Optimal dividend strategies for a compound Poisson process under transaction costs and power utility

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Abstract:
We characterize the value function of maximizing the total discounted utility of dividend payments for a compound Poisson insurance risk model when strictly positive transaction costs are included, leading to an impulse control problem. We show how to derive an optimal strategy for exponential claim amounts and construct a numerical procedure to deal with general claim amount distributions.

1 Control problem with transaction costs

Let \((\Omega, \mathcal{F}, P)\) be a probability space carrying all the stochastic quantities defined below. The uncontrolled risk reserve process \(R = (R_t)_{t \geq 0}\) of an insurance portfolio is given by the Cramér-Lundberg model

\[ R_t = x + ct - \sum_{n=1}^{N_t} Y_i, \]

where \(N = (N_t)_{t \geq 0}\) is a homogeneous Poisson process with intensity \(\lambda > 0\). The sequence of claim amounts is \(\{Y_i\} \overset{i.i.d.}{\sim} F_Y\), where \(F_Y\) is a probability distribution on \((0, \infty)\) with a continuous distribution function. The deterministic components are the initial capital \(x\) and the premium intensity \(c\).

As an extension of the classical model, the insurance company is allowed to pay out dividends to its shareholders, but with the constraint that for every payment a strictly positive amount \(K > 0\), the transaction cost, has to be paid. This has the consequence that a reduction of the reserve of size \(z\) yields a dividend payment of \(z - K\) only. In addition a possible further modification (cf. [4]) is the introduction of an additional proportional transaction cost given by some factor \(k \in (0, 1)\). Then the value of a payment of size \(z\) is reduced to \(kz - K\). This can also be interpreted as a tax payment on the dividend with rate \(1 - k\), which has to be paid immediately at the payment. One immediately observes that only payments of size greater than \(a \coloneqq K/k\) are reasonable and only a finite number of actions in bounded time intervals will be feasible. The following definition is due to Korn [12] and fixes the class of appropriate control strategies.

**Definition 1.1.** An impulse control \(S = \{(\tau_i, Z_i)\}_{i \in \mathbb{N}}\) is a sequence of increasing intervention times \(\tau_i\) and associated control actions \(Z_i\), which fulfills the following four conditions:

- \(0 \leq \tau_i \leq \tau_{i+1}\) a.s. for all \(i \in \mathbb{N}\),

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\begin{itemize}
  \item $\tau_i$ is a stopping time with respect to the filtration $\mathcal{F}_t = \sigma\{R^{S}_{s-} \mid s \leq t\}$ for $t \geq 0$,
  \item $Z_i$ is measurable with respect to $\mathcal{F}_{\tau_i}$,
  \item $P(\lim_{i \to \infty} \tau_i \leq T) = 0$ for all $T \geq 0$.
\end{itemize}

Let $S^*$ denote the set of impulse controls and $S_n^*$ the set for which at most $n$ interventions are used, $\tau_{n+1} = \infty$.

The controlled process $R^S = (R^{S}_t)_{t \geq 0}$ for $S \in S^*$, is defined by

$$R^{S}_t = x + ct - \sum_{n=1}^{N_t} Y_n - \sum_{i=1}^{\infty} I_{(\tau_i < t)} Z_i.$$  

In particular we have $R^{S}_{\tau_i} = R^{S}_{\tau_i -} - Z_i$. The time of ruin of the controlled reserve is denoted by $\tau^{S} = \inf\{t > 0 \mid R^{S}_t < 0\}$ for a strategy $S \in S^*$.

Following [4], we model the utility of the shareholders by a power function. Then, in our context, the value of a strategy $S$ is of the form

$$V_S(x) = \mathbb{E}_x \left( \sum_{i=1}^{\infty} e^{-\delta \tau_i} u(Z_i) I_{(\tau_i < \tau)} \right),$$

where $\delta > 0$ is a constant discount factor and $u(z) = \frac{1}{\gamma} (kz - K)^\gamma$ with $\gamma \in (0, 1]$.

As mentioned above we need some further constraints on $S^*$ to formulate a reasonable optimization problem. Because of the transaction costs we need $Z_i \geq a$ and of course $R^{S}_{\tau_i} = R^{S}_{\tau_i -} - Z_i \geq 0$, thus $Z_i \in (a, R^{S}_{\tau_i -})$. In the terminology of [6], $K(x) = \{y \mid y \in (a, x)\}$ denotes the set of applicable actions at some point $x \geq 0$ with $K(x) = \emptyset$ for $x \in [0, a]$.

We call the set of admissible controls $S$ and $S_n$, respectively. The maximization problem consists of identifying

$$V(x) = \sup_{S \in S} V_S(x). \quad (1)$$

In [6], Davis develops a general theory of optimal impulse control for PDMP’s and refers to articles by Lenhart [14] and Gątarek [7] on that topic. Cadenillas et al. [4] look at the maximization problem (1) for a mean-reverting diffusion process and solve the problem by calculating an explicit solution to the associated quasi-variational inequalities. Other examples from finance dealing with portfolio optimization under various types of transaction costs are Korn [11, 12] and Irle and Saš [9]. Although similar in flavor, these results are not directly applicable in the present situation, since either the bounds on the value functions or the class of considered processes differ substantially from our setup. The papers by Bensoussan et al. [3] and Benkherouf & Bensoussan [2] deal with inventory models with diffusion and compound Poisson demand with ordering policies. While upward impulses used in these papers, a dividend payment as described above is a downward impulse, so that these models can not be interpreted as a dividend problem in an insurance context. They rather correspond to the problem of minimizing capital injections, where the investor is allowed to give upward impulses to the reserve process in order to avoid ruin, see Kulenko & Schmidli [13]. Furthermore such inventory models do not include the complication of a ruin event which affects the dividend problem.

For diffusion risk reserve processes dividend maximization problems including transaction costs are for example investigated by Jeanblanc-Picqué & Shiryaev [10], Paulsen [16, 17] or He & Liang [8]. In [15] Loeffen studies the dividend maximization problem including transaction costs and linear utility for spectrally-negative Lévy processes, including the compound Poisson reserve
process, with probabilistic means and gives a condition when a strategy of a simple form is the optimal one.

In this paper we tackle such a control problem by analytical means and develop a concise theoretical framework for its solution. In Section 2, basic properties such as linear boundedness and continuity of the value function are derived. These results are used in Section 3 to characterize the value function as a fixed point of an associated optimal stopping operator. In Section 4 the problem is studied from the viewpoint of quasi-variational inequalities. These inequalities are then used in Section 5 to derive an optimal strategy in the case of exponentially distributed claim amounts and a fixed parameter set. Finally the construction of a numerical procedure for getting approximations of the optimal value for general claim amount distributions is illustrated.

2 Properties of the Value Function

In this section some basic properties of the value function are proved. We can derive immediately that \(V\) is increasing, because for \(0 \leq y \leq x\) we have that \(\mathcal{K}(y) \subset \mathcal{K}(x)\), so that by the definition of the value function \(V\) and the utility function \(u\) we get that with identical intervention times always a higher pay-off can be realized. The definition of the set of admissible strategies \(\mathcal{S}\) implies \(V \geq 0\) for all \(x \geq 0\). The next lemma identifies further properties of \(V\).

**Lemma 2.1.** \(V\) is locally Lipschitz-continuous and bounded by a linear function.

**Proof.** Let \(x > y \geq 0\) and \(S^x = \{(\tau_i, Z_i)\}_{i \in \mathbb{N}}\) be an \(\epsilon\)-optimal strategy for initial capital \(x\) \((V(x) \leq V_{S^x}(x) + \epsilon)\). We define the strategy \(S_y\) for an initial capital \(y\) in the following way. As long as \(R^{S_y}\) stays below \(x\), pay out nothing and if \(x\) is reached, apply strategy \(S^x\).

Let \(\theta_x\) denote the first time when \(R^{S_y}\) hits \(x\). On \(\{\theta_x \leq \tau^{S_y}\}\), we get \(S_y = \{(\tau_i + \theta_x, Z_i)\}_{i \in \mathbb{N}}\) and because the sum of stopping times is a stopping time, \(S_y\) is admissible. We have

\[
0 \leq V(x) - V(y) \leq V(x) - V_{S_y}(y) \leq V(x) - (V(x) - \epsilon)\mathbb{E}_y \left( e^{-\delta \theta_x} I_{\{\theta_x < \tau^{S_y}\}} \right) \\
= V(x)(W(x) - W(y)) + \epsilon W(y).
\]

The function \(W\) is defined as \(W(y) := \mathbb{E}_y \left( e^{-\delta \theta_x} I_{\{\theta_x < \tau^{S_y}\}} \right)\) and it is a differentiable solution (cf. [20]) to

\[
0 = cW'(y) + \lambda \int_0^y W(y - z)dF(y) - (\delta + \lambda)W(y).
\]

Further, by definition \(W(y) \leq W(x) = 1\).

Therefore we get from (2) that \(0 \leq V(x) - V(y) \leq V(x)C(x)(x - y) + \epsilon\), where \(C(x)\) is a local constant depending on \(W'(x)\). Because this inequality holds for every \(\epsilon > 0\), we have \(V(x) - V(y) \leq C^*(x)(x - y)\), where \(C^*(x)\) is another local constant.

From that we obtain that \(V\) is locally Lipschitz continuous on \([0, \infty)\), and consequently that \(V\) is differentiable almost everywhere. Because of the Lipschitz continuity on compact sets, the above calculations give bounds for \(V'(x)\) which is therefore integrable on compact sets. This yields that \(V\) is absolutely continuous, see [21].

The next step is to determine a linear bound for \(V\). The idea is to use a maximizing strategy for the upper bound \(x + ct \geq R_t\) which will serve as a bound for \(V\). Because we are in a deterministic setup, such a strategy will be of the simple structure that a level \(b > a\) is fixed, and whenever this level is reached, the whole available amount is paid as a dividend. The value of such a strategy is given by

\[
u(x)I_{\{x \geq b\}} + u(b) \sum_{n=1}^{\infty} \left( e^{-\frac{\delta t}{\lambda}} \right)^n \leq u(x) + h(b),
\]
where \( h(b) = \frac{1}{n}(kb - K)^\gamma - \frac{1}{e^{\frac{K}{n}} - 1} > 0 \), \( \forall b > a \). Observe that the time needed to reach \( b \) for the \( n \)th time is \( \frac{b}{n} \).

For \( b > a \), the function \( h \) fulfills
\[
h'(b) = \frac{(bk - K)^{\gamma - 1}(e^{(db)/c}(ck\gamma - \delta(bk - K)) - ck\gamma)}{\gamma(c(e^{(db)/c} - 1))^2} = 0
\]

so that \( h \) has exactly one maximum which we will denote by \( b^* > a \). Because of the boundary values, this has to be a global maximum. We get \( V(x) \leq mx + n + h(b^*) \), because \( u \) itself is linearly bounded.

Let us define an operator \( M \) acting on a function \( f \) by
\[
Mf(x) = \sup_{y \in K(x)} \{u(y) + f(x - y)\}. \tag{3}
\]

It gives the optimal value of an intervention at some surplus height \( x \geq 0 \). Because of the definition of the set of admissible strategies \( S \), we have \( K(x) = \emptyset \) for \( 0 \leq x \leq a \), in which case we define \( Mf = 0 \). The definitions of the set \( S \) and \( V \) by (1) give that
\[
V(x) \geq MV(x)
\]
holds for all \( x \geq 0 \) and indicate that at points \( x' \) where it would be optimal to intervene, \( V(x') = MV(x') \) should hold. The next lemma presents basic properties of \( Mf : [0, \infty) \to [0, \infty) \) for a suitable function \( f \).

**Lemma 2.2.** Let \( f \) be an increasing, continuous and linearly bounded function, then \( Mf \), as defined in (3), is (as a function in \( x \in [0, \infty) \)) also bounded by a linear function, increasing and continuous.

**Proof.** Let \( x > a \) and \( m, n \in \mathbb{R} \) such that \( f(x) \leq mx + n \). We have,
\[
Mf(x) = \sup_{y \in (a, x]} \{u(y) + f(x - y)\} \leq \sup_{y \in (a, x]} \{u(y) + m(x - y) + n\} \leq mx + u(x) + n
\]
\[
\leq m_1x + n_1, \text{ for some } m_1, n_1 \in \mathbb{R},
\]
where we use that also \( u(x) \) is linearly bounded by \( kx + u \left(\frac{K+1}{K}\right) \) for every \( x \). The case \( x \leq a \) needs no special treatment because \( Mf(x) = 0 \).

Let \( x > \bar{x} > a \) and let \( y^* \in (a, \bar{x}] \) such that \( \sup_{y \in (a, x]} \{u(y) + f(x - y)\} \leq u(y^*) + f(x - y^*) + \epsilon \) for some \( \epsilon > 0 \). Then
\[
Mf(x) - Mf(\bar{x}) = \sup_{y \in (a, x]} \{u(y) + f(x - y)\} - \sup_{y \in (a, \bar{x}]} \{u(y) + f(\bar{x} - y)\}
\]
\[
\geq \sup_{y \in (a, x]} \{u(y) + f(x - y)\} - \sup_{y \in (a, \bar{x}]} \{u(y) + f(\bar{x} - y)\}
\]
\[
\geq u(y^*) + f(x - y^*) - u(y^*) - f(\bar{x} - y^*) \geq 0,
\]
where the last inequality holds because \( \epsilon > 0 \) can be chosen arbitrarily small.

For proving the continuity, fix some \( \epsilon > 0 \). Because \( f \) and \( u \) are continuous, we can choose \( \delta_1 > 0 \) such that for \( 0 < x - \bar{x} < \delta_1/2 \) \((x, \bar{x} > a)\) and \( |y' - \bar{y}| < \delta_1/2 \) we have that \( |u(y') - u(\bar{y})| < \epsilon/3 \) and \( |f(x - y') - f(\bar{x} - y)| < \epsilon/3 \) hold. We will state the definitions of \( y' \) and \( \bar{y} \) immediately (the
last inequality is based on the continuity of $f$, the fact that $|x - y' + (-\dot{x} + \dot{y})| < \delta_1$ and an appropriate choice of $\delta_1$.

We choose $y' \in (a, x]$ such that $Mf(x) \leq u(y') + f(x - y') + \epsilon/3$. Because $x - \dot{x} < \delta_1/2$ we can choose a $\tilde{y} \in (a, \tilde{x}]$ with $|y' - \tilde{y}| < \delta_1/2$. Then we have

$$0 \leq Mf(x) - Mf(\tilde{x}) \leq \frac{\epsilon}{3} + u(y') + f(x - y') - u(\tilde{y}) - f(\tilde{x} - \tilde{y}) < \epsilon.$$ 

The continuity of $f$ and $u$ hence ensures the continuity of $Mf$ for $x \in [0, a]$.

\[ \square \]

3 Characterization of $V$

In this section we will characterize $V$ as a fixed point to a related optimal stopping operator. The proof uses a construction which allows to interpret $V$ as a gain when applying iterated optimal stopping.

Such a characterization is inspired by Davis [6], where such a fixed point characterization is shown for a minimization problem with a bounded pay-off function $u$. Korn [11] uses this approach to characterize the solution of a portfolio optimization problem with an underlying diffusion process.

Let $T$ denote the set of a.s. finite stopping times with respect to $\{\mathcal{F}_t\}_{t\geq 0}$.

Remark 3.1. In the sequel of this section, an optimal stopping operator with stopping times from the class $T$ is defined. But notice that when looking at the original problem, the stopping times (part of admissible strategies) do not have to be in $T$ (like the use of $\tau_{n+1}$ in the definition of $S_n$ demonstrates). However, this has no consequences, since the linear bounds of $V$ and $MV$ indicate that, due to discounting, the use of an unbounded stopping time results in a profit equal to zero.

The following operator gives the value of the related optimal stopping problem,

$$\mathcal{M}f(x) = \sup_{\theta \in T} \mathbb{E}_x \left( e^{-\delta \theta} Mf(R_\theta) \right). \tag{4}$$ 

At first we have to develop some basic properties of the operator $\mathcal{M}$.

Lemma 3.1. Let $f$ be an increasing, linearly bounded and continuous function, then the same holds for $\mathcal{M}f$ as a function in $x \in [0, \infty)$.

Proof. We start with deriving the linear bound, using $Mf(x) \leq mx + n$ and $R_\theta \leq x + c\theta$,

$$0 \leq \mathcal{M}f(x) = \sup_{\theta \in T} \mathbb{E}_x \left( e^{-\delta \theta} Mf(R_\theta) \right)$$

$$\leq \sup_{\theta \in T} \mathbb{E}_x \left( e^{-\delta \theta} (mx + c\theta + n) \right) \leq \mathbb{E}_x \left( e^{-\delta \theta} (mx + n) \right) + \sup_{\theta \in T} \mathbb{E}_x \left( e^{-\delta \theta} c\theta \right)$$

$$\leq mx + n + \frac{c}{\epsilon \delta}.$$ 

The monotonicity of $\mathcal{M}f$ follows by the monotonicity of $Mf$.

Now let $0 < x - x' = \delta_1$ and let $R^x$ and $R^{x'}$ be the uncontrolled processes starting at the indicated points with times of ruin $\tau(x)$, $\tau(x')$. If we look pathwise on these processes we get that $R^x_t(\omega) - R^{x'}_t(\omega) = \delta_1$ for each $\omega \in \Omega$ as long as $t < \tau(x') \leq \tau(x)$. Now let $\theta^*$ be an $\epsilon$-optimal stopping time for initial capital $x$. Then we derive

$$0 \leq \mathcal{M}f(x) - \mathcal{M}f(x') \leq \mathbb{E} \left( e^{-\delta \theta^*} Mf(R^x_{\theta^*}) - e^{-\delta \theta^*} Mf(R^{x'}_{\theta^*}) \right) + \epsilon. \tag{5}$$

At this point we have to distinguish three cases:
1. \( \theta^* \geq \tau(x) \geq \tau(x') \): here we have \( Mf(R^x_{\theta^*}) = Mf(R^{x'}_{\theta^*}) = 0 \),

2. \( \theta^* < \tau(x') \leq \tau(x) \): here \( R^x_{\theta^*} - R^{x'}_{\theta^*} = \delta_1 \),

3. \( \tau(x') \leq \theta^* < \tau(x) \): here we have that \( Mf(R^{x'}_{\theta^*}) = 0 \) and \( R^x_{\tau(x')} \leq \delta_1 \) because \( R^{x'}_{\tau(x')} = R^x_{\theta^*} \).

The above observations show that we only have to take care of the Cases 2 and 3. Case 2 is described by the set of all paths

\[
P_{\delta_1} = \{ \omega \in \Omega \mid R^x_t(\omega) \geq \delta_1 \text{ for all } t \in [0, \theta^*] \}.
\]

Case 3 then is described by

\[
P = \{ \omega \in \Omega \mid R^x_t(\omega) \geq 0 \forall t \in [0, \theta^*] \text{ and } \exists s \in [0, \theta^*] \text{ such that } R^x_s < \delta_1 \}.
\]

Let \( t^* \) be the smallest time such that \( R^x_{t^*} < \delta_1 \), then \( t^* = \tau(x') \). We derive that the right-hand side of (5) is smaller or equal to

\[
\mathbb{E} \left( I_{\delta_1} \left( Mf(R^x_{\theta^*}) - Mf(R^{x'}_{\theta^*}) \right) \right) + \mathbb{E} \left( I_{\delta_1} \left( Mf(R^x_{\theta^*}) - Mf(R^{x'}_{\theta^*}) \right) \right) + \epsilon \\
\leq \mathbb{E} \left( I_{\delta_1} \left( Mf(R^x_{\theta^*}) - Mf(R^{x'}_{\theta^*}) \right) \right) + \mathbb{E} \left( I_{\delta_1} \left( Mf(R^x_{\theta^*}) - Mf(R^{x'}_{\theta^*}) \right) \right) + \epsilon
\leq \mathbb{E} \left( I_{\delta_1} \left( Mf(R^x_{\theta^*}) - Mf(R^{x'}_{\theta^*}) \right) \right) + Mf(\delta_1) + \epsilon. \tag{6}
\]

The last inequality holds due to the strong Markov property of the uncontrolled process \( R^x \) and because on \( P \) we have that \( R^x_{\tau(x')} < \delta_1 \). Now we can use that \( Mf \) is continuous and an appropriate choice of \( \delta_1 \) (i.e. letting \( \delta_1 \to 0 \)) gives the continuity of \( Mf \) in \( x' \). \( \square \)

Now we are able to prove the main result of this section. It verifies the proposed connection between \( V \) and the implicit optimal stopping problem.

**Theorem 3.2.** Let \( v_0(x) = 0 \) for all \( x \in [0, \infty) \) and define functions \( v_n : [0, \infty) \to [0, \infty) \) by \( v_n(x) = Mv_{n-1}(x) \) for \( x \in [0, \infty) \). Then the following holds:

\[
\lim_{n \to \infty} v_n(x) = V(x),
\]

\[
V(x) = MV(x). \tag{7}
\]

**Proof.** Note that \( v_0 = 0 \) is equal to the payoff of a strategy with no intervention. Now look at \( v_1(x) \),

\[
v_1(x) = Mv_0(x) = \sup_{\theta \in T} \mathbb{E}_x \left( e^{-\delta \theta} \sup_{y \in [a, R_\theta]} \{ u(y) + v_0(R_\theta - y) \} \right)
\geq \sup_{\theta \in T} \mathbb{E}_x \left( e^{-\delta \theta} u(R_\theta)I_{\{R_\theta > a\}} \right) \geq v_0.
\]

If only one intervention is allowed, followed by a gain equal to zero, it is optimal to pay out the whole reserve at the best stopping time, therefore \( v_1(x) = \sup_{S \in S_1} V_S(x) \).

Define the sequence of functions \( \{ v_n \}_{n \in \mathbb{N}} \) by \( v_n(x) = Mv_{n-1}(x) \). The properties of the operator \( M \) (see Lemma 3.1) and \( v_0 \) guarantee monotonicity, continuity and linear boundedness of the
functions \( v_n \) for all \( n \in \mathbb{N} \) by induction. Assume that \( v_{n-1}(x) = \sup_{S \in S_{n-1}} V_S(x) \), then

\[
v_n(x) = \sup_{\theta \in T} \mathbb{E}_x \left( e^{-\delta \theta} \sup_{y \in (a, R^\theta]} \{ u(y) + v_{n-1}(R^\theta - y) \} \right) = \sup_{\theta \in T} \mathbb{E}_x \left( e^{-\delta \theta} \sup_{y \in (a, R^\theta]} \left( u(y) + \sup_{S \in S_{n-1}} V_S(R^\theta - y) \right) \right). \tag{8}
\]

In (8), at most \( n \) decisions are taken, so that clearly \( v_n(x) \leq \sup_{S \in S_n} V_S(x) \). On the other hand, let \( S^* = \{(\tau_s^i, Z_s^i)\}_{1 \leq i \leq n} \in S_n \) be an \( \epsilon \)-optimal strategy within \( S_n \), then we get

\[
\sup_{S \in S_n} V_S(x) < \epsilon + V_{S^*}(x) = \epsilon + \mathbb{E}_x \left( e^{-\delta \tau_{1}^*} u(Z_1^*) I_{\{\tau_1^* < s^*\}} + e^{-\delta \tau_{2}^*} \sum_{i=2}^{n} e^{-\delta (\tau_i^* - \tau_{i-1}^*)} u(Z_i^*) I_{\{\tau_i^* < s^*\}} \right)
\leq \epsilon + \mathbb{E}_x \left( e^{-\delta \tau_{1}^*} u(Z_1^*) I_{\{\tau_1^* < s^*\}} + e^{-\delta \tau_{n-1}^*} v_{n-1}(R_{\tau_1^*} - Z_1^*) \right) \leq \epsilon + v_n(x).
\]

Consequently \( v_n(x) = \sup_{S \in S_n} V_S(x) \) holds for all \( n \in \mathbb{N} \) by induction. Because \( v_n(x) \leq V(x) \) and the monotonicity of \( \{v_n(x)\}_{n \in \mathbb{N}} \) for all \( x \in [0, \infty) \), we get that the sequence converges pointwise to some function, \( \lim_{n \to \infty} v_n(x) = v^*(x) \).

As a last step we have to show that \( v^*(x) = V(x) \). For that purpose, let \( S = \{\tau_i, Z_i\}_{i \in \mathbb{N}} \in S \) and define \( S_n = \{\tau_i, Z_i\}_{1 \leq i \leq n} \in S_n \), so that \( S_n \) is equal to \( S \) up to the \( n \)th intervention and hence

\[
\left| V_S(x) - V_{S_n}(x) \right| = \mathbb{E}_x \left( \sum_{i=n+1}^{\infty} e^{-\delta \tau^*_i} u(Z_i^*) I_{\{\tau_i^* < s^*\}} \right).
\]

Because the sum converges when starting at \( i = 1 \) and since \( V \) is linearly bounded, we get that for \( n \to \infty \) the right-hand side converges to zero. Finally we have

\[
V(x) = \sup_{S \in S} V_S(x) = \sup_{S \in \bigcup_{n \in \mathbb{N}} S_n} V_S(x).
\]

For the fixed point property, just observe

\[
v_{n+1}(x) = \sup_{\theta \in T} \mathbb{E}_x \left( e^{-\delta \theta} M v_n(R^\theta) \right), \tag{9}
\]

\[
MV(x) = \sup_{y \in (a, x]} \{ u(y) + \sup_{n \in \mathbb{N}} v_n(x - y) \} \tag{10}
\]

\[
= \sup_{n \in \mathbb{N}} M v_n(x).
\]

Because \( v_n \leq v_{n+1} \), we have that \( M v_n \) is increasing in \( n \), and dominated convergence in (9) yields \( V(x) = MV(x) \). \( \square \)

**Remark 3.2.** Let \( w \) be a positive, absolutely continuous and linearly bounded function. Suppose \( w \) is another fixed point of \( M \). We have \( w \geq v_0 = 0 \) and assuming \( w \geq v_n \) we get by induction and the monotonicity of \( M \),

\[
w - v_{n+1} = M w - M v_n \geq 0.
\]

Therefore \( w \geq v_n \) holds for all \( n \in \mathbb{N} \) and consequently \( w \geq V \).

This theorem provides a characterization of the value function \( V \) defined by (1) as the smallest fixed point of an optimal stopping operator. The constructive nature of the proof shows that the optimal impulse control can be approximated by iterated solutions of optimal stopping problems.
4 The problem from an QVI point of view

We observed above, that at points $x \geq 0$, where it would be optimal to intervene, we should have $MV(x) - V(x) = 0$. On the other hand, if it would be optimal not to intervene in an open interval around a point $x$, conditioning on the first claim occurrence in a small time interval $[0, h]$ and letting $h \to 0$ should result in

$$LV(x) = cV'(x) + \lambda \left( \int_0^x V(x - y) \, dF_Y(y) - V(x) \right) - \delta V(x) = 0.$$  

These observations motivate heuristically the so-called quasi-variational-inequalities (QVI):

$$LV \leq 0,$$

$$MV - V \leq 0,$$

$$(LV)(MV - V) = 0,$$

or equivalently

$$\max \{LV, MV - V\} = 0. \quad (11)$$

This equation will allow for a better computability of $V$ than the characterization through iterated optimal stopping. But first we have to determine if and in which sense $V$ is a solution to (11).

**Proposition 4.1.** The value function $V$ is differentiable on $(0, a)$ and fulfills the QVI (11) a.e.

**Proof.** From the definition of $V$ in (1) we immediately get $MV(x) - V(x) \leq 0$. Further, Theorem 3.2 shows that

$$V(x) = \sup_{\theta \in \mathcal{P}} \mathbb{E}_x \left( e^{-\delta \theta} \sup_{y \in [a, R_\theta]} \{u(y) + V(R_\theta - y)\} \right).$$

Assume that around some point $x$ it is not optimal to intervene and $LV(x) > 0$. Because $V$ and $LV$ are continuous we can choose $r > 0$ and $\xi > 0$ such that in $B = (x - r, x + r)$ it is still not optimal to intervene and

$$V(x + r) \leq V(x) + \xi,$$

$$LV(y) > \delta \xi \quad y \in B.$$

Define $\tau^* = \{t > 0 \mid R_t \notin B\}$, and since we know that $V$ is in the domain of the generator of the uncontrolled reservoir (see for instance [19]) we can write

$$\mathbb{E}_x \left( e^{-\delta \tau^*} V(R_{\tau^*}) \right) = V(x) + \mathbb{E}_x \left( \int_0^{\tau^*} e^{-\delta s} LV(R_{s-}) \, ds \right).$$

Using the above assumptions we derive

$$\mathbb{E}_x \left( e^{-\delta \tau^*} V(R_{\tau^*}) \right) > V(x) + \xi \mathbb{E}_x \left( e^{-\delta \tau^*} \right)$$

and because of $R_{\tau^*} \leq x + r$ and the monotonicity of $V$ we get

$$\mathbb{E}_x \left( e^{-\delta \tau^*} (V(x) + \xi) \right) > V(x) + \xi \mathbb{E}_x \left( e^{-\delta \tau^*} \right).$$
From $P(\tau^* > 0) > 0$ we derive the contradiction

$$\mathbb{E}_x \left( e^{-\delta \tau^*} \right) > 1.$$ \hspace{1cm} (8)

We have proven that if at some point $x$ it is not optimal to intervene $\mathcal{L}V(x) \leq 0$ holds.

Now we want to determine the behaviour of $V$ on $(0, a)$ and indeed we can follow the classical proof of the fact that the probability of ruin is differentiable as long as $F_Y$ is continuous (see [19]).

Let $x \in (0, a)$ and choose some $h > 0$ small enough such that $x + ch < a$. Because of the definition of the set $S$ of admissible strategies, there will be no intervention before time $h$. We get, by conditioning on the first claim occurrence,

$$V(x) = e^{-(\delta + \lambda)h}V(x + ch) + \int_0^h \lambda e^{-(\delta + \lambda)t} \int_0^{x+ct} V(x + ct - y) dF_Y(y) dt,$$

which, after some manipulations, results in

$$c \frac{V(x + ch) - V(x)}{ch} = 1 - e^{-(\delta + \lambda)h} \frac{1}{h} V(x + ch) - \frac{1}{h} \int_0^h \lambda e^{-(\delta + \lambda)t} \int_0^{x+ct} V(x + ct - y) dF_Y(y) dt.$$

Because $V$ is continuous and linearly bounded, the limit $h \to 0$ of the right-hand side exists, therefore $V$ is differentiable from the right with

$$cV'(x+) = (\delta + \lambda)V(x) - \lambda \int_0^x V(x - y)dF_Y(y).$$

Let $h > 0$ such that $0 < x - ch$, then we can proceed as before to derive

$$cV'(x-) = (\delta + \lambda)V(x) - \lambda \int_0^{x-} V(x - y)dF_Y(y).$$

If $F_Y$ admits a continuous distribution function, we get that $V'$ exists on $(0, a)$ and that $\mathcal{L}V(x) = 0$ for $x \in (0, a)$.

From now on suppose that the density of $V$ is fixed and denoted by $V'$.

Again $V$, because is absolutely continuous and linearly bounded, is in the domain of the generator of the uncontrolled reserve. This has the consequence that for a bounded stopping time $\theta$ the following equality holds:

$$V(x) = \mathbb{E}_x \left( e^{-\delta \theta} V(R_{\theta}) \right. \left. \right.$$

$$\left. - \int_0^\theta e^{-\delta s} \left[ cV'(R_{s-}) - (\delta + \lambda)V(R_{s-}) + \lambda \int_0^{R_{s-}} V(R_{s-} - y) dF_Y(y) \right] ds \right).$$

Combining this expression for $V(x)$ with its fixed point property (7), we get

$$0 = \sup_{\theta \in T} \mathbb{E}_x \left( e^{-\delta \theta} (MV(R_{\theta}) - V(R_{\theta})) \right. \left. \right.$$

$$\left. + \int_0^\theta e^{-\delta s} \left[ cV'(R_{s-}) - (\delta + \lambda)V(R_{s-}) + \lambda \int_0^{R_{s-}} V(R_{s-} - y) dF_Y(y) \right] ds \right). \hspace{1cm} (12)$$

Of course the first summand in the above equation is smaller or equal to zero (at optimal intervention times it is optimal to jump). Further we know from above that $\mathcal{L}V \leq 0$. Therefore we obtain - in the case where immediate stopping is not optimal - that $\mathcal{L}V = 0$ a.e. $x > 0$, such
that the second summand is equal to zero.
Summarizing, at points where it is optimal to pay out optimal lump sums (at optimal stopping times) the relation \( V(x) = MV(x) \) holds. Between such actions (after the \( i \)th intervention, the fixed point argument can be applied again starting at \( R_{\tau_i}^{S_i} \)), the process follows the dynamics of the uncontrolled reserve and because of the above observations \( \mathcal{L}V(x) = 0 \) is fulfilled a.e. Therefore \( V \) fulfills (11) a.e.. □

For the moment, let \( g \) be an increasing, absolutely continuous and linearly bounded solution to the QVI (11), as \( V \) is. We can define in an abstract way the following admissible impulse control:

**Definition 4.1** (QVI control). The strategy \( S^g \in \mathcal{S} \) defined by

\[
\tau_i^g = \inf \left\{ t > 0 \mid Mg(R_t^{S_i^g}) = g(R_t^{S_i^g}) \right\},
\]

\[
Z_i^g = \arg\max \left\{ u(z) + g(R_{\tau_{i-1}^g}^{S_i^g} - z) \mid z \in (a, R_{\tau_{i-1}^g}^{S_i^g}) \right\}
\]

and for \( n \geq 2 \)

\[
\tau_n^g = \inf \left\{ t > \tau_{n-1}^g \mid Mg(R_t^{S_i^g}) = g(R_t^{S_i^g}) \right\},
\]

\[
Z_n^g = \arg\max \left\{ u(z) + g(R_{\tau_{n-1}^g}^{S_i^g} - z) \mid z \in (a, R_{\tau_{n-1}^g}^{S_i^g}) \right\}.
\]

is called QVI control.

**Remark 4.1.** We see from the definition of the QVI control that the \( i \)th intervention takes place at time \( \tau_i^g \) which only takes the knowledge of \( R_{\tau_i}^{S_i} \) into account (an intervention has to be decided before observing a possible claim at the same time). Note that \( g \) and \( Mg \) are continuous and \( R^{S_i} \) has càdlàg paths between interventions. Because of the relation to the optimal stopping problem given by (4), we have that \( \tau_{i-1} < \tau_i \), as otherwise the decision at \( \tau_{i-1} \) cannot be optimal (this relation is pointed out in the proof of the following proposition). Finally, the prescribed choice of \( Z_i^g > a \) (a maximizer exists because \( u \) and \( V \) are continuous) and the fact that at most \( (x + \lambda T)/a \) interventions in the time interval \([0, T]\) can be done, ensure the admissibility of such a strategy \( S^g \).

**Proposition 4.2.** The strategy \( S^V \) is optimal and \( V \geq V_S \) for all \( S \in \mathcal{S} \). Further, we have that \( V \) is the smallest increasing, absolutely continuous and linearly bounded solution to the QVI (11).

**Proof.** Let \( R^S = (R_t^S)_{t \geq 0} \) denote the free reserve controlled by a strategy \( S = \{(\tau_i, Z_i)_{i \in \mathbb{N}}\} \in \mathcal{S} \) and let \( \tau^S \) be the associated time of ruin. By following the dynamics of the uncontrolled reserve between interventions and taking the dividend payments into account, we get the following

\[
\mathbb{E}_x \left( e^{-\delta(t \wedge \tau^S)}V(R_{t \wedge \tau^S}^S) \right) = V(x)
\]

\[
+ \mathbb{E}_x \left( \sum_{i=1}^{\infty} I_{t_i < (t \wedge \tau^S)} \int_{\tau_{i-1}}^{\tau_i} e^{-\delta s} \left( cV'(R_{s-}^S) - (\delta + \lambda)V(R_{s-}^S) + \lambda \int_0^{R_{s-}^S} V(R_{s-}^S - y) dF_Y(y) \right) ds \right)
\]

\[
+ \mathbb{E}_x \left( \sum_{i=1}^{\infty} e^{-\delta \tau_i} \left( V(R_{\tau_i}^S - Z_i) - V(R_{\tau_i}^S) \right) I_{t_i < (t \wedge \tau^S)} \right). \quad (13)
\]

Because \( V \) solves the QVI (11) \((\mathcal{L}V \leq 0)\) and for a general strategy \( S \) we have \( V(R_{\tau_i}^S) \geq u(Z_i) + V(R_{\tau_i}^S - Z_i) \), we arrive at

\[
\mathbb{E}_x \left( e^{-\delta(t \wedge \tau^S)}V(R_{t \wedge \tau^S}^S) \right) + \mathbb{E}_x \left( \sum_{i=1}^{\infty} e^{-\delta \tau_i} u(Z_i) I_{t_i < (t \wedge \tau^S)} \right) \leq V(x).
\]
Letting $t \to \infty$ and using the linear boundedness of $V$, we get by bounded and monotone convergence

$$V_S(x) \leq V(x).$$

Looking at the strategy $S^V$, we get $V(R_{\tau_i}^S) = u(Z_{\tau_i}^V) + V(R_{\tau_i}^S - Z_{\tau_i}^V)$ and for $s \in (\tau_{i-1}^V, \tau_i^V)$ we have

$$\mathcal{L}V(R_{s-}^S) = 0.$$

From (13) we obtain as above that $V_S(x) = V(x)$ which proves the optimality of this strategy. Similarly, as done for $V$ in the derivation of (12), one can obtain for every fixed point $g$ of the operator $\mathcal{M}$ (4), which is dominating $V$, that it is a solution to (11). On the other hand, we have to prove that a solution to (11) is a fixed point of (4) and therefore dominates $V$.

From now on let $g$ be an increasing, absolutely continuous and linearly bounded solution to the QVI (11) and $S^g$ the associated QVI control. We observe that for $s < \tau_i^g$ the reserve controlled by $S^g$ follows the dynamics of the uncontrolled reserve and that $g$ is in the domain of its generator,

$$g(x) = \mathbb{E}_x \left( e^{-\delta \tau_i^g} g(R_{\tau_i^g}) - \int_0^{\tau_i^g} e^{-\delta s} \mathcal{L} g(R_{s-}) ds \right). \quad (14)$$

We have by construction of $S^g$ that the integral part of the right hand side is equal to zero. Now, in view of (4),

$$\mathcal{M}g(x) = \sup_{\theta \in T} \mathbb{E}_x \left( e^{-\delta \theta} M g(R_{\theta}) \right).$$

Observe from (11) that in general $M g(R_{\theta}) \leq g(R_{\theta})$. But for $\theta = \tau_i^g$ we have $M g(R_{\tau_i^g}) = g(R_{\tau_i^g})$.

Therefore we conclude from (14) that

$$g(x) = \mathbb{E}_x \left( e^{-\delta \tau_i^g} M g(R_{\tau_i^g}) \right) = \sup_{\theta \in T} \mathbb{E}_x \left( e^{-\delta \theta} M g(R_{\theta}) \right)$$

holds, i.e. $g$ is a fixed point of $\mathcal{M}$, and therefore dominates $V$. We have proven that $V$ is the smallest solution to the QVI (11) in the set of increasing, absolutely continuous and linearly bounded functions. \hfill \Box

5 Computedations

5.1 Strategies of the form $b = \{b_1, b_2\}$

From other results on impulse control problems for diffusion processes (e.g. [4], [10] and [17]) one knows that strategies of the following type could be optimal: below a certain level $b_2 > a$ the process is not controlled, when hitting this level (from below) a fixed amount is paid out as dividend. This fixed amount can be represented as another level $0 \leq b_1 < b_2 - a$ such that the payment is of size $b_2 - b_1$. Let the value of such a strategy $b = \{b_1, b_2\}$ be $V_b(x)$. We get for $x < b_2$ by the Markov property and by the continuously increasing paths of the uncontrolled reserve process that

$$V_b(x) = \mathbb{E}_x \left( e^{-\delta \theta(b_2)} \right) V_b(b_2),$$
where \( \theta(w) = \inf\{ t > 0 \mid R_t = w \text{ and } R_s \geq 0 \text{ for } 0 \leq s \leq t \} \). From e.g. [20] we know that \( f(x) = \mathbb{E}_x (e^{-\delta \theta(b_2)}) \) is the unique solution to \( \mathcal{L}f = 0 \) and \( f(b_2) = 1 \). Using \( V_b(b_2) = u(b_2 - b_1) + \gamma V_1(b_1) \) we get

\[
V_b(b_2) = \frac{u(b_2 - b_1)}{1 - f(b_1)}.
\]

It remains to deal with the case \( x > b_2 \). In the related literature usually the continuation \( u(x - b_1) + \gamma V_1(b_1) \) is chosen. But in our utility framework, and from an optimization point of view (and some numerical experiments), the choice \( \sup_{\{x - b_2 < x < a\}} u(z) + V_b(x - z) \) seems to be more appropriate. Fixing \( b \) we have

\[
V_b(x) = \begin{cases} 
  f(x) \frac{u(b_2 - b_1)}{1 - f(b_1)}, & x \leq b_2, \\
  \sup_{\{x - b_2 < x < a\}} u(z) + f(x - z) \frac{u(b_2 - b_1)}{1 - f(b_1)}, & x > b_2.
\end{cases}
\]

In [15], Loeffen derives similar expressions for values of such strategies, for the case \( \gamma = 1 \), in terms of scale functions of spectrally negative Lévy processes. In order to find a maximizing strategy within this restricted class, he proposes to maximize \( \frac{u(b_2 - b_1)}{1 - f(b_1)} \) over admissible pairs \( b = \{b_1, b_2\} \). Furthermore he can state that if the Lévy measure has a log-convex density, then there exists a strategy \( b^* = \{b_1^*, b_2^*\} \) which is optimal in the set of all admissible strategies.

In the case of a mean-reverting diffusion risk reserve process and \( \gamma \in (0, 1) \), additional strong assumptions for the optimality of a \( b = \{b_1, b_2\} \) strategy are needed, see [4]. Paulsen [17] proves for a general diffusion risk reserve model that if an optimal strategy exists, it is of the \( b = \{b_1, b_2\} \) type.

For exponentially distributed claim amounts it is possible to compute \( V_b(x) \) for some \( b = \{b_1, b_2\} \), up to the supremum continuation, explicitly. Because of the results from [15] we know that for \( \gamma = 1 \) and exponentially distributed claim amounts a certain \( b = \{b_1, b_2\} \) strategy is the optimal one. For \( 0 < \gamma < 1 \) such a result is not proven up to now, but one expects that the general methods used in [15] should also be applicable for this case.

We get for the following set of parameters \( c = 2.5, \lambda = 1, \delta = 0.03, k = 0.99, K = 0.1, \gamma = 0.7 \) and \( Exp(2) \) distributed claims that the strategy given by \( b^* = (4.61, 4.95) \) is indeed optimal (it is possible to verify that \( V_{b^*} \) fulfills the QVI (11)).

Figure 1 shows the value function \( V_{b^*} \) for these parameters and Figure 2 shows the difference of the possible continuations from \( b_2^* \) onwards. One observes that the supremum continuation improves the common one.
5.2 A numerical scheme - policy iteration

For getting (approximate) solutions to the optimization problem for claim amount distributions beyond the exponential distribution, we implemented a policy iteration algorithm (also known as Howard algorithm) for this specific problem. The idea is mainly based on Rogers’s Optimal Investment notes [18] and Chancelier et al. [5].

At first we need to fix a discretization width \( h > 0 \) and some \( N \in \mathbb{N} \) for approximating \( V \) on the interval \([0, hN]\). Note that a strategy divides the set \( \{0, h, \ldots, Nh\} \) into two parts \( A \) (points at which no dividends are paid) and \( B \) (points at which a certain dividend is paid). Let us denote \( B \) as active set and \( A \) as non-active set. The starting point of the algorithm is an admissible strategy \( S_0 \in \mathcal{S} \) (with associated \( A_0 \) and \( B_0 \)) and its associate value \( v_0 \), which we can in the best case compute explicitly (a simple type of strategy as introduced above seems to be appropriate in most cases). If

\[
\max \{ \mathcal{L}v_0, Mv_0 - v_0 \} = 0
\]

holds, one is finished. If not, we go through each point \( ih \in A_0, \) \( 0 \leq i \leq N \) and check if \( v_0(ih) \leq Mv_0(ih) \) holds. If this is the case, this point is moved to \( B_1 \) and in the end we set \( A_1 = \{0, h, \ldots, hN\} \setminus B_1 \). In the following step we compute \( v_1 \) as a solution to

\[
cf'(x) - (\delta + \lambda)v_0(x) + \lambda \int_0^x f(x - y) dF_Y(y) = 0, \tag{15}
\]

for \( x \in A \) and set \( v_1(x) = Mv_0(x) \) for \( x \in B_1 \) (note that as suggested by Rogers for solving a classical control problem, using \( -(\delta + \lambda)v_0(x) \) instead of \( -(\delta + \lambda)f(x) \) in (15) makes the procedure for solving the integro-differential equation more stable). This algorithm gives us an admissible strategy \( S_1 \), taking the local maximizers from \( Mv_0 \) and its value \( v_1 \) into account. Iterating this procedure (maybe up to a number \( n \) of iterations such that \( \sup_{x \in \{0, h, \ldots, hN\}} (v_n(x) - v_{n-1}(x)) \) is smaller than a specified level) gives us an approximation of \( V \) and some sort of recommendable admissible strategy. For verifying this procedure one can plug the constructed solution into the QVI (11) for getting some sort of numerical verification, but notice that because of the numerical local maximizers and the non-local structure of the first part of the QVI, this can become a vague statement.

Of course it will be hard, or even impossible to determine the optimal strategy exactly, but one will at least get a good guess of the structure of the optimal policy and an approximation for \( V \). The constructed sequence of values of admissible strategies is increasing and bounded by the value function of the optimization problem, which itself is linearly bounded, and therefore converges. Because at every step an admissible strategy is constructed, the procedure converges (formally) to the right (the smallest) solution of (11).

By these means we are able to demonstrate that as in the classical dividend maximization problem for the compound Poisson model (see [1] or [20]), it is possible that the optimal strategy is of a band structure, saying that \( B \) and \( A \) consist of several non-connected intervals. Figure 3 and 4 show the value function (full line: policy iterated one, dotted line: best \( b = \{b_1, b_2\} \) strategy) and its associated band type strategy for the following set of parameters: \( c = 21.4, \lambda = 10, \delta = 0.1, k = 0.95, K = 0.05, \gamma = 0.9 \) and Erlang(2,1) distributed claim amounts. For the linear utility case this parameter set turns out to admit no \( b = \{b_1, b_2\} \) type optimal strategy, see [15].

In Figure 4 positive values mark the action set whereas zero values mark the non action set. The depicted heights of the action points are the heights of the (approximative) optimal dividend payments.
Figure 3: Value functions of iterated and best simple strategy

Figure 4: Strategy obtained by iteration
References


