline{C}_t = c$ for $t < \widehat{T}$; in the event that $T_1 \leq \widehat{T}$ (and so $X^C_{\widehat{T}} \leq \delta$), the strategy for $t \geq \widehat{T}$ consists of paying dividends at constant rate c until ruin; and in the event that $T_1 > \widehat{T}$ (and so $X^C_{\widehat{T}} > \delta$), we define $\overline{C}_t = \widehat{C}^i_{t-T_1}$ for $t \geq \widehat{T}$ in the case that $X^C_{T_1} \in [y_i, y_{i+1})$. Note that with this definition the strategy $\overline{C}$ turns out to be admissible and $C_s - \overline{C}_s \leq 0$ for $s \leq \widehat{T}$.

Let us call τ and $\overline{\tau}$ the ruin times of the processes X^C_t and $X^{\overline{C}}_t$, respectively. In order to prove the result, we consider different cases depending on the value of $\widehat{T}$:

$$\begin{array}{ll} & V(x,p) - V(x,c) \\ \leq & J(x;C) - J(x;\overline{C}) + \varepsilon \\ = & \mathbb{E}\left[I_{\{\widehat{T}\geq\overline{\tau}\}}(J(x;C) - J(x;\overline{C}))\right] + \mathbb{E}\left[I_{\{\widehat{T}<\overline{\tau},\widehat{T}>T_2\}}(J(x;C) - J(x;\overline{C}))\right] \\ & +\mathbb{E}\left[I_{\{\widehat{T}<\overline{\tau},\widehat{T}\leq T_2\wedge T_1\}}(J(x;C) - J(x;\overline{C}))\right] + \varepsilon \\ & +\mathbb{E}\left[I_{\{\widehat{T}<\overline{\tau},\widehat{T}\in[T_1,T_2]\}}(I_{\{T_1\neq\tau_j,\ 1\leq j\}} + \sum\limits_{j=1}^{\infty} I_{\{T_1=\tau_j\}})\left(J(x;C) - J(x;\overline{C})\right)\right]. \end{array} \tag{66}$$

In the event $\widehat{T} \geq \overline{\tau}$, using $\tau \geq \overline{\tau}$ and Lemma 8.5, we can show that

$$\begin{array}{l} \mathbb{E}\left[I_{\{\widehat{T}\geq\overline{\tau}\}}(J(x;C) - J(x;\overline{C}))\right] \\ \leq \overline{K}\left[1 + \frac{1}{x} + \frac{e^{-\frac{1}{1-\alpha}}}{(xq(1-\alpha))^{1/(1-\alpha)}} + \frac{x}{(c-p)^{1-\alpha}} + \frac{1}{q}\right](c-p). \end{array} \tag{67}$$

In the event that $\widehat{T} < \overline{\tau}$ and $\widehat{T} > T_2$, by the definition of T_2,

$$\mathbb{E}\left[I_{\{\widehat{T}<\overline{\tau},\widehat{T}>T_2\}}(J(x;C) - J(x;\overline{C}))\right] \leq \mathbb{E}\left[\int_{T_2}^{\infty} e^{-qs}\overline{c}ds\right] \leq \varepsilon. \tag{68}$$

In the event that $\widehat{T} < \overline{\tau}, \widehat{T} \leq T_2 \wedge T_1$, it holds that $X^C_{\widehat{T}} \geq \delta$ and

$$0 \leq X^C_{\widehat{T}} - X^{\overline{C}}_{\widehat{T}} \leq (c-p)\widehat{T} < (c-p)T_2 \leq \min\left\{\varepsilon, \delta^{3/2}\right\} < \delta/2.$$

Therefore, since $V(\cdot, c)$ is non-decreasing and $X^{\overline{C}}_{\widehat{T}} \in [\delta/2, x)$, we obtain from Lemma 8.6

$$\begin{array}{ll} & \mathbb{E}\left[I_{\{\widehat{T}<\overline{\tau},\widehat{T}\leq T_2\wedge T_1\}}(J(x;C) - J(x;\overline{C}))\right] \\ \leq & \widetilde{K}\left(1 + \frac{2}{\delta} + \frac{e^{-\frac{1}{1-\alpha}}}{(q(1-\alpha)\delta/2)^{1/(1-\alpha)}}\right)\delta^{3/2} + \widetilde{K}x(c-p)^{\alpha}T_2 + \varepsilon. \end{array} \tag{69}$$

In the event that $\widehat{T} < \overline{\tau}$ and $\widehat{T} \geq T_1$, the strategy is $\overline{C}_t = c$ for all t. If T_1 does not coincide with the arrival of a claim, then $X^C_{T_1} = \delta$ (and so $X^{\overline{C}}_{T_1} \geq \delta/2$). Then we can write, using (64), Proposition 3.2 and Lemma 8.5,

$$\begin{array}{ll} & \mathbb{E}\left[I_{\{\widehat{T}<\overline{\tau},\widehat{T}\in[T_1,T_2]\}}I_{\{T_1\neq\tau_j,\ 1\leq j\}}(J(x;C) - J(x;\overline{C}))\right] \\ \leq & \mathbb{E}\left[I_{\{\widehat{T}<\overline{\tau},\widehat{T}\in[T_1,T_2]\}}I_{\{T_1\neq\tau_j,\ 1\leq j\}}e^{-qT_1}(V(\delta,p) - J^c_{\delta/2})\right] \\ \leq & (V(\delta,p) - J^p_{\delta}) + \left(J^p_{\delta} - J^p_{\delta/2}\right) + \left(J^p_{\delta/2} - J^c_{\delta/2}\right) \\ \leq & \varepsilon + K_1\frac{\varepsilon}{2K_1} + \overline{K}(1 + \frac{2}{\delta} + \frac{e^{-\frac{1}{1-\alpha}}}{(\delta/2q(1-\alpha))^{1/(1-\alpha)}})\delta^{3/2} + \overline{K}\delta/2(c-p)^{\alpha}. \end{array} \tag{70}$$

Finally, in the event that $\widehat{T} < \overline{\tau}$, $\widehat{T} \geq T_1$ and T_1 coincides with the j-th claim arrival, then $X^C_{T_1} = X^C_{\tau_j} \in (0, \delta)$ and $X^C_{\tau_j^-} \geq \delta$. Hence,

$$0 < X^C_{T_1} = X^C_{\tau_j} = X^C_{\tau_j^-} - U_j < \delta.$$

Therefore, $X^C_{\tau_j^-} > U_j > X^C_{\tau_j^-} - \delta \geq 0$. Since $F(X^C_{\tau_j^-}) - F(X^C_{\tau_j^-} - \delta) \leq K\delta$ and, by the compound Poisson assumptions, we obtain

$$\mathbb{E}\left[I_{\left\{\widehat{T}<\overline{\tau}, \widehat{T}\in[T_1,T_2]\right\}} I_{\left\{T_1=\tau_j\right\}} e^{-q\tau_j} \right] \leq K\delta \mathbb{E}\left[e^{-q\tau_j} \right].$$

So, by (50) and Proposition 3.3,

$$\begin{array}{ll} \sum\limits_{j=1}^{\infty} \mathbb{E}\left[I_{\left\{\widehat{T}<\overline{\tau}, \widehat{T}\in[T_1,T_2]\right\}} I_{\left\{T_1=\tau_j\right\}} (J(x;C) - J(x;\overline{C})) \right] & \\ \leq \quad K\delta V(\delta, c) \sum_{j=1}^{\infty} \mathbb{E}\left[e^{-q\tau_j} \right] & \\ \leq \quad \frac{K\overline{c}\beta}{q^2}\delta. & \end{array} \tag{71}$$

Using (65)–(71) with $\alpha = 1/5$, and so $1/(1-\alpha) = 5/4 < 3/2$, we get the result. ■