DIVIDENDS: FROM REFRACTING TO RATCHETING

HANSJÖRG ALBRECHER†, NICOLE BÄUERLE*, AND MARTIN BLADT△

Abstract. In this paper we consider an alternative dividend payment strategy in risk theory, where the dividend rate can never decrease. This addresses a concern that has often been raised in connection with the practical relevance of optimal classical dividend payment strategies of barrier and threshold type. We study the case where once during the lifetime of the risk process the dividend rate can be increased and derive corresponding formulae for the resulting expected discounted dividend payments until ruin. We first consider a general spectrally-negative Lévy risk model, and then refine the analysis for a diffusion approximation and a compound Poisson risk model. It is shown that for the diffusion approximation the optimal barrier for the ratcheting strategy is characterized by an unexpected relation to the case of refracted dividend payments. Finally, numerical illustrations for the diffusion case indicate that with such a simple ratcheting dividend strategy the expected value of discounted dividends can already get quite close to the respective value of the refracted dividend strategy, the latter being known to be optimal among all admissible dividend strategies.

1. Introduction

Starting with de Finetti’s work [12], the study of optimal dividend payout strategies in collective risk theory has been a very active field of research over the last 60 years. It is nowadays well-known that in order to maximize the expected aggregate discounted dividends until ruin, it is optimal to pay dividends according to a band strategy, which in a number of cases collapses to a barrier strategy, see e.g. [14], [25], [24] and [9]. When this optimal control problem is considered with an upper bound on the dividend rate, then in many cases a refracting dividend strategy (or, synonymously, a threshold strategy) is optimal (i.e., no dividend payments up to a certain barrier level, and dividend payments at maximal allowed rate above that barrier), see for instance [19], [5] for diffusions and [15] and [22] for the compound Poisson process. Since then numerous extensions and variations of the dividend problem have been considered, many of which leading to intricate and interesting mathematical problems, see e.g. [3] and [7] for surveys on the topic. Among these variations, some also address the issue that the theoretically optimal re-fraction strategy may not be realistic when it comes to implementation in practice. One constraint is that dividends can only be paid out in a discrete rather than a continuous-time fashion, see [1] for a contribution in this direction for random discrete payment times. Another problem with the threshold strategy is the strong variability of payment patterns across time. In [8] it was proposed for a diffusion process to consider dividend payments that are proportional to the current surplus level, which leads to much smoother dividend streams. Recently, the respective analysis of performance measures was extended to the compound Poisson model in [2]. In [10] and [11] the aim was to maximize risk-sensitive dividend payments in discrete time which, in case of exponential utility, results in optimizing a weighted criterion of expectation and variance of the dividends.

In this paper we would like to consider another constraint that is often mentioned in discussions with practitioners. Concretely, it is at times considered desirable to have a dividend payment stream that does not decrease over time (which we will refer to as
a ratcheting strategy, see e.g. [13]). One reason is that a decrease may have a negative psychological impact on shareholders and the considered value of the company in general. Sticking to the analytically much more tractable situation of a continuous-time model, the over-all mathematical question could then be to find the optimal dividend payment pattern with a non-decreasing dividend rate. This general optimal control problem is a considerable challenge, as one immediately is put into a two-dimensional setting with one variable keeping track of the current dividend rate, so that the usually advantageous Markovian structure of the surplus process is lost even for simple processes. We hence in this paper will consider only a first step towards improving the understanding of this general problem, namely to allow for a constant dividend rate throughout the lifetime of the risk process, which once during the lifetime can be increased to a higher level. The task then is to find the optimal rates before and after that change, and the surplus level at which this change should take place, so that the expected over-all discounted dividend payments are maximized. The only constraint in this context will be that the dividend rate must never exceed the drift rate of the original process. This setting leads to quite explicit results and will give some insight into the nature of the problem. A particular focus will be given on the comparison of the resulting optimal ratcheting strategy with the best refracting strategy, which is known to be optimal among all admissible dividend strategies in the absence of such a ratcheting constraint.

We will first deal with the general case of spectrally-negative Lévy processes in Section 2. After collecting some necessary preliminaries, we adapt an existing formula for refracting strategies that also allows for dividend payments below the barrier. We then derive a general formula for the expected discounted dividend payments until ruin according to a ratcheting strategy in terms of the scale function of the underlying Lévy process, and derive a criterion for the optimal ratcheting barrier. Subsequently, in Section 3 we refine the analysis for the case of a pure Brownian motion with drift (diffusion approximation). While many of the respective formulas can in fact be obtained from Section 2 by substituting the simple form of the respective scale function, we derive a number of formulas in that section in a self-contained, sometimes more direct, way. This adds another perspective to the problems under study, and also allows to read parts of that section independently of the general Lévy fluctuation theory that underlies the approach in Section 2. We then give some numerical illustrations on the performance of both the refracting and ratcheting strategy, which somewhat surprisingly indicate that the best ratcheting procedure is not far behind the best refracting (and hence over-all best) dividend strategy. We also derive a somewhat surprising criterion for the optimal ratcheting barrier in terms of a matching with a refracting strategy. In Section 4 we then proceed to work out the formulas of Section 2 to the particular case of a Cramér-Lundberg risk model with hyper-exponential claim sizes. We also show that the optimality criterion for the diffusion case mentioned above no longer holds in the compound Poisson setting, the reason being the non-differentiability of the refraction value function at the barrier level. One natural concern for the implementation of a ratcheting strategy of the above kind is that after the switching, the higher dividend rate will always stay, even when the surplus level gets low. Correspondingly one may expect that ruin is more likely or happens earlier, particularly when the optimal barrier level is chosen according to the profitability criterion of maximal expected dividend payments. Also, the optimal racheting barrier is in general higher than the optimal refraction barrier. In Section 5 we discuss this issue further and quantify it in terms of expected ruin time, given that ruin occurs. The numerical results in fact indicate that, conditional on ruin to occur in finite time, the
expected time to ruin is larger for the ratcheting strategy. Finally, Section 6 contains some conclusions.

2. The Spectrally-Negative Lévy Risk Model

Let us consider a spectrally-negative Lévy process $Y = \{Y_t\}_{t \geq 0}$, i.e. a Lévy process with only negative jumps, and which is not a.s. a non-increasing process. We assume that $Y_0 = x \geq 0$ and the drift of this process is positive, and call such a $Y$ a Lévy risk process. Our focus in this paper will be on risk processes constantly paying out dividends at rate $c_1 \geq 0$ (so possibly $c_1 = 0$), and in certain periods (to be specified) at an increased rate $c_1 + c_2$, with $c_2 > 0$. Correspondingly, the Lévy processes of interest are

$$X_t = Y_t - c_1 t, \quad \tilde{X}_t = Y_t - (c_1 + c_2) t.$$

for all $t \geq 0$. Define the Laplace exponent

$$\psi(\theta) := \log \mathbb{E} e^{\theta X_1},$$

which is finite for at least all $\theta \geq 0$, and denote by $\Phi(\delta)$ the largest root of the equation $\psi(\theta) = \delta$. The $\delta$-scale functions $W(x)$ and $Z(x)$ of the process $X$ are defined for any $\delta \geq 0$ as

$$\int_0^\infty e^{-ux} W(x) \, dx = \frac{1}{\psi(u) - \delta}, \quad u > \Phi(\delta),$$

$$Z(x) = 1 + \delta \int_0^x W(y) dy.$$

The respective $\delta$-scale functions for the process $\tilde{X}$ will be denoted by $\mathcal{W}(x)$ and $\mathcal{Z}(x)$, respectively, and $\phi(\delta)$ shall be the corresponding largest root of $\psi(\theta) - c_2 \theta = \delta$. Define now for a fixed $b \geq 0$ the first passage times

$$\tau^-_0 := \inf\{t \geq 0 : X_t < 0\}, \quad \tau^+_b := \inf\{t \geq 0 : X_t > b\}.$$

We will be working on a canonical probability space for all the processes involved, consisting of the space of all right-continuous functions with left-sided limits, with a probability law denoted by $\mathbb{P}_x$, and associated conditional expectation $\mathbb{E}_x$, given that the process starts at $x \geq 0$. It is well-known from Lévy fluctuation theory that one has

$$(1) \quad \mathbb{E}_x \left( e^{-\delta \tau^+_b} ; \tau^+_b < \tau^-_0 \right) = \frac{W(x)}{W(b)},$$

and

$$(2) \quad \mathbb{E}_x \left( e^{-\delta \tau^-_0} ; \tau^-_0 < \tau^+_b \right) = Z(x) - Z(b) \frac{W(x)}{W(b)},$$

for any $0 \leq x \leq b$, see for instance [20, Ch.8].

2.1. Refracting Strategy. For reasons of comparison, let us now first recollect a formula from [21] for the expected sum of discounted dividends until ruin under a refraction strategy. For any threshold level $b \geq 0$, the respective modified Lévy risk process is given by

$$U_t = X_t - c_2 \int_0^t 1_{\{U_s > b\}} \, ds, \quad t \geq 0.$$

The interpretation is that dividends are paid at rate $c_1$ while the original process $Y$ is below the threshold $b$, and at rate $c_1 + c_2$ above the threshold. Note that the existence
of the process (3) is in fact not as straightforward as one may expect, and one has to require
\[ c_2 \in \left( 0, \gamma + \int_{(0,1)} x \Pi(dx) \right), \]
whenever \( X \) has paths of bounded variation, where \( \gamma \) is the canonical drift coefficient in the Lévy-Khintchine representation of \( X \), and \( \Pi \) is the corresponding Lévy measure, see Theorem 1 of [21] for details.

Define
\[ \tau = \inf \{ t > 0 : U_t < 0 \} \]
as the time of ruin of the process \( U \) defined in (3). A slight adaptation of Eq. (10.25) of [21] (which corresponds to the case \( c_1 = 0 \)) then leads to a formula for the expected sum of discounted dividends until ruin under the threshold strategy (3), for general \( x, b \geq 0 \):
\[
V(x, c_1, c_2, b) := \mathbb{E}_x \left[ \int_0^\tau e^{-\delta s} c_2 1_{\{M_s \geq b\}} ds \right] + c_1 \mathbb{E}_x \left[ \int_0^\tau e^{-\delta s} ds \right]
\]
\[
= \frac{c_2}{\delta} (1 - Z(x - b)) + \frac{W(x) + c_2 \int_b^x W(x - y) W'(y) dy}{\phi(\delta) \int_0^\infty e^{-\phi(\delta)y} W'(y + b) dy}
\]
\[
+ \frac{c_1}{\delta} \left[ 1 - Z(x) - \delta c_2 \int_b^x W(x - y) W(y) dy \right]
\]
\[
+ \frac{c_1}{\delta} \left[ \frac{W(x) + c_2 \int_b^x W(x - y) W'(y) dy}{e^{-\phi(\delta)b} \int_0^\infty e^{-\phi(\delta)y} W'(y + b) dy} \right] \delta \int_b^\infty e^{-\phi(\delta)y} W(y) dy.
\]
This is a somewhat involved, but completely explicit expression for \( V(x, c_1, c_2, b) \), which can be evaluated whenever the scale function of the underlying Lévy process is available.

2.2. Ratchet Strategy. We now turn to the study of the following ratcheting strategy: Dividends are paid at a fixed constant rate \( c_1 \geq 0 \) until the first time the surplus process hits a barrier \( b \), at which point the dividend rate is increased (ratcheted) to \( c_1 + c_2 \) for a fixed constant \( c_2 > 0 \), and stays at this higher level until the time of ruin. The modified Lévy risk process under this ratcheting strategy is then given by
\[
U_t^R = Y_t - \int_0^t (c_1 + c_2 1_{\{M_s \geq b\}}) ds = X_t - c_2 \int_0^t 1_{\{M_s \geq b\}} ds,
\]
where \( M_t = \sup_{0 \leq s \leq t} Y_s \). In contrast to the refracting case, the existence of \( U_t^R \) is straightforward for any \( c_2 > 0 \). Such a ratcheting strategy takes into account the fact that shareholders prefer to not experience a decrease in the rate of their dividend stream. This strategy is no longer Markovian, but depends on the history of the process. Define by
\[ \tau^R = \inf \{ t > 0 : U_t^R < 0 \} \]
the time of ruin and by
\[
V^R(x, c_1, c_2, b) = \mathbb{E}_x \left[ \int_0^{\tau^R} e^{-\delta s} (c_1 + c_2 1_{\{M_s \geq b\}}) ds \right]
\]
the expected value of the aggregate discounted dividend payments under such a ratcheting strategy.
\textbf{Theorem 2.1.} The expected value of the aggregate discounted dividend payments until ruin under a ratcheting strategy for a Lévy risk model is given by

\begin{equation}
V^R(x, c_1, c_2, b) = \begin{cases}
\frac{c_1 + c_2}{\delta} [1 - Z(x) + \delta \phi(b) W(x)], & 0 \leq b \leq x, \\
\frac{c_1 + c_2}{\delta} [1 - Z(b) + \delta \phi(b) W(b)] \frac{W(x)}{W(b)}, & 0 \leq x < b.
\end{cases}
\end{equation}

\textit{Proof.} Consider first the case $x \geq b$. Then the higher dividend rate $c_1 + c_2$ is paid out on from the beginning until ruin, i.e.

\[ V^R(x, c_1, c_2, b) = \mathbb{E}_x \left[ \int_0^{\tau^-} e^{-\delta s} (c_1 + c_2) ds \right] = \frac{c_1 + c_2}{\delta} [1 - \mathbb{E}_x (e^{-\delta \tau^-})], \]

where

\[ \tau^- = \inf \{ t \geq 0 : \tilde{X}_t < 0 \} \]

is the ruin time of the risk process when the original drift is reduced by $c_1 + c_2$. But then the result follows from (2) and the fact that $\lim_{b \to \infty} Z(b)/W(b) = \delta \cdot \lim_{b \to \infty} W(b)/W(b) = \delta/\phi(b)$.

For $x < b$, we have to distinguish whether the process will reach $b$ before ruin or not, and in the former case we apply the strong Markov property at that point in time, on from which the process dynamics change to the drift being reduced by $c_1 + c_2$. We thus have

\[ V^R(x, c_1, c_2, b) = c_1 \mathbb{E}_x \left[ \int_0^{\tau_0^+ \wedge \tau_R} e^{-\delta s} ds \right] + \mathbb{E}_x (e^{-\delta \tau_0^+}; \tau_0^+ < \tau^-) \cdot V^R(b, c_1, c_2, b), \]

and $V^R(b, c_1, c_2, b)$ is given above. The result then follows from

\[ \delta \mathbb{E}_x \left[ \int_0^{\tau_0^+ \wedge \tau_R} e^{-\delta s} ds \right] = 1 - \mathbb{E}_x \left[ e^{-\delta (\tau_0^+ \wedge \tau_R)} \right] = 1 - \mathbb{E}_x \left[ e^{-\delta \tau_0^+}; \tau_0^+ < \tau^- \right] - \mathbb{E}_x \left[ e^{-\delta \tau_0^-}; \tau_0^- < \tau_0^+ \right], \]

and again using (1) and (2). \hfill \Box

It is interesting to try to identify the barrier level $b$, for which $V^R(x, c_1, c_2, b)$ is maximized. The natural criterion for that purpose is to look for a solution of

\begin{equation}
\frac{\partial V^R(x, c_1, c_2, b)}{\partial b} = 0
\end{equation}

One should keep in mind, however, that the necessity and sufficiency of this criterion depends on the analytical properties of $V^R$, which are inherited from the scale function $W$. Correspondingly, it is not possible to characterize such a barrier level in full generality, but for most cases of practical interest the scale function structure is such that the above derivative condition for the optimal barrier is the relevant one (see also [23]). Also, if the optimal barrier is positive, it represents a necessary condition. For simplicity we will hence in the following refer to a barrier that fulfills (8) as optimal. Theorem 2.1 can now be used to derive a criterion for a barrier level $b$ to be optimal for the ratcheting strategy:

\textbf{Proposition 2.2. (Optimal barrier)} For fixed $c_1, c_2$, the barrier $b$ that maximizes (7) does not depend on $x$ and is characterized by the equation

\begin{equation}
\frac{d}{db} \left( \frac{W(b)}{W(b)} \right) = 0,
\end{equation}

\[ \frac{d}{db} \left( \frac{W(b)}{W(b)} \right) = 0, \]
where
\[ W(x) = \frac{c_1 Z(x) + c_2}{c_1 + c_2} - Z(x) + \frac{\delta}{\phi(\delta)} W(x). \]

**Proof.** By the nature of the ratcheting strategy, for every initial capital \( x \), a barrier level \( b < x \) is equivalent to a barrier level \( b = x \), so that we can w.l.o.g. consider the case \( 0 \leq x \leq b \) only. Differentiation of expression (7) with respect to \( b \) and equating it to zero shows that all terms depending on \( x \) disappear and that

\[ \frac{W(b)}{W'(b)} = \frac{W(b)}{W''(b)} \]

or equivalently Equation (9) must hold. □

One can easily see from (7) that the function \( V^R(x, c_1, c_2, b) \) is continuous at \( x = b \). On the other hand, the derivative of \( V^R \) with respect to \( x \) does not have to be continuous in that point. The following result shows that this derivative is, however, continuous in the optimal barrier level. There is hence an alternative way to identify the optimal barrier:

**Theorem 2.3.** (Smooth pasting) In the ratcheting dividend problem, the optimal barrier \( b^R \) is exactly the one which makes the value function continuously differentiable.

**Proof.** Taking the derivative of \( V^R \) of (6) with respect to \( x \) on both sides of the barrier \( b \), evaluating at \( b \) and equating both expressions yields after some algebra precisely the criterion (9). □

In the next sections we will now refine the analysis for the case of Brownian motion and for a compound Poisson process with hyper-exponential jumps.

3. **BROWNIAN APPROXIMATION**

Consider now a risk process

\[ Y_t = x + \mu t + \sigma B_t, \quad t \geq 0, \]

where \( \mu > 0 \) is a constant drift, \( \sigma > 0 \) and \( (B_t)_{t \geq 0} \) denotes standard Brownian motion. Clearly, this is a special case of the Lévy risk model considered in Section 2, and by substituting the corresponding scale function (which in this case is a linear combination of two exponential terms), one can retrieve the respective formulas for the refracting and ratcheting strategies in this more particular setting.

We prefer, however, to give here a self-contained, more direct derivation for the diffusion case, and then use the resulting formulas for a more detailed analysis of the performance of the ratcheting dividend strategy. In particular, we will also establish an unexpected connection between the refracted and the ratcheting case which does not hold for general Lévy risk processes.

3.1. **Refracting Strategy.** Consider a diffusion risk reserve process with continuous dividend payout

\[ dU_t = (\mu - c_t)dt + \sigma dB_t, \]

where dividends are paid at rate

\[ c_t = c_1 1_{\{U_t \leq b\}} + (c_1 + c_2) 1_{\{U_t > b\}} \]

with \( b, c_1, c_2 \geq 0 \). I.e., whenever the risk reserve is above level \( b \) we pay at rate \( c_1 + c_2 > 0 \), otherwise only at rate \( c_1 \). Thus the payment rate changes at \( b \) which could be physically
understood as a refraction. We first derive the corresponding value of this strategy, measured in terms of the expected aggregate discounted dividend payments

\[ V(x, c_1, c_2, b) := \mathbb{E}_x \left[ \int_0^\tau e^{-\delta s} c_s ds \right], \]

where as before

\[ \tau := \inf \{ t \geq 0 : X_t = 0 \}, \]

\( \delta \geq 0 \) is a discount rate and \( \mathbb{E}_x \) is the conditional expectation given that \( X_0 = x \). Here \( X_t = Y_t - c_1 t \). For \( c_1 = 0 \), a formula for \( V \) is well-known, see e.g. [16]. In the following we establish the corresponding extension for \( c_1 > 0 \).

Denote by \( \theta_1 > 0 > \theta_2 \) the roots of

\[
\frac{1}{2} z^2 + (\mu - c_1) z - \delta = 0, \tag{12}
\]

and by \( \tilde{\theta}_1 > 0 > \tilde{\theta}_2 \) the roots of

\[
\frac{1}{2} z^2 + (\mu - c_1 - c_2) z - \delta = 0. \tag{13}
\]

Moreover, let \( \kappa := ((\mu - c_1)^2 + 2\sigma^2\delta)^{-\frac{1}{2}} \) and

\[
W(x) := \kappa (e^{\theta_1 x} - e^{\theta_2 x}), \quad x \geq 0.
\]

Note that \( W(x) \) is the scale function of the process \((X_t)_{t \geq 0}\).

**Theorem 3.1.** The value function under the fixed threshold strategy (11) in the diffusion case is given by

\[
V(x, c_1, c_2, b) = \begin{cases} 
B \cdot W(x) + \frac{c_1}{\delta} (1 - e^{\theta_2 x}), & 0 \leq x \leq b \\
\frac{c_1 + c_2}{\delta} + De^{\tilde{\theta}_2 x}, & x \geq b 
\end{cases}
\]

where

\[
B := \frac{1}{\delta} \cdot \frac{c_1 e^{\theta_2 b} (\theta_2 - \tilde{\theta}_2) - c_2 \tilde{\theta}_2}{W'(b) - \tilde{\theta}_2 W(b)}, \tag{14}
\]

\[
D := Be^{-\tilde{\theta}_2 b} W(b) - \frac{c_1}{\delta} e^{(\theta_2 - \tilde{\theta}_2) b} - \frac{c_2}{\delta} e^{-\tilde{\theta}_2 b}. \tag{15}
\]

**Proof.** In what follows we write \( V(x) \) instead of \( V(x, c_1, c_2, b) \) since \( c_1, c_2, b \) are fixed. First note that we can decompose the value function into

\[
V(x) = c_1 \mathbb{E}_x \left[ \int_0^\tau e^{-\delta s} ds \right] + V(x, \mu - c_1),
\]

where \( V(x, \mu - c_1) \) is the value function of the expected discounted dividends of a process \( X_t \) with drift \( \mu - c_1 \) where nothing is paid below the barrier \( b \) and above \( b \) the rate \( c_2 \) is paid. This is the usual refracting case (see e.g. [17]). From this observation it follows that \( V \in C^1 \), i.e. \( V \) is continuously differentiable.

Now, since the process

\[
\int_0^{t \wedge \tau} e^{-\delta s} c_s ds + e^{-\delta (t \wedge \tau)} V(X_{t \wedge \tau})
\]
is a martingale, the drift of the process has to vanish and the value function has to satisfy the following differential equations below and above the barrier:

\begin{equation}
\frac{1}{2} \sigma^2 V''(x) + (\mu - c_1) V'(x) - \delta V(x) + c_1 = 0, \quad 0 \leq x \leq b,
\end{equation}

\begin{equation}
\frac{1}{2} \sigma^2 V''(x) + (\mu - c_1 - c_2) V'(x) - \delta V(x) + c_1 + c_2 = 0, \quad x \geq b,
\end{equation}

with boundary conditions \( V(0) = 0 \) and \( \lim_{x \to \infty} V(x) = \frac{c_1 + c_2}{\delta} \). The general solution of (16) is

\[
V(x) = \frac{c_1}{\delta} + B_1 e^{\theta_1 x} + B_2 e^{\theta_2 x}
\]

with constants \( B_1, B_2 \in \mathbb{R} \). Using \( V(0) = 0 \) we obtain \( B_2 = -\frac{c_1}{\delta} - B_1 \). The general solution of (17) is

\[
V(x) = \frac{c_1 + c_2}{\delta} + D_1 e^{\tilde{\theta}_1 x} + D_2 e^{\tilde{\theta}_2 x}
\]

with constants \( D_1, D_2 \in \mathbb{R} \). Using the second boundary condition we obtain \( D_1 = 0 \). The smooth fit condition that \( V \) and \( V' \) are continuous at \( b \) yields

\begin{align*}
B_1 e^{\theta_1 b} - \left( \frac{c_1}{\delta} + B_1 \right) e^{\theta_2 b} &= \frac{c_2}{\delta} + D_2 e^{\tilde{\theta}_2 b} \\
B_1 \theta_1 e^{\theta_1 b} - \theta_2 \left( \frac{c_1}{\delta} + B_1 \right) e^{\theta_2 b} &= D_2 \tilde{\theta}_2 e^{\tilde{\theta}_2 b}.
\end{align*}

The first equation implies

\begin{equation}
D_2 e^{\tilde{\theta}_2 b} = B_1 (e^{\theta_1 b} - e^{\theta_2 b}) - \frac{c_1}{\delta} e^{\theta_2 b} - \frac{c_2}{\delta}
\end{equation}

which can be inserted into the second equation to obtain

\[
\frac{B_1}{\kappa} W'(b) = \frac{B_1}{\kappa} \tilde{\theta}_2 W(b) + \frac{c_1}{\delta} e^{\theta_2 b (\theta_2 - \tilde{\theta}_2)} - \frac{c_2}{\delta} \tilde{\theta}_2.
\]

This implies that \( B_1 = \kappa B \) with \( B \) as in (14). Inserting the expression for \( B_1 \) into (20) yields \( D := D_2 \) in (15).

**Remark 3.1.** The special case \( c_1 = 0, c_2 = c > 0 \) has been studied intensively before. In this case we obtain the formula

\[
V(x, b) := V(x, 0, c, b) = \begin{cases}
\frac{\kappa}{\delta} W(x) e^{\tilde{\theta}_2 (\theta_2 W(b) - W'(b))}, & 0 \leq x \leq b \\
\frac{\kappa}{\delta} \left( 1 + \frac{W'(b)}{\theta_2 W(b) - W'(b)} e^{\theta_2 (x-b)} \right), & x \geq b.
\end{cases}
\]

This result can e.g. be found in [16] as equation (2.22), (2.23). Taking the derivative w.r.t. \( b \) and equating to zero establishes the optimal barrier \( b^* \) that maximizes \( V(x, b) \) as

\[
b^* = \frac{1}{\theta_1 - \theta_2} \log \left( \frac{\theta_2 (\theta_2 - \tilde{\theta}_2)}{\theta_1 (\theta_1 - \tilde{\theta}_2)} \right)
\]

if \( \mu + \frac{\sigma^2}{2} \tilde{\theta}_2 > 0 \) and \( b^* = 0 \) otherwise (see e.g. [6]). In fact, in [16], various other characterizations of \( b^* \) have been shown: First \( b^* \) can be characterized as the unique \( b \) s.t. \( V(x, b) \) is twice continuously differentiable in \( x \). Moreover it is the unique \( b \) s.t. the value function \( V(x, b) \) coincides with the value function of the dividends according to a horizontal barrier strategy with barrier \( b \), i.e. the case \( c = \mu \).
3.2. **Ratchet Strategy.** Let us now look into the value function for the diffusion case under the ratcheting strategy:

**Theorem 3.2.** The value function under the ratchet dividend strategy with barrier \( b \) for the diffusion case is given by

\[
V^R(x, c_1, c_2, b) = \begin{cases} 
\frac{c_1}{\delta}(1 - e^{\theta x}) + \frac{1}{\delta} \left( e^{\theta_1 x} - e^{\theta_2 x} \right) \left( c_2 + c_1 e^{\theta_2 b} - (c_1 + c_2) e^{\theta_1 b} \right), & 0 \leq x \leq b, \\
\frac{c_1 + c_2}{\delta}(1 - e^{\theta_2 x}), & x \geq b.
\end{cases}
\]

**Proof.** In the diffusion case the scale functions are given by

\[
W(x) = \kappa(e^{\theta_1 x} - e^{\theta_2 x}), \quad \tilde{W}(x) = \tilde{\kappa}(e^{\theta_1 x} - e^{\theta_2 x}),
\]

with \( \kappa = (\mu - c_1)^2 + 2\sigma^2 \delta \) and \( \tilde{\kappa} := (\mu - c_1 - c_2)^2 + 2\sigma^2 \delta \). Substituting these expressions into (7) in Theorem 2.1, together with some algebraic manipulations, gives the result. Note that here \( \phi(\delta) = \tilde{\theta}_1 \) and \( Z(x) = \frac{\tilde{\delta}}{\delta} W(x) + e^{\theta_2 x} \).

For \( x \geq b \) there is also a direct way: Since in this case we start immediately to pay out at rate \( c_1 + c_2 \), the value function is

\[
V^R(x, c_1, c_2, b) = \frac{c_1 + c_2}{\delta} (1 - E_x(e^{-\tilde{\delta}_0}))
\]

But by the same arguments as in Theorem 3.1, the quantity \( m(x) := E_x(e^{-\tilde{\delta}_0}) \) satisfies, for any \( x > 0 \), the differential equation

\[
\frac{1}{2} \sigma^2 m''(x) + (\mu - c_1 - c_2) m'(x) - \delta m(x) = 0,
\]

with boundary condition \( m(0) = 1 \) and \( \lim_{x \to \infty} m(x) = 0 \). Hence \( m(x) = A e^{\delta_1 x} + B e^{\delta_2 x} \) with \( A = 0 \) and \( B = 1 \), giving \( V^R(x, c_1, c_2, b) = \frac{c_1 + c_2}{\delta} (1 - e^{\theta_2 x}) \) for \( x \geq b \).

Note that \( V^R \) is continuous in \( x \).

**Remark 3.2.** For \( c_1 = 0, c_2 = c > 0 \), (21) simplifies to the formula

\[
V^R(x, 0, c, b) = \begin{cases} 
\frac{c}{\delta} e^{\delta_1 x} - e^{\delta_2 x}(1 - e^{\delta_2 b}), & 0 \leq x \leq b, \\
\frac{c}{\delta}(1 - e^{\delta_2 x}), & x \geq b.
\end{cases}
\]

From Theorem 2.3 we already know that the barrier \( b^R \) which maximizes the payout in the ratcheting case, i.e. \( V^R(x, c_1, c_2, b^R) := \sup \limits_b V^R(x, c_1, c_2, b) \) is the one for which the value function is continuously differentiable. It turns out, that for the diffusion case another somewhat surprising characterization of the optimal barrier \( b^R \) can be found:

**Theorem 3.3.** In the ratcheting dividend problem, the optimal barrier \( b^R \) is exactly the one for which the value function coincides with the value function in the refracting case, i.e.

\[
V^R(x, c_1, c_2, b^R) = V(x, c_1, c_2, b^R), \quad x \geq 0.
\]

**Proof.** Inspecting the value function of the ratcheting problem we see that we can write it as

\[
V^R(x, c_1, c_2, b) = \begin{cases} 
\gamma W(x) + \frac{\alpha}{\delta} (1 - e^{\theta x}), & 0 \leq x \leq b \\
\frac{c_1 + c_2}{\delta}(1 - e^{\theta_2 x}), & x \geq b,
\end{cases}
\]
with a suitable constant $\gamma$. The value function in the refracting problem is given by

$$V(x, c_1, c_2, b) = \left\{ \begin{array}{ll}
B \cdot W(x) + \frac{c_1}{\delta} (1 - e^{\theta_2 x}) & , 0 \leq x \leq b \\
\frac{c_1 + c_2}{\delta} + De^{\theta_2 x} & , x \geq b
\end{array} \right.$$ 

with suitable constants $B$ and $D$. Since $V^R \in C^1$, if we plug in $b = b^R$ and since $V \in C^1$ for all $b \geq 0$ we deduce that the following equations hold:

$$\begin{align*}
(22) & \quad \gamma W(b^R) - \frac{c_1}{\delta} e^{\theta_2 b^R} = \frac{c_2}{\delta} - \frac{c_1 + c_2}{\delta} e^{\theta_2 b^R} \\
(23) & \quad \gamma W'(b^R) - \frac{c_1}{\delta} \theta_2 e^{\theta_2 b^R} = -\frac{c_1 + c_2}{\delta} \theta_2 e^{\theta_2 b^R} \\
(24) & \quad B \cdot W(b^R) - \frac{c_1}{\delta} e^{\theta_2 b^R} = \frac{c_2}{\delta} + De^{\theta_2 b^R} \\
(25) & \quad B \cdot W'(b^R) - \frac{c_1}{\delta} \theta_2 e^{\theta_2 b^R} = D \theta_2 e^{\theta_2 b^R}.
\end{align*}$$

Subtracting (24) from (22) we obtain

$$W(b^R)(\gamma - B) = -e^{\theta_2 b^R} \left( \frac{c_1 + c_2}{\delta} + D \right)$$

and subtracting (25) from (23) we obtain

$$W'(b^R)(\gamma - B) = -\theta_2 e^{\theta_2 b^R} \left( \frac{c_1 + c_2}{\delta} + D \right).$$

These last two equations yield that

$$W'(b^R)(\gamma - B) = -\theta_2 e^{\theta_2 b^R} \left( \frac{c_1 + c_2}{\delta} + D \right) = 0.$$ 

Since $W(x) = \kappa(e^{\theta_1 x} - e^{\theta_2 x}) > 0$ and $W'(x) = \kappa(\theta_1 e^{\theta_1 x} - \theta_2 e^{\theta_2 x}) > 0$ for $x \geq 0$ (note that $\theta_2 < 0$ and $\theta_3 < 0$), we obtain that $B = \gamma$, but in view of (24) and (22) the latter implies that $D = -\frac{c_1 + c_2}{\delta}$. Hence at this barrier level $b^R$ both value functions coincide.

**Example 3.1.** Consider the case where $c_1 = 0, c_2 = 5$ and the parameters of the risk reserve process are given by $\mu = 10, \sigma = 4$ and the discount rate is $\delta = 0.999$. In Figure 1 we see on the left-hand side the value function $V(x, 0, 5, b)$ as a function of the initial state and the barrier level. Note that $x \mapsto V(x, 0, 5, b)$ is here differentiable for all $b$. On the right-hand side we see the value function $V^R(x, 0, 5, b)$ as a function of the initial state and the barrier level. The mapping $x \mapsto V(x, 0, 5, b)$ has a kink for all but one $b$ (note that the unusually high value of $\delta$ was chosen here to visually amplify this phenomenon).

In Figure 2 we see both functions in one picture (left). There is exactly one $b$ for which these functions coincide. In the picture on the right-hand side one can see the difference $V^R(x, 0, 5, b) - V(x, 0, 5, b)$. The optimal barrier is $b^R = 2.41$ in this case.

In Figure 3 the two value functions can be seen as a function of $b$ for fixed $x = 0.5$. One nicely observes that the value $b^R$ which maximizes $V^R$ indeed coincides with the one where both values coincide.

In the example above, when comparing the ratcheting strategy with the refraction strategy we fixed $c_1 = 0$, i.e. no dividend payments before the first hitting time of the barrier. While $c_1 = 0$ is optimal for the refraction strategy (since the resulting strategy is known to maximize the expected discounted dividend payments until ruin among all admissible dividend strategies), it may not be a fair way to compare the performance of the two types of dividend strategies, since for the ratcheting case it may very well be possible that a positive $c_1$ is preferable. Let us therefore compare the best refraction
strategy with the best ratcheting strategy. For that purpose, one can determine the optimal threshold value $b^*(c_1, c_2)$ for the refraction strategy and the optimal barrier value $b^R(c_1, c_2)$ for the ratcheting case for each fixed $c_1$ and $c_2$. In a second step, an optimization of

\begin{align}
V^R(x, c_1, c_2, b^R(c_1, c_2)) \quad \text{and} \quad V(x, c_1, c_2, b^*(c_1, c_2))
\end{align}

with respect to $c_1$ and $c_2$ in some constrained region of choice will yield the (trivariate) optimum of $V^R$ and $V$ for fixed initial surplus value $x$. Unfortunately, even for the diffusion case, identifying the optimal trivariate choice of $c_1, c_2, b$ for a given $x$ in an analytic way seems out of reach since $c_1, c_2$ enter into the equations in an intricate way through the roots of the Laplace equation, and the characterization of the optimal ratcheting barrier
level also does not lead to an explicit formula. We will, however, illustrate the behaviour of (28) numerically in the following example.

**Example 3.2.** Let $\mu = 10$, $\sigma = 6$, $\delta = 0.1$ and $x = 1$. Figure 4 depicts the two functions in (28) as well as their difference, for all dividend rates in the region

\[(c_1, c_2) \in [0, 5]^2\]

(note that $c_1 + c_2 = 10$ corresponds to the drift $\mu$ of the original risk process, in which case the refracting strategy turns into a horizontal dividend barrier, which in the absence of any constraint on $c_1, c_2$ is the optimal dividend strategy). The two functions agree for $c_2 = 0$, and are strictly increasing in $c_2$ for fixed $c_1$. However, for fixed $c_2$ they are both not monotone in $c_1$ (note that increasing $c_1$ also increases the larger dividend rate $c_1 + c_2$). The maximum absolute difference between the two functions is achieved for the largest values of $c_1$ and $c_2$.

In order to compare the performance of the ratcheting strategy with optimal ratcheting barrier $b^R$ and optimal choices of $c^R_1$ and $c^R_2$ to the threshold strategy with optimal threshold $b^*$ (with $c_1 = 0$ and $c_2$ being as large as feasible, which is known to be optimal in that case), we plot those two functions for each given upper bound $k = c_1 + c_2$ on the maximal dividend rate in Figure 5 (again for $x = 1$, $\mu = 10$, $\sigma = 6$, $\delta = 0.1$). Note that each curve depicts the respective overall best possible performance among ratcheting and refraction strategies for a given upper bound $k$, $k \in (0, 10)$. It is quite remarkable that the performance of the ratcheting strategy is so close to the refracting strategy (which is the overall optimal strategy), albeit the type of ratcheting is very simple (only based on one switch). Note that here the optimal choice of $c^R_1$ also turns out to be zero, and the optimal choice of $c^R_2$ is $k$, just as for the refracting case. This may lead to the conjecture that $c^R_1 = 0$ and $c^R_2 = k$ is always optimal for ratcheting. In view of the intricate structure, a proof of this conjecture seems, however, difficult.

If the barrier is not optimal, one also observes non-monotonicity in $c_2$, cf. Figure 6, where $V^R$ and $V$ are plotted as a function of $c_1, c_2$ for a fixed and non-optimal $b = 1$. 

---

![Figure 3](image-url)  
**Figure 3.** Value functions $V(0.5, 0, 5, b)$ (solid curve) and $V^R(0.5, 0, 5, b)$ (dashed curve) with $\mu = 10, \sigma = 4, \delta = 0.999, c = 5$ as functions of $b$. 

---
Figure 4. Plot of the functions (28) (left) and their difference (right) for $x = 1, \mu = 10, \sigma = 6, \delta = 0.1$.

Figure 5. The optimal value of $V$ (solid) and $V^R$ (dashed) as a function of the upper limit $k = c_1 + c_2$ for $x = 1, \mu = 10, \sigma = 6, \delta = 0.1$.

Figure 6. Plot of $V$ and $V^R$ for a fixed, non-optimal $b = 1$ and $x = 1, \mu = 10, \sigma = 6, \delta = 0.1$.

4. The Cramér-Lundberg model with hyper-exponential claims
In the compound Poisson case, we have the usual Crámer-Lundberg setting, where

$$Y_t = x + ct - \sum_{i=1}^{N_t} Z_i$$
where \( N = \{ N_t \}_{t \geq 0} \) is a Poisson process with intensity \( \lambda > 0 \) and \( Z_i \) are i.i.d. non-negative random variables which are independent of \( N \). Assume also that \( c_1 + c_2 \leq c \). We focus on the case where \( Z_1 \) is hyper-exponential, i.e. a mixture of exponentials with

\[
\psi(\theta) = c\theta - \lambda + \lambda \sum_{k=1}^{n} \frac{A_k}{1 + \theta/\alpha_k}, \quad A_k \geq 0, \quad \sum_{k=1}^{n} A_k = 1, \quad \alpha_k > 0, \quad n \in \mathbb{N},
\]

for \( \theta > -\min_k \{ \alpha_k \} \). The scale functions of the process \( X_t = Y_t - c_1 t \) are then given by

\[
W(x) = \sum_{k=0}^{n} D_k e^{\theta_k x}, \quad Z(x) = 1 + \delta \sum_{k=0}^{n} D_k (e^{\theta_k x} - 1),
\]

where \( \theta_k, k = 1, \ldots, n + 1 \) are the \( n + 1 \) roots, in decreasing order, of the function

\[
f(\theta) = (c - c_1)\theta + \lambda \sum_{k=1}^{n} \frac{A_k}{1 + \theta/\alpha_k} - \delta,
\]

and

\[
D_k^{-1} = \frac{df}{d\theta_k}(\theta_k).
\]

Similarly we obtain \( W, Z \) in terms of \( \tilde{\theta}_k \) and \( \tilde{D}_k \) by replacing \( c_1 \) by \( c := c_1 + c_2 \) in the above formulae. Substituting these expressions into the formulas derived in Section 2, it follows that

\begin{equation}
V^R(x, c_1, c_2, b) = \begin{cases}
\frac{\xi}{\delta} + c \sum_{k=1}^{n+1} \xi_k e^{\theta_k x}, & 0 \leq b \leq x \\
\frac{c_1}{\delta} - c_1 \sum_{k=1}^{n+1} \theta_k e^{\theta_k x} + HW(x), & 0 \leq x < b,
\end{cases}
\end{equation}

where

\[
G_k = \tilde{D}_k \left( \frac{1}{\tilde{\theta}_1} - \frac{1}{\tilde{\theta}_k} \right),
\]

\[
H = \frac{1}{W(b)} \left[ \frac{c_2}{\delta} + c \sum_{k=1}^{n+1} \tilde{D}_k \left( \frac{1}{\tilde{\theta}_1} - \frac{1}{\tilde{\theta}_k} \right) e^{\delta_k b} + c_1 \sum_{k=1}^{n+1} \tilde{D}_k e^{\theta_k b} \right]
\]

Similarly, for the refraction strategy, it is given by

\begin{equation}
V(x, c_1, c_2, b) = \begin{cases}
\frac{\xi}{\delta} + c_1 \sum_{j=1}^{n+1} \xi_j e^{\theta_j x} + c_1 \sum_{j=1}^{n+1} \chi_j e^{\theta_j x} + c_1 \sum_{j=1}^{n+1} \chi_j (\chi_{ij} e^{\theta_j (x-b)} - e^{\theta_j (x-b)}), & 0 \leq b \leq x \\
\frac{c_1}{\delta} + (\eta + c_1 \xi) W(x) + c_1 \sum_{j=1}^{n+1} \chi_j e^{\theta_j x} + c_1 \sum_{j=1}^{n+1} \chi_j (\chi_{ij} e^{\theta_j (x-b)} - e^{\theta_j (x-b)}), & 0 \leq x < b,
\end{cases}
\end{equation}

where

\[
\eta = \left( \sum_{k=1}^{n+1} \frac{D_k \theta_k}{\theta_1 - \theta_k} e^{\theta_k b} \right)^{-1}, \quad \zeta_j = \left\{ \eta \sum_{k=1}^{n+1} \frac{D_k \theta_k}{\theta_j - \theta_k} e^{\theta_k b} - \frac{\tilde{D}_j}{\theta_j} \right\} e^{-\theta_j b},
\]

\[
\xi = e^{\delta b} \tilde{\theta}_1 \eta \sum_{k=1}^{n+1} \frac{D_k}{\theta_1 - \theta_k} e^{\theta_k b}, \quad \chi_j = -D_j/\theta_j, \quad c_1 = \frac{D_j \tilde{D}_j}{\theta_j - \theta_i} e^{\theta_i b}, \quad \chi_{ij} = c_2 \frac{D_j \tilde{D}_j}{\theta_j - \theta_i} e^{\theta_i b} \xi.
\]

**Remark 4.1.** In the compound Poisson case it generally does not hold anymore that

\[
V^R(x, c_1, c_2, b^R) = V(x, c_1, c_2, b^R), \quad x \geq 0,
\]
where $b^R$ is the optimal barrier under the ratcheting strategy. To see this, consider $c_1 = 0$ and $n = 1$. In that case, (29) and (30) simplify to

$$V^R(x,c_1,c_2,b) = \begin{cases} \frac{c_2}{\delta} + c_2 G_2 e^{\delta x}, & 0 \leq b \leq x \\ H \cdot W(x), & 0 \leq x < b, \end{cases}$$

and

$$V(x,c_1,c_2,b) = \begin{cases} \frac{c_2}{\delta} + c_2 \xi e^{\delta x}, & 0 \leq b \leq x \\ \eta W(x), & 0 \leq x < b, \end{cases}$$

Arguing as in the diffusion case, the two analogous equations to (26) and (27) would then be

$$(\eta - H)W(b^R) = c_2(\zeta_2 - G_2)e^{\delta b^R}$$

and

$$(\eta - H)W'(b^R) = c_2(\zeta_2 - G_2)\tilde{\theta}_2 e^{\delta b^R}$$

and one may be inclined to think that the identity then follows in the same way. However, in contrast to the diffusion case, the derivative of $V$ is not continuous in $b$ unless $b$ is the optimal barrier $b^*$ of the threshold case (cf. [15]), which differs from $b^R$. Hence equation (34) is in fact not valid and the identity does no longer hold.

It is instructive to study the weak limit of the compound Poisson process with exponential claims towards a Brownian motion (which can be achieved by driving up the intensity and reducing the mean claim size at the appropriate speed). Concretely, fix the mean and variance

$$\mu = c - \frac{\lambda}{\alpha}, \quad \sigma^2 = \frac{2\lambda}{\alpha^2}. \tag{35}$$

Thus, for every choice of $\alpha$ we set $\lambda(\alpha) = \alpha^2 \sigma^2 / 2$ and $c(\alpha) = \mu + \alpha \sigma^2 / 2$. Then, as $\alpha \to \infty$ one reaches the diffusion case in the limit and we have

$$D_1 \to -\kappa, \quad D_2 \to \kappa, \quad G_1 = 0, \quad G_2 \to \delta^{-1},$$

and the corresponding roots have the form

$$\theta_{1,2} = \frac{\frac{\delta}{\alpha} - \mu + c_1 \pm \sqrt{\left(\frac{\delta}{\alpha} - \mu + c_1\right)^2 + 2\sigma^2 \delta + \frac{4(\mu - c_1)\delta}{\alpha}}}{\sigma^2 + \frac{2(\mu - c_1)}{\alpha}},$$

which coincide with the ones of the Brownian case defined in Section 3. This implies that $V^R$ and $V$ indeed converge to the ones of the Brownian case as $\alpha \to \infty$. We can hence observe the optimality condition of Theorem 3.3 to gradually come into place and being valid in the limit. Indeed, in the limit $V$ is continuously differentiable at any barrier level $b$. This transition is illustrated in Figure 7, where $V^R(1,0,8,b)$ (green) and $V(1,0,8,b)$ (purple) are plotted as a function of $b$. The solid line corresponds to the diffusion case with $\mu = 10$, $\sigma = 6$ and $\delta = 0.1$, the dashed-and-dotted and the dashed lines correspond to the compound Poisson case with $\alpha = 5, \ 10$, respectively, where $\lambda, c$ are chosen according to (35) in each case. The crossing of the solid lines at the maximum of the green curve exemplifies Theorem 3.3. The other crossings do not share this property.

5. THE EXPECTED TIME TO RUIN

The fact that an optimal ratcheting strategy can perform nearly as well in some cases as the optimal refracting strategy is remarkable, especially since shareholders are guaranteed payments from a certain point onwards (until the time of ruin). The drawback is that the optimal ratcheting barrier is in general higher than the optimal refracting.
barrier (see for instance Figure 7), so that in the ratcheting case shareholders will have to wait longer until receiving the increased payments (or any payments at all if \(c_1 = 0\)). Furthermore, the distribution of the time until ruin and hence the length of the overall period of dividend payments will differ, but due to discounting this difference becomes less relevant the larger the time of ruin is. In this section we intend to quantify this tradeoff.

Whenever \(c_1 + c_2\) is smaller than the drift of the original risk process, there is a positive probability to not have ruin at all, so we will confine our analysis here to the time of ruin given that ruin occurs (a more refined analysis could look into occupation time distributions of certain surplus ranges). Consider an initial surplus \(x\) below barrier \(b\).

The expected time of ruin, given it occurs in finite time, under a ratcheting strategy is then given by

\[
E_x[\tau^R; \tau^R < \infty] = E_x[\tau_0^-; \tau_0^- < \tau_b^+] + E_x[\tau_b^+ + \widehat{\tau}_0^-; \tau_b^+ < \tau_0^-; \tau^R < \infty],
\]

where \(\widehat{\tau}_0^-\) is the time to ruin, starting from \(b\) under the increased dividend rate \(c_1 + c_2\). This leads to

\[
E_x[\tau^R; \tau^R < \infty] = E_x[\tau_0^-; \tau_0^- < \tau_b^+] + P_b(\tau_0^- < \infty)E_x[\tau_b^+; \tau_b^+ < \tau_0^-] \\
+ P_x(\tau_b^+ < \tau_0^-)E_b[\tau_0^-; \tau_0^- < \infty].
\]

All these quantities can be recovered from the identities (1) and (2).

For the refracted strategy, for \(x \leq b\) one has directly from Theorem 5(ii) of [21] that

\[
E_x(e^{-\delta\tau_0^-}; \tau_0^- < \infty) = Z(x) - \left[ \frac{W(x)}{e^{-\phi(\delta)b} \int_0^\infty e^{-\phi(\delta)y} W'(y + b)dy} \right] \delta \int_b^\infty e^{-\phi(\delta)y} W(y)dy,
\]

from which the expected ruin time can be obtained by differentiation w.r.t. \(\delta\) and evaluating the result at \(\delta = 0\).
Example 5.1. In the compound Poisson case with exponential claims and the safety loading condition $c - c_1 - c_2 > \lambda/\alpha$, we have for the first term

$$E_x[\tau_0^-; \tau_0^- < \tau_b^+] =$$

$$- (c - c_1)\lambda e^{\frac{\lambda c}{\lambda - c_1}} \left( \lambda(ax + 1)e^{\frac{\lambda c}{\lambda - c_1}} + \alpha e^{ab}(ab(c - c_1) - \alpha(c - c_1)x + b\lambda + (c - c_1)) \right)$$

$$+ \lambda e^{ab-\alpha x + \frac{\lambda b\lambda e^{\frac{\lambda c}{\lambda - c_1}}}{\lambda(c - c_1)(\alpha(c - c_1) - \lambda)} \left( (c - c_1)(\alpha(c - c_1) - \lambda) \left( \alpha(c - c_1)e^{ab} - \lambda e^{\frac{\lambda c}{\lambda - c_1}} \right)^2 \right),$$

for the second

$$P_b[\tau_b^- < \infty] = \frac{\lambda}{\alpha(c - c_1 - c_2)} e^{\frac{\lambda c}{\lambda - c_1} - \alpha x} b,$$

$$E_x[\tau_b^+; \tau_b^- < \tau_0^-] =$$

$$e^{\ln(x) - \frac{\lambda x}{\lambda - c_1} (b - x)} \alpha(c - c_1) e^{\frac{\lambda x}{\lambda - c_1} (a x + 2) + \lambda x} + e^{\alpha x} \left( a^2(c - c_1)^2 e^{ab(x - b)} - e^{\frac{\lambda x}{\lambda - c_1} - \lambda b(c - c_1)(ax + 2)} \right)$$

$$\left( c - c_1)(\alpha(c - c_1) - \lambda) \left( a(c - c_1)e^{ab} - e^{\frac{\lambda b\lambda e^{\frac{\lambda c}{\lambda - c_1}}}{\lambda(c - c_1)}} \right)^2, \right),$$

and for the third

$$P_x[\tau_b^+ < \tau_0^-] = \frac{\alpha(c - c_1)}{\alpha(c - c_1) - \lambda e^{\frac{\lambda c}{\lambda - c_1} - \alpha x}}$$

$$E_b[\tau_0^-; \tau_0^- < \infty] = \frac{\alpha(c - c_1) - \lambda e^{\frac{\lambda c}{\lambda - c_1} - \alpha x}}{(c - c_1 - c_2)(\alpha(c - c_1 - c_2) - \lambda)} \cdot \frac{\lambda}{\alpha(c - c_1 - c_2)} e^{\frac{\lambda c}{\lambda - c_1} - \alpha x} b.$$

Similar formulae can be derived in a simpler manner for the ruin probabilities both in the ratcheting and refracted case. Taking the ratio then yields

$$E_x[\tau^R; \tau^R < \infty] = \frac{E_x[\tau^R; \tau^R < \infty]}{P_x[\tau^R < \infty]} \quad \text{and} \quad E_x[\tau; \tau < \infty] = \frac{E_x[\tau; \tau < \infty]}{P_x[\tau < \infty]},$$

cf. also [18, p.59]. Figure 8 depicts the behaviour of these two quantities as a function of $b$ for initial capital $x = 1$ and parameters $c = 6$, $c_1 = c_2 = 2$, $\lambda = \alpha = 1$. One observes that the expected ruin time (given ruin occurs in finite time) is, for the same barrier, typically larger for the ratcheting case, which at first sight may look counter-intuitive, since the refraction strategy increases the drift again when the process is below $b$. However, this indicates that in the refraction case those sample paths that do not lead to ruin quickly, will more likely escape ruin also later, so the conditioning on the event of ruin is essential here.

Finally, we compare these properties of sample paths for the refracting and ratcheting strategies when the respective optimal barrier is chosen. Figure 9 shows the probability to reach the optimal barrier before ruin as well as the expected time to reach the optimal barrier, given that it is reached before ruin for the two strategies, as a function of initial capital $x$. Note that for the used parameters, the respective optimal barrier levels are $b^R = 4.604602$ and $b^* = 2.7523496$. One observes that the despite the higher value of $b^R$, the probability to reach that level (and hence the probability to increase the dividend rate) is not much less than for the respective refraction strategy, whereas the expected time to get there roughly doubles.
Remark 5.1. Observe that, as a corollary of the first formula in Example 5.1, by taking the limit to the diffusion process (10) with drift $\mu$ and variance $\sigma^2$ (using the parametrization (35)) as well as taking $b \to \infty$, one retrieves the simple expression

$$E_x(\tau^-_0 | \tau^-_0 < \infty) = \frac{x}{\mu}.$$ 

Since this formula is interesting in its own right, and seems not to have been considered in actuarial circles before, we derive it here also directly using an alternative approach. Consider $S_t = x - Y_t = -\mu t - \sigma B_t$, with drift $-\mu$. By exponential tilting by $\theta$ we have that in the new measure $\tilde{P}$ the process $S$ has drift $-\mu + \theta \sigma^2$ and the likelihood ratio

$$\exp(\theta S_t - t \psi(\theta)) = \exp((\mu/\sigma^2) S_t - t \mu^2/(2\sigma^2))$$

is a martingale wrt $\tilde{P}_x$, the law of $Y_t$ starting at zero, where $\psi$ is the Laplace exponent given as in the beginning of Section 2 (here with respect to $\tilde{P}_x$). Inserting now $\theta = \mu/\sigma^2$ simplifies to zero drift and

$$\exp(\theta S_t - t \psi(\theta)) = \exp((\mu/\sigma^2) S_t - t \mu^2/(2\sigma^2)).$$

Optional stopping holds for this martingale if and only if the stopping times are finite (see e.g. [4, Ch.IV.4]), and the time of ruin $\tau^-_0$ is such a time, since under $\tilde{P}_x$ the drift is
zero. Hence we have
\[ \tilde{E}_x(\exp((\mu/\sigma^2)S_{\tau^-} - \tau_0^- \mu^2/(2\sigma^2))) = 1. \]
But \( S_{\tau^-} = x \), and
\[ \exp((\mu/\sigma^2)x - \tau_0^- \mu^2/(2\sigma^2)) \]
is bounded around any finite neighbourhood of \( \mu \), hence uniformly integrable for any sequence \( \mu_n \to \mu \) so we may take the derivative with respect to \( \mu \) and get
\[ \tilde{E}_x\left( \left\{ \frac{x}{\sigma^2} - \frac{\tau_0^- \mu}{\sigma^2} \right\} \exp((\mu/\sigma^2)x - \tau_0^- \mu^2/(2\sigma^2)) \right) = 0. \]
This translates in the original measure to
\[ E_x\left( \left\{ \frac{x}{\sigma^2} - \frac{\tau_0^- \mu}{\sigma^2} \right\}; \tau^- < \infty \right) = 0, \]
and a rearrangement yields indeed
\[ \frac{x}{\mu} = \frac{E_x(\tau^-_0; \tau^-_0 < \infty)}{P_x(\tau^-_0 < \infty)} = E_x(\tau^-_0 | \tau^-_0 < \infty). \]

6. Conclusion and Future Research

In this paper we considered a ratcheting dividend strategy in an insurance risk theory context, where the dividend rate can be raised once during the lifetime of the surplus process. We derived analytical formulas for the expected discounted dividend payments until ruin for a general Lévy risk model, and refined the results for a diffusion approximation and a compound Poisson model with hyper-exponential claims. The numerical illustrations indicate that the performance of such a ratcheting strategy is in fact not far behind the optimal refraction strategy, and also in terms of expected ruin time the resulting performance seems rather competitive.

There are many possible directions for extensions and generalizations from here. In a future paper we will consider the case of multiple barriers, where the ratcheting strategy will mean a gradual increase of the dividend rate. Another question of interest is to analytically show that the performance of the ratcheting strategy is monotone in the choice of the dividend rate increase \( c_2 \) at the switching time. Finally, to solve the general stochastic control problem of identifying the optimal ratcheting strategy (which possibly leads to a continuous function \( c(x) \) as a function of first hitting of the surplus level \( x \)) will be an interesting challenge for future research.

Acknowledgement. The first author gratefully acknowledges financial support by the Swiss National Science Foundation Project 200021_168993.

References


(H. Albrecher) DEPARTMENT OF ACTUARIAL SCIENCE, FACULTY OF BUSINESS AND ECONOMICS AND SWISS FINANCE INSTITUTE, UNIVERSITY OF LAUSANNE, CH-1015 LAUSANNE, SWITZERLAND

E-mail address: hansjoerg.albrecher@unil.ch

(N. Bäuerle) DEPARTMENT OF MATHEMATICS, KARLSRUHE INSTITUTE OF TECHNOLOGY, D-76128 KARLSRUHE, GERMANY

E-mail address: nicole.baueerle@kit.edu

(M. Bladt) DEPARTMENT OF ACTUARIAL SCIENCE, FACULTY OF BUSINESS AND ECONOMICS, UNIVERSITY OF LAUSANNE, CH-1015 LAUSANNE, SWITZERLAND

E-mail address: martin.bladt@unil.ch