Optimal Liquidation Strategies in Illiquid Markets

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Abstract

In this paper, we study the economic relevance of optimal liquidation strategies by calibrating a recent and realistic microstructure model with data from the Paris Stock Exchange. We distinguish the case of parameters which are constant through the day from time-varying ones. An optimization problem incorporating this realistic microstructure model is presented and solved. Our model endogenizes the number of trades required before the position is liquidated. A comparative static exercise demonstrates the realism of our model. We find that a sell decision taken in the morning will be liquidated by the early afternoon. If price impacts increase over the day, the liquidation will take place more rapidly.

Keywords: Optimal execution strategy, liquidity risk, price impact, high frequency data, microstructure.

JEL classification: G12.

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1 Introduction

In this paper, we determine the optimal liquidation strategy of an investor who wishes to liquidate a large position on a given asset, whenever her decisions affect the liquidity of the asset. Such a strategy proves to be useful for financial institutions or mutual funds that may buy or sell quantities that represent significant fractions of the market capitalization of the firm. The investor is thus confronted with a trade-off between the discount that a large trade may have and the risk taken by a given execution strategy. If she decides to liquidate the whole position immediately she will know the liquidation value, but she may have a huge discount in the selling price. On the other hand, if she decides to liquidate in a very smooth way by selling small quantities, she might have a small discount but may face a huge risk because most of the trades are executed after prices have moved. From this trade-off results a liquidation strategy, which involves the volume and the time of liquidation.

This paper contributes to the discussion by finding that using actual data and a realistic microstructure description of the market, an optimal execution strategy that takes into account microstructure elements of the price dynamics may be implemented. This strategy incorporates (time-varying) permanent and transitory price impacts of the order flow, and of the direction of the order flow. The strategy also endogenizes the number of periods over which the position should be sold. The investor maximizes the expected total income of liquidation penalized by its variance, where the penalization factor represents the investor risk aversion. The original problem is quadratic, and we are able to express it as a linear optimization problem, which is solved numerically. We study the characteristics of the solution and we establish that the optimal number of liquidation periods is decreasing in the permanent fixed price impact and in the standard deviation of the process associated with the arrival of public information, and is increasing in the variable unitary cost of trade and in the fixed cost per order. We study how the solution to the optimization problem responds to several patterns of the permanent price impact of the order flow, and we establish that the investor liquidates larger quantities when the price-impact of the order flow is smaller. We illustrate the methodology using a randomly selected stock, i.e. Orange, a company traded on the French stock market. For this stock, we estimate the parameters of the price process when they are assumed constant throughout the day, and when they are changing hourly. We determine the optimal liquidation strategy whenever the parameters are assumed
Various papers discuss how to establish an optimal liquidation strategy. Almgren and Chriss (2001) find a pseudo closed-form solution of the optimal trading strategy of the investor who has to liquidate a large position on a security, assuming that the price dynamics responds to permanent and transitory linear impacts. In that paper the total time of liquidation is determined exogenously, and the time between trades is constant. This specification does not incorporate some important microstructure features, such as a price impact that changes over time or an order book that needs some time to regenerate after a sale.

Bertsimas and Lo (1998) derive dynamic trading strategies using stochastic dynamic programming techniques, which minimize the expected cost of purchasing a large block of shares. Contrary to Almgren and Chriss (2000), they do not incorporate the volatility of the total cost of trading in the optimization problem.

Assuming a similar framework as Almgren and Chriss (2000), Dubil (2002), and Hisata and Yamai (2000) obtain the optimal total time of liquidation, when a constant speed of liquidation is assumed. Both approaches study the risk of a liquidation strategy in a value-at-risk framework. The main differences between the two papers is the specification of the price impact function. Hisata and Yamai (2000) find a closed-form solution for the optimal liquidation time, when the market impact is a square root function of the trading volume. In addition, they also study the case of a stochastic market impact model and a portfolio model. In these cases the optimal time of liquidation is found numerically.

Huberman and Stanzl (2000), assuming a slightly different price dynamics than Almgren and Chriss (2000), find a closed-form solution to the optimal static selling strategy when the price impact of the trade size is constant, and a recursive solution when it changes through time. This approach minimizes the expected value of the total revenue of the liquidation minus the product of the risk aversion and the variance of the total revenue of liquidation. The optimization problem is solved using dynamic programming.

Mönch (2004) incorporates two microstructure features in the price function, the U-shape of intraday market liquidity and the resiliency of the order book. Then, assuming

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1For the empirical work we use data purchased from the Paris Stock Exchange. The date covers January and February 2003. The average number of trades was 1599 per day for Orange, for an average daily traded volume of 5'789'074 Euros. We used also several other series for our empirical investigation but decided to report only the results for Orange since they are representative for the other stocks.
a price dynamics that takes only permanent price impacts into account, he numerically determines the optimal liquidation strategy. The permanent price impact is modeled as the product of two deterministic functions, one depending on the time-of-the-day, and the other depending on the trade size. Contrary to other papers, it does not restrict the time between trades to be constant.

Our paper differs from those of Almgren and Chriss (2000), Dubil (2002), and Hisata and Yamai (2000) in different respects. First, it does not restrict the price impact of the trade direction and the trade size to be constant through the time. This extension is achieved by modeling the parameters representing these impacts as time varying. Second, using tick-by-tick data of the Paris Stock Exchange, we are able to estimate the parameters that characterize the price process. It is worth emphasizing that in this context, only Mönch (2004) has done the estimation of the parameters of the model he proposes. He assumes two particular functional forms for the price and traded size functions. Then, using tick-by-tick data of stocks traded in the Helsinki Stock Exchange, and the same specification for all the stocks, he estimates the parameters through least squares. In our model, we use a more advanced model stemming from the microstructure literature. The model that we estimate was proposed by Sadka (2006). This is a rich model incorporating many features that have been exhibited in the microstructure literature. Indeed, it crystallizes earlier contributions of Glosten and Harris (1988), Brennan and Subrahmanyam (1996), Madhavan, Richardson, and Roomans (1997), as well as of Huang and Stoll (1997).

The rest of the paper is organized as follows: In Section 2, the optimization problem and its solution are presented. We also introduce the microstructure model describing the dynamics of the price evolution, and we present the methodology used to estimate it. In Section 3, we estimate the model for Orange, when the parameters describing the price dynamics are constant through the time of the day, and when they change hourly. Then, for both cases we determine the optimal liquidation strategy and study the sensitivity of the solution to changes in the level of the parameters. We also study how the optimal liquidation strategy behaves when the value of the parameters evolves according to several pre-established patterns. Section 4 provides some concluding comments.
2 Theoretical problem

We begin with a presentation of the main building blocks of the model: the trading strategy, the optimization problem, and the price dynamics. The definition of trading strategy and the representation of the optimization problem are closely related to those adopted by Almgren and Chriss (2001) and Huberman and Stanzl (2000). Assume that a large investor has a position $X$ on a security, which she wants to liquidate totally by time $T$. A liquidation strategy is defined by a sequence of positive numbers $(x_{t_0}, \ldots, x_{t_N})$, where $x_{t_i} \geq 0$, for $i = 0, \ldots, N$, is the number of units the investor plans to hold at time $t_i$. The strategy is implemented at time $t_0$, where $t_0 = 0 < t_1 < \cdots < t_N = T$, and $x_{t_i} \geq x_{t_j}$ if $i < j$. The times $t_i$ are assumed to be discretely spaced.\(^2\) Given that initially the investor holds $X$ units of the security and she needs to liquidate it totally by time $T$, it follows that $x_{t_0} = X$ and $x_{t_N} = 0$. The liquidation trajectory $(x_{t_0}, \ldots, x_{t_N})$ can be expressed equivalently as a trade list $(n_{t_1}, \ldots, n_{t_N})$, where $n_{t_k}$ is the number of units sold immediately before time $t_k$. Hence, the variables $x_{t_k}$ and $n_{t_k}$ are related by

$$x_{t_k} = X - \sum_{j=1}^{k} n_{t_j}, \text{ for } k = 0, \ldots, N.$$ 

This equation states that the investor’s holding at time $t_k$ is the initial holding, $X$, minus the sum of the quantities sold up to time $t_k$, $\sum_{j=1}^{k} n_{t_j}$. Another implication from the definition of the variables $n_{t_k}$ and $x_{t_k}$ is that $n_{t_k} = x_{t_{k-1}} - x_{t_k}$, meaning that the quantity sold between the times $t_{k-1}$ and $t_k$ is the difference between the holding quantities at these times. The liquidation strategy is based on the information available at time $t_0$.\(^3\) The investor’s optimization problem consists in finding the liquidation strategy $(x_{t_0}, \ldots, x_{t_{N-1}})$, or equivalently $(n_{t_1}, \ldots, n_{t_N})$, that maximizes the expected total income of liquidation, penalized by its variance, with a penalization factor denoted

\(^2\)Since the optimal liquidation strategies may yield to a liquidation of 0 units in some periods, and since we endogenize the number of liquidation periods, our assumption that the time between liquidation periods is constant does not constitute a limitation of our model.

\(^3\)At time $t_0$, we define deterministic strategies, which determine the sold quantities for the next $N$ periods of time. Ideally, one would like to use adaptive strategies whereby the arrival of news affects the trading strategy. For the moment, no theoretical framework for such strategies appears to exist.
by \( \eta \). This leads us to the following optimization program:

\[
\max_{\{n_1, \ldots, n_N\}} E \left[ \sum_{i=1}^{N} n_t p_t \right] - \eta V \left[ \sum_{i=1}^{N} n_t p_t \right],
\]

s.t. \( \sum_{i=1}^{N} n_t = X \), and \( 0 \leq n_t \leq X \), for \( i = 1, \ldots, N \).

The optimization problem described by Equation (1) assumes a specific number of liquidation periods \( N \). To endogenize the number of liquidation periods we solve the optimization problem for \( N = 1, \ldots, N^{\text{max}} \), where \( N^{\text{max}} \) is a large number, and we choose the optimal \( N \) as the value that maximizes the objective function in Equation (1).

It is clear from Equation (1) that the solution to the optimization problem depends on the dynamics of the trading price, \( p_{t_k} \). We use the trading price specification to incorporate in the optimization several microstructure features, such as the time-varying impact of the direction and magnitude of the order flow. The price is established in such a way that larger discounts are suffered when larger quantities are sold and, that more risk is faced as the liquidation strategy takes more execution time. In the following section, we discuss the dynamics followed by the trading price and its particular form in the case of a pure selling strategy.

### 2.1 The trading price dynamics

From now on, we drop the index \( k \) from the time \( t_k \), and we denote it only by \( t \). Time \( t \) represents the event time of the trade, and \( t - 1 \) represents the time of the trade occurring immediately before. The dynamics of the transaction price, \( p_t \), is inspired by the models of Glosten and Harris (1988), Brennan and Subrahmanyam (1996), Madhavan, Richardson, and Roomans (1997), Huang and Stoll (1997), and Sadka (2006), who relate the price movements to the dynamics of the fundamental value of the stock, denoted by \( m_t \). We define \( m_t \) by,

\[
m_t = m_{t-1} + \psi_t (D_t - E_{t-1} [D_t]) + \lambda_t (n_t - E_{t-1} [n_t]) D_t + y_t,
\]

where \( y_t \sim iid(0, \sigma^2_y) \).

The variable \( m_t \) represents the expected value of the security, conditional on the information available at time \( t \). For a given transaction occurring at time \( t \), the indicator
variable $D_t$ represents the direction of the order flow, which is defined as $+1$ if the trade is buy-initiated and $-1$ if it is sell-initiated. The variable $n_t$ represents the order flow (i.e., the traded volume), and the variable $y_t$ is an exogenous shock that represents the arrival of new information.\footnote{To infer the direction of the order flow, $D_t$, we used the data corresponding to trades and quotes, and the following rule: If the trade price is larger than or equal to the last best-ask price, the trade is classified as buy-initiated (i.e., $D_t = +1$); if the trade price is smaller than or equal to the last best-bid price, the trade is classified as sell-initiated (i.e., $D_t = -1$), and if the trade is between the last best-bid and the last best-ask, the trade direction is assigned to 0. The aforementioned methodology consistently classifies more than 96 percent of the trades as sell-initiated or buy-initiated, and only for less than 4 percent of the trades the direction of the order flow is undefined. To evaluate the quality of this inference methodology, we selected several stocks and we compared the direction of the order flow obtained from the inference method with the actual direction of the order flow content in the order book. For all the studied stocks the performance of the inference methodology was very good. It is worth clarifying that the classification methodology of Lee and Ready (1991) does not apply to our case because the Paris Stock Exchange is an order-driven market, while the Lee and Ready (1991) methodology was developed for quote-driven markets.}

Equation (2) models the permanent effect on price of both, private information, $D_t$ and $n_t$, and public information, $y_t$. The intuition behind this representation is that some traders have some private information about the price of the security, and this information is reflected in the decisions they make. These decisions include the decision of buying or selling (i.e., the value of $D_t$), and the quantity they decide to buy or sell (i.e., the value of $n_t$). Hasbrouck (1991 a,b) and Foster and Viswanathan (1993), among others, have documented the presence of predictability in the order flow.\footnote{The predictability of prices is related to several empirical issues. For instance, the decisions of some investors of break down large trades into small orders to reduce the price impact can create serial correlation in the order flow. Also, trades that are done following the decisions of other traders can create a chain of sells or buys.} To exclude the predictable part of the trade, which of course does not contain new information, in Equation (2) we do not assume that the direction and magnitude of the order flow have an impact on prices, but their innovation processes, represented respectively by, $(D_t - E_{t-1}[D_t])$ and $(n_t - E_{t-1}[n_t])$. In Equation (2), the parameters $\psi_t$ and $\lambda_t$ measure the (permanent) fixed and variable impacts on price.

The dynamics of the transaction price is completed by the transitory impact of the order flow and the direction of the order flow, and is given by:

$$p_t = m_t + (\psi_t + \lambda_t n_t) D_t + \xi_t, \quad (3)$$
where $\xi_t \sim iid(0, \sigma^2_t)$. This equation embodies all the other transient microstructure imperfections that can generate deviations between the expected value of the security and its transaction price. They include for example the effect of stochastic rounding error induced by price discreteness. Equation (3) models the transitory effect on prices of the trades characteristics. The parameter $\psi_t$ can be interpreted as a fixed cost per order, and the parameter $\lambda_t$ as a variable (unitary) cost. Intuitively, these costs should affect the price only at the current time and in a transitory manner. It is worth mentioning that in Equations (2) and (3) the parameters, $\psi_t, \psi_t, \lambda_t$, and $\lambda_t$, are time-varying functions.\footnote{This parameterization relaxes the assumption that the permanent and transitory impacts of the trade characteristics on the price are constant through the time-of-the-day, as it is assumed by Almgren and Chriss (2000), Dubil (2000), Hisata and Yamai (2000), and Sadka (2004).}

One particular pattern of this time-varying impact is the well-illustrated feature in microstructure theory known as the U-shape form of variables related with the order book.\footnote{Empirical evidence of this fact is found in Hasbrouck (1991b) and Mönch (2004) and the references mentioned therein.}

To explore empirically the time-varying behavior of the parameters in our model, we estimate them for different hours of the day. Figure (1) displays the hourly estimated values of the parameters for Orange. We observe that in general they are not constant through the time-of-the-day, suggesting that this fact needs to be captured by the model.

Intuitively, the value of the parameters $\psi_t$ and $\lambda_t$ should be positive. Indeed, a sell order (i.e., $D_t = -1$) typically reflects the belief that the price is high (the security is overvalued) and it will decrease. Conversely, a buy order (i.e., $D_t = +1$) reflects the belief that the price is low (the security is undervalued) and it will increase. The same interpretation applies to the parameter $\lambda_t$ which measures the impact on prices of the directed innovation of the order flow.

In relation to the sign of the transient price-impact parameters, the parameter $\bar{\psi}_t$ represents a fixed cost and is expected to be positive. The parameter $\bar{\lambda}_t$ represents a unitary cost and its sign should be negative reflecting that as the total traded quantity increases the unitary effect should decrease.

The estimation of the parameters is based on the whole set of trades (i.e., trades that were buy-initiated as well as trades that were sell-initiated), while in the optimal liquidation strategy we focus only on one investor who follows a selling strategy. Consequently, we assume that the predictable part of the series (captured by the terms
$E_{t-1}[D_t]$ and $E_{t-1}[n_t]$) is negligible for the optimization problem. In other words, the predictable part captures the effect of all the other trades. For this pure selling case (i.e., $D_t = -1$), and building on the microstructure model, the fundamental value of the stock is thus given by

$$m_t = m_{t-1} - \psi_t - \lambda_t n_t + \sigma \tau^{1/2} \omega_t,$$

where $w_t \sim iid N(0, 1)$, and the decision variable $n_t$ denotes the quantity the investor decides to sell at time $\tau t = \frac{T}{N} t$ for $t = 1, \ldots, N$.

The trading price is given by

$$p_t = m_t - \psi_t - \lambda_t n_t + \xi_t,$$

which corresponds to Equation (3) for a pure selling strategy (i.e., $D_t = -1$).

We use the price dynamics defined by Equations (4) and (5) to determine the optimal liquidation strategy, which corresponds to the solution to the optimization problem described by Equation (1).

Before determining the optimal liquidation strategy, we need to estimate the parameters of our model. In the next section, we describe in detail the estimation procedure. In Sections 3.1 and 3.2 we obtain the estimates using actual data of the Paris Stock Exchange, when the parameters are constant through the time-of-the-day, and when they are continuous functions constructed as the spline interpolation of hourly step functions, respectively.

### 2.2 Estimation of the microstructure model

Before estimation, some filtering of the information is done: (i) the first trade of each day is eliminated, since the process generating the opening price (an auction) is different from the process generating the prices during the day, and (ii) original volumes are seasonally adjusted. The rationale for this filter is that traded volume can seen as the sum of a stochastic part and a deterministic part. The latter is usually called the time-of-the-day effect. This effect reflects the systematic variations of the traded volume during a normal day. Empirical evidence suggests that traded volume is in general larger just after the opening and just before closing than in the middle of the day. We verified such features for our data. The goal of the seasonal adjustment procedure is to filter out this seasonal effect in order to keep only the stochastic effect. To correct for this effect, we adjust the
raw traded volume series by the method proposed by Engle and Russell (1998) under its multiplicative form.\footnote{We regress the logarithm of the raw volume (traded quantity) on time-of-the-day dummies. Specifically, the day is divided in $K$ sub-periods, and we consider the regression $\log n_t = \sum_{k=1}^{K} a_k x_{kt} + \varepsilon_t = a'x_t + \varepsilon_t$, where $x_{kt} = 1$, if the time of the trade $t$ belongs to the intraday sub-period $k$ for $k = 1, \ldots, K$, and 0 otherwise. The seasonally adjusted series is defined by $\tilde{n}_t = n_t \exp(-\tilde{a}'x_t)$, where $\tilde{a}$ denotes the OLS estimate of $a$.}

### 2.2.1 Estimation - Step 1

The estimation is done in two steps. In the first step we adjust and estimate parametric processes for the direction of the order book, $D_t$, and for the volume of the order flow, $n_t$. These estimated processes are used to infer the series $E_{t-1}[D_t]$ and $E_{t-1}[n_t]$. These series are needed for the Generalized Method of Moments (GMM) estimation of Equations (2) and (3).\footnote{In Equations (2) and (3), $n_t$ represents the traded volume after seasonal adjustment.}

For the first step, there is no consensus as to which model best describes the dynamics of the direction of the order book and the order flow. For instance, Hasbrouck (1991 a,b) expresses quote-midpoints and order flows through a vector autoregressive process of order 5. Based on this early model, Brennan and Subrahmanyam (1996) use five lags of prices and order flows to estimate the unexpected order flow. Huang and Stoll (1997) model the direction of the order book as an autoregressive process of order one. Sadka (2006) estimates an $AR(5)$ process for the (signed) order flow.

We assume that the order flow, $n_t$, follows an autoregressive process, and that the direction of the order flow, $D_t$, follows a Markov Chain process with state-space $S = \{-1,0,1\}$.

Therefore, the order flow is described by: $n_t = c + \varphi_1 n_{t-1} + \cdots + \varphi_p n_{t-p} + \varepsilon_t$, with $\varepsilon_t \sim iid(0, \sigma^2)$. Once the parameters of the processes $n_t$ (i.e., $\Gamma = (c, \varphi_1, \ldots, \varphi_p)$) and $D_t$ (i.e., the transition probabilities $\pi_{i,j} = \text{prob}\{D_t = j \mid D_{t-1} = i\}$ for $i,j = -1,0,1$) are estimated, we calculate

\[
\hat{E}_{t-1}[n_t] = \hat{c} + \hat{\varphi}_1 n_{t-1} + \cdots + \hat{\varphi}_p n_{t-p},
\]

and

\[
\hat{E}_{t-1}[D_t|D_{t-1} = i] = -\pi_{i,-1} + \pi_{i,1}, \quad \text{for} \quad i = -1,0,1.
\]
Estimations of an AR model for \( n_t \) revealed different patterns depending on the stock used. Using the Schwartz (1978) criterion, we found that Orange’s order flow is best described by an AR(6) model.\(^\text{10}\)

### 2.2.2 Estimation - Step 2

We estimate by GMM the remaining set of parameters, \( \psi_t, \bar{\psi}_t, \lambda_t, \bar{\lambda}_t, \sigma^2_{y_t}, \) and \( \sigma^2_{\xi_t} \), describing the dynamics of the expected and traded prices in Equations (2) and (3). To obtain the orthogonality conditions, we perform some manipulation of Equations (2) and (3). First, we take the first difference of \( p_t \) in Equation (3), yielding

\[
p_t - p_{t-1} = m_t - m_{t-1} + \psi_t D_t - \bar{\psi}_{t-1} D_{t-1} + \bar{\lambda}_t n_t D_t - \bar{\lambda}_{t-1} n_{t-1} D_{t-1} + \xi_t - \xi_{t-1}.
\]

Plugging \( m_t - m_{t-1} \) from Equation (2), and grouping some terms, we obtain that the dynamics of the actual trade price is given by

\[
p_t - p_{t-1} = \psi_t (D_t - E_{t-1}[D_t]) + \lambda_t (n_t - E_{t-1}[n_t]) D_t + \bar{\psi}_t D_t - \bar{\psi}_{t-1} D_{t-1} + \lambda_t n_t D_t - \lambda_{t-1} n_{t-1} D_{t-1} + y_t + \xi_t - \xi_{t-1}.
\]

If the parameters \( \psi_t, \bar{\psi}_t, \lambda_t, \bar{\lambda}_t, \sigma^2_{y_t}, \) and \( \sigma^2_{\xi_t} \) are constant, then from Equation (6) and defining \( u_t = y_t + \xi_t - \xi_{t-1} \), we have that

\[
u_t = y_t + \xi_t - \xi_{t-1} = p_t - p_{t-1} - (\psi + \bar{\psi}) D_t + \psi E_{t-1}[D_t] - (\lambda + \bar{\lambda}) n_t D_t + \lambda E_{t-1}[n_t] D_t + \bar{\psi} D_{t-1} + \bar{\lambda} n_{t-1} D_{t-1}.
\]

From this equation, we notice that the model implies linearity between the observable variables, and that the parameters can be identified from the following orthogonality conditions.

\(^{10}\)For some other series such as Suez or Sodexo-Allianz, no autocorrelation could be detected at all. Others, such as Alcatel, required even 10 lags in the AR process. This autocorrelation is not related to the size of the company. Nor are these stocks very illiquid.
conditions

\[ E \begin{pmatrix} u_t \\ u_tD_t \\ u_tE_{t-1}[D_t] \\ u_tn_tD_t \\ u_tE_{t-1}[n_tD_t] \\ u_tD_{t-1} \\ u_tn_{t-1}D_{t-1} \\ u^2_t - (\sigma_y^2 + 2\sigma_x^2) \\ u_tu_{t-1} + \sigma_x^2 \end{pmatrix} = 0. \] (8)

The first 7 orthogonality conditions correspond to the ordinary least squares normal equations, while the last two conditions correspond, respectively, to the variance and covariance terms. Specifically, from Equation (7) we have that

\[ V[u_t] = V[y_t] + V[\xi_t] + V[\xi_{t-1}] = \sigma_y^2 + 2\sigma_x^2, \]

and

\[ Cov[u_t, u_{t-1}] = Cov[y_t + \xi_t - \xi_{t-1} + y_{t-1} + \xi_{t-1} - \xi_{t-2}] = -Cov[\xi_{t-1}, \xi_{t-1}] = -\sigma_x^2. \]

2.3 The optimal liquidation strategy

In this section, we derive the optimal liquidation strategy by solving the optimization problem described by Equation (1).

Proposition 1 The solution to the optimization problem described by Equation (1), when the price function is given by Equation (5), corresponds to the solution to the linear system

\[ c_k + \sum_{t=1}^{k-1} (-\lambda_t - b_t)n_t + (d_k - b_k)n_k - \sum_{t=k+1}^{N} b_kn_t = 0 \quad \text{for } t = 1, \ldots, N, \]

where \( c_t = -\left( \sum_{j=1}^{t} \psi_j + \bar{\psi}_t \right), b_t = 2\eta\tau \left( \sum_{j=1}^{t} \sigma_j^2 \right), d_t = -2 \left( \lambda_t + \bar{\lambda}_t \eta \sigma^2_{t, t} \right), \) and \( (n_1, \ldots, n_N) \) satisfies the constraints of the original problem.

Rather than expressing the solution in terms of \( n_t \), it is possible to express it as a function of \( x_t \). This is expressed in the following proposition.
Proposition 2 The solution to the optimization problem described by Equation (1), when the price function is given by Equation (5), corresponds to the solution to the linear system

\[ a_t + b_t x_{t-1} + 2c_t x_t + b_{t+1} x_{t+1} = 0 \quad \text{for } t = 1, \ldots, N - 1, \]

where \( a_t = (\overline{\psi}_t - \overline{\psi}_{t+1} - \psi_{t+1}) \), \( b_t = \lambda_t + 2\overline{\lambda}_t + 2\eta\sigma^2_{\xi_t} \), \( c_t = -\lambda_{t+1} - \overline{\lambda}_{t+1} - \overline{\lambda}_t - \eta\sigma^2_{\psi_{t+1}} - \eta\sigma^2_{\xi_{t+1}} - \eta\sigma^2_{\xi_t} \), \( n_t = x_{t-1} - x_t \), and \((x_0, \ldots, x_N)\) satisfies the constraints of the original problem.

The proof of Proposition 1 is relegated to an Appendix. The proof of Proposition 2 is analogous and is not presented here. We use Proposition 1 to determine the optimal liquidation strategy. Indeed, the proof of Proposition 1 reveals that problem (1) is equivalent to solving the Phase-I of the simplex algorithm, see Nocedal and Wright (1999).11

It is worth noting that Equation (1) assumes a specific number of liquidation periods. To endogenize the optimal number of liquidation periods, in a first step, we obtain the optimal liquidation strategy, \((n^*_1, \ldots, n^*_N)\), for \(N = 1, \ldots, N^{\text{max}}\). Then, in a second step, we choose \(N^{\text{opt}}\) as the value of \(N\) that maximizes the objective function and we retain \((n^*_1, \ldots, n^*_{N^{\text{opt}}})\) as the optimal liquidation strategy.

3 Empirical results

In this section, we use high frequency data to estimate parameters \(\psi, \overline{\psi}, \lambda, \overline{\lambda}, \sigma_y,\) and \(\sigma_\xi\).12 Then, following the procedure described in Section 2.3, we determine the optimal liquidation strategy. Even though the estimation is done for many stocks, we only present the results for Orange. The optimal liquidation strategy is established in three cases: (1) When the value of the parameters is constant, (2) when the parameter \(\lambda_t\) follows an hypothetical increasing function of time, and (3) when each parameter

11There exist efficient algorithms that perform such computations. We used the implementation available in the MATLAB optimization toolbox.

12The high frequency data is taken from the Paris Stock Exchange database. The original data contains information about all the buy and sell contracts, including date, price, and traded quantity, with a precision of 1 second. It also contains information about quotes, including date, best bid, best ask, depth at the best bid (maximum quantity to be bought at the best bid), and depth at the best ask (maximum quantity offered at the best ask).
follows a continuous time function, which is constructed as the spline’s approximation of the hourly estimates. For the first case, we also implement a sensitivity analysis to study how the optimal number of liquidation periods, and the speed of liquidation are affected by changes in the value of the parameters.

A first issue might be applications, i.e. orders that are traded in an upstairs market. Once such deals are made, they must be channeled through the electronic system. We argue that the fact that a large trade took place is informative. For this reason, we did not exclude applications from our research.\footnote{For many studied stocks, we establish that applications represent less than 1 percent of the total number of trades. However, they represent between 5 and 12 percent of the total traded volume. This fact evidences that applications are important and that they should be left in the analysis.}

3.1 Case 1: Constant parameters

3.1.1 Estimation

We first assume that the parameters, $\psi_t$, $\bar{\psi}_t$, $\lambda_t$, $\bar{\lambda}_t$, $\sigma_{y_t}^2$, and $\sigma_{z_t}^2$, governing the price evolution in Equations (2) and (3), are constant through the time-of-the-day.

Using the two-step procedure described in Section 2.2, in the first step, we adjust and estimate an autoregressive process for the order flow, $n_t$, and we assume that the direction of the order flow, $D_t$, evolves according to a Markov Chain. In the second step, the other parameters (i.e., $\psi$, $\bar{\psi}$, $\lambda$, $\bar{\lambda}$, $\sigma_{y_t}^2$, and $\sigma_{z_t}^2$) are estimated by GMM, where the set of orthogonality conditions is given by Equation (8). We wish to report that for the various stocks under consideration, all the parameter estimates had the right sign. The magnitude of the coefficients was also of a comparable magnitude.\footnote{All prices were scaled as to yield comparable parameter estimates.} Last, a general observation is that parameter significance is very similar across the stocks considered.

Table 1 reports the parameter estimates for Orange. From this table, we observe that the permanent impact of the innovation of the direction of the order book, $\psi$, and the transitory impact of the direction of the order book, $\bar{\psi}$, are positive. We observe that both parameters are always significant. The parameter $\psi$ can be interpreted as the expected change in the price after an unexpected change in the direction of the order flow (i.e., in $(n_t - E_{t-1}[n_i])D_t$), and $\bar{\psi}$ is a fixed cost by order whose effect on price is only transitory. We observe that the signs of the estimates are in concordance with the microstructure interpretation that claims that trades obey to private information of...
some traders, who decide to buy a security because they think it is undervalued, and who decide to sell it because they think it is overvalued.

For the permanent effect of the directed innovation of the order flow and the transitory effect of the directed order flow, which are captured respectively by $\lambda$ and $\bar{\lambda}$, we observe that the former is positive and significant, meaning that a sell order has a negative impact on the price and a buy order has a positive impact on it. The parameter $\bar{\lambda}$ is interpreted as the (transitory) marginal cost per unit. Its estimated value is negative, meaning that the marginal cost of an additional unit is decreasing (i.e., as the order quantity increases, the unitary cost decreases).

### 3.1.2 Optimization

We now determine the optimal liquidation strategy for Orange. We assume that the investor wishes to liquidate 6966 shares (i.e., $X = 6966$), within one trading day (i.e., $T = 8.5$ hours). The benchmark risk aversion parameter is $\eta = 4$, and the value of the parameters in Equations (4) and (5) are the GMM estimates (i.e., $\psi = 0.0022479, \bar{\psi} = 0.0047696, \lambda = 0.000045, \bar{\lambda} = -0.0000307, \sigma_y = 0.0058662$, and $\sigma_\xi = 0.0040590$). This specification constitutes our benchmark. Figure 2 displays the optimal selling strategy. We observe that the optimal number of liquidation periods is 5, and the optimal selling strategy consists of selling 5291 units at 10:42 a.m., 1280 units at 12:24 a.m., and so on. In addition, the optimal selling strategy $(n_1^*, \ldots, n_5^*)$ is a decreasing function of time.

Figure 3 displays with the continuous line the cumulative quantity sold against time. This figure permits to visualize the selling speed and will be used for comparison purposes. Indeed, at a specific time, the slope of the curve gives a measure of the liquidation speed. For this particular optimal liquidation strategy, we observe that the slope of the curve is decreasing, meaning that the investor liquidates faster at the beginning of the day than at the end. This figure also shows that if the market opens at 9 a.m., then by 13 a.m., 95% of the initial position would have been liquidated.

To compare this example to other parameterizations, we compute the duration $D$ of the liquidation strategy (see Mönch, 2004). This duration is defined as the average weighted time needed to liquidate a position, where the weights are given by the

---

15 This value corresponds to a liquidation of around 10% of the average traded total daily volume.

16 These values correspond to the values reported in Table 1 given the initial price, $p_0 = 6.54$. 
proportion of the liquidated quantity, i.e.,

\[ D = \sum_{i=1}^{N} (t_i - 9) \frac{n_i}{X}, \]

where \( t_i \) is the time at which the \( i^{th} \) order is put, \( n_i \) is its size, and \( X \) is the initial position. The duration is an indicator of the speed at which the order is liquidated. It also represents the time when the same amount of shares has been sold as there remain shares to sell. For the current parametrization the duration of the liquidation strategy is 2 hours and 13 minutes.

In the following subsections, we perform a comparative static exercise and study how the optimal selling strategy changes when the risk aversion or the price-impact parameters are moved away from the benchmark. This exercise permits to verify the intuition we have about the effect of the movements of the parameters on the characteristics of the optimal selling strategy.

3.1.3 Sensitivity Analysis

**Sensitivity to the risk aversion \( \eta \).** We change the value of the parameter \( \eta \) with respect to the benchmark, from \( \eta = 4 \) to \( \eta = 8 \). The investor becomes more impatient and she is expected to liquidate her position faster to avoid the risk involved in a potential price change. This fact is verified in Figure 3. Comparing the slopes of the curves denoted by GMM estimates and \( \eta = 8 \), we notice that the investor liquidates faster until (around) 12 a.m., time by which more than 95 percent of the initial position has already been liquidated, and slower in the afternoon. This liquidation strategy permits the investor to avoid part of the exposition she faces because of changes in price. Another consequence of the change in the value of the parameter \( \eta \) is that the optimal number of liquidation periods increases from 5 to 6, implying that the first trades happen at earlier times. The first sell order takes place earlier during the day, presumably to avoid the risk of price changes.

At 11:40 a.m., 95 percent of the shares to be sold are liquidated for the benchmark. For the strategy with a risk aversion of \( \eta = 8 \), this only occurs at 12:40 a.m. Clearly, this strategy also started later. The duration decreases by 20 minutes relative to the benchmark, passing from 2 hours 13 minutes to 1 hour and 53 minutes. It means that on average the speed of liquidation increases.
Sensitivity to the parameter $\psi$. We expect that the larger the value of the permanent impact parameter $\psi$, the smaller the optimal number of liquidation periods. The intuition is that each time the investor puts an order, the price is expected to decrease permanently by $\psi$ units. Thus, if the value of the parameter increases, the investor has more incentives to avoid its negative price-impact, by putting less orders. To verify it, the parameter $\psi$ is decreased with respect to the benchmark, passing from $\psi = 0.0022479$ to $\psi = 0.001123932$. As a consequence, the optimal number of liquidation periods increases from 5 to 6.

Figure 4 displays the optimal number of liquidation periods as a function of $\psi$. We observe that the optimal number of liquidation periods is decreasing in $\psi$, and its sensitivity to changes in the parameter value is larger as $\psi$ is smaller. From this graph we also observe that the optimal number of liquidation periods is not very sensitive to changes in the parameter $\psi$.

From Figure 3, we observe that the decrease in the value of the parameter $\psi$ implies that the liquidation speed slightly increases at the beginning and decreases towards the end (see the slopes of the curves). The duration is again smaller than in the benchmark, to 1 hour and 51 minutes. It means that on average the liquidation speed increases since larger lots will be sold.

At the new parametrization the value of the objective function is lower. This is in part a consequence of the increase of the fixed costs which are incurred each time an order is put.

Sensitivity to the parameter $\bar{\psi}$. The parameter $\bar{\psi}$ represents a transitory fixed cost. When this parameter is constant over time, its effect on the objective function is represented by the term $- \sum_{t=1}^{n} \bar{\psi}_{t}(x_{t-1} - x_{t}) = - \sum_{t=1}^{n} \bar{\psi}_{t}n_{t} = -\bar{\psi}X$. Thus, $\bar{\psi}$ affects the objective function through a constant term, implying that changes in its value do not alter the optimal liquidation strategy. It only changes the level of the utility. Numerical experiments corroborate this point.

Sensitivity to the parameter $\lambda$. An increase in the value of the parameter $\lambda$ means that the permanent effect of each unit sold has a larger (negative) impact on price. If the unitary impact is larger, the investor might prefer to liquidate smaller quantities, packaging the total initial position in more trades. It means she should increase the number of liquidation periods. To verify it, the parameter $\lambda$ is increased with respect
to the benchmark, passing from $\lambda = 0.000045$ to $\lambda = 0.00009$. As a consequence, the optimal number of liquidation periods rises from 5 to 7 and, in each of the first periods the investor liquidates around 100 units less than in the benchmark.

Figure 5 displays the optimal number of liquidation periods as function of $\lambda$. We observe that they are increasing in $\lambda$. We also observe that the number of liquidation periods is much more sensible to the parameter $\lambda$ than to the parameter $\psi$.

Figure 3 displays the percentage that has been sold under the optimal liquidation strategy over time, in both cases, when $\lambda = 0.000045$ (benchmark) and when $\lambda = 0.00009$. We observe that at the beginning and at the end of the liquidation period, the investor liquidates slower than in the benchmark, while in the middle she liquidates slightly faster. The duration decreases by 14 minutes with respect to the benchmark, implying that on average the liquidation speed slightly increases in the current parameterization.

**Sensitivity to the parameter $\overline{\lambda}$.** An increase in the value of the parameter $\overline{\lambda}$ causes a reduction of its positive impact on price. This is because $\overline{\lambda}$ is negative and has a negative effect in Equation (4). If the unitary (positive) impact on price is smaller, the investor might prefer to liquidate smaller quantities and to increase the number of liquidation periods. To verify this intuition, we increase the value of the parameter $\overline{\lambda}$ with respect to the benchmark, passing from $\overline{\lambda} = -0.0000307$ to $\overline{\lambda} = -0.000015$. As a consequence, the optimal number of liquidation periods increases from 5 to 6. From Figure 6, we verify that the optimal number of liquidation periods is increasing in $\overline{\lambda}$. Figure 3 displays the percentage that has been sold over time. We observe that it is almost the same as in the benchmark case, implying that the speed of liquidation is not very sensitive to changes in the value of the parameter $\overline{\lambda}$. The duration is now smaller than in the benchmark, slightly below 2 hours. It means that on average the liquidation speed increases, but the change is small as mentioned.

**Sensitivity to $\sigma_y$**. This parameter captures the variability of the expected price in Equation (4), and thus, part of the variability of the total income of liquidation, $\sum_{i=1}^{N} n_i p_i$, in Equation (1). When the parameters $\psi_t, \overline{\psi}_t, \lambda_t, \overline{\lambda}_t, \sigma_{y_t},$ and $\sigma_{\xi_t}$ are constant through the time-of-the-day, as is the case we are studying now, we obtain that plugging $m_t - m_{t-1}$ from Equation (4) in the first difference of Equation (5), yields
\[ \hat{p}_t = p_{t-1} - \psi - \lambda n_t + \sigma_y \omega_t - \tilde{\lambda}(n_t - n_{t-1}) + \xi_t - \xi_{t-1}. \]

Taking variances on both sides, and substituting iteratively \( p_t \) by lagged terms, we have that \( Var[p_t] = t \sigma^2_y. \)

Thus, the effect of increasing the value of \( \sigma^2_y \) is to increase the variance of prices, and as a consequence the variance of the total income of liquidation. This tends to decrease the objective function, so that the investor prefers to decrease the number of liquidation periods and increase the liquidation quantity each period in order to decrease the variance. To illustrate this point, we decrease the value of the parameter \( \sigma_y \) from 0.0058662 (benchmark) to 0.0029331. As a consequence, the optimal number of liquidation periods increases from 5 to 8. Figure 7 displays the evolution of the optimal number of liquidation periods as a function of \( \sigma_y \). We observe that they are decreasing in \( \sigma_y \).

From Figure 3, we see that the liquidation speed decreases during the first half of the day, where approximately 90 percent of the initial position is liquidated. The duration decreases by 15 minutes with respect to the benchmark to 1 hour and 58 minutes. It means that on average the liquidation speed increases.

**Sensitivity to \( \sigma_\xi \).** This parameter appears in the objective function through the variance term, affecting it through the term \(-n \sum_{t=1}^{m} n_t^2 \sigma_\xi^2 \). If the investor decides to liquidate aggressively at the beginning of the day (\( n_t \) is large), it would increase considerably the term \( \sum_{t=1}^{m} n_t^2 \sigma_\xi^2 \) and consequently would decrease the objective function. Therefore, a higher value of \( \sigma_\xi \) should increase the number of liquidation periods and smooth the sold quantities, decreasing the size of the trades at the beginning of the day and increasing them at the end of the day. The effect is the opposite to the effect of \( \sigma_y \), since \( \sigma_y \) affects the hold quantities and not the sold quantities.

We increase the parameter \( \sigma_\xi \) from \( \sigma_\xi = 0.00406 \) (benchmark) to \( \sigma_\xi = 0.00609 \), keeping the other parameters at the benchmark level. We observe that the number of liquidation periods increases from 5 to 6, and that in the first periods the investor liquidates less quantities.

Figure 8 displays the evolution of the optimal number of liquidation periods as a function of \( \sigma_\xi \), revealing that the function is increasing in \( \sigma_\xi \). From Figure 3, we conclude that the liquidation speed is much less sensitive to variations in the parameter \( \sigma_\xi \) than to variations in the parameter \( \sigma_y \). The duration decreases by 30 minutes with respect to the benchmark to 1 hour and 43 minutes, implying that on average the speed

---

\(^{17}\)From the analogy between Equations (2) and (4), we have that \( \sigma^2_y \equiv \sigma^2_T. \)

\(^{18}\)\( V[p_t] = V[p_{t-1}] + \sigma_y^2 = (V[p_{t-1}] + \sigma_y^2) + \sigma_y^2 = \cdots = t \sigma_y^2 + V[p_0] = t \sigma_y^2. \)
of liquidation increases.

Summarizing, \( N^{\text{opt}} \) is decreasing in \( \psi \), invariant to \( \overline{\psi} \), increasing in \( \lambda \), increasing in \( \overline{\lambda} \), decreasing in \( \sigma_y \), and increasing in \( \sigma_\xi \).

### 3.2 Case 2: The parameter \( \lambda_t \) is a function of time

In this section we study how the solution to the optimization problem behaves when the price-impact function \( \lambda_t \) is time varying. This parameter is specially important because it measures the permanent price impact of the order flow, and because it appears in the parameterization of the evolution of most of the related literature. To perform the study, we determine the optimal liquidation when the time varying parameter \( \lambda_t \) follows the increasing hypothetical pattern displayed in the top panel of Figure 9, while the other parameters stay at the benchmark levels. This pattern corresponds to the specification 

\[
\lambda_t = a + bt + ct^2,
\]

where \( a = 0.00025759, b = -0.00000051, c = 0.000001 \), and \( t \) is measured in hours with respect to 9 a.m. Intuitively, we expect that the investor liquidates larger quantities at the beginning of the day, when the price-impact of the order flow is smaller, than late in the afternoon when it is larger.

The bottom panel of Figure 9 displays the corresponding optimal liquidation strategy. We observe that the optimization recommends a liquidation of the position in 5 trades as in the benchmark case. But the main effect of introducing an increasing function for the parameter \( \lambda_t \) is an increase of the liquidation speed. Now, the total position is liquidated in the first hour, during which time the permanent variable impact on price of the order flow is small.

We conclude that taking into account an even relatively shallow variation of \( \lambda \) over the day may have important consequences for the market.

### 3.3 Case 3: All the parameters follow a continuous time function

Empirical evidence has shown that in general the price impact function is not constant over a given trading day. To illustrate this point with our data, we estimate the parameters hour by hour (i.e., the parameters are estimated by the GMM taking as sample only the trades and quotes that occur in a particular hour). As already mentioned, Figure 1 displays the estimated parameters in the case of Orange, and their 95 percent confidence
intervals, for different non-overlapping intervals. We observe that, in general, there are significant differences in the value of the parameters as the time-of-the-day changes. The parameters $\psi$ and $\sigma_\varepsilon$ exhibit U-shapes, their values being significantly greater in periods near the opening and the closing of the trading day, rather than in the middle of the trading day.

The value of the parameter $\lambda$ is significantly greater in the hours just before the closing, than in hours just after the opening or around the middle of the day. It means that the impact on price of a unit sold at the end of the day is larger than the impact on price of the same unit but sold at the beginning or in the middle of the day. If this was the only parameter affecting the price, we would expect that the investor should prefer to liquidate larger quantities at the beginning and in the middle of the day when the price discount is smaller, than at the end of the day when it is larger.

The value of the parameter $\bar{\lambda}$ is significantly smaller near the ending, than near the opening or in the middle of the trading day. The parameter $\sigma_y$ is very stable and its hourly GMM estimates are not significantly different through the time-of-the-day.

The parameters $\psi$ and $\bar{\psi}$ in our model, can be interpreted, respectively, as the parameters $\theta$ and $\phi$ in Madhavan, Richardson, and Roomans (1997). They also estimate them for different non-overlapping intervals. They established a similar U-shape pattern for their parameter $\theta$, and a decreasing pattern for their parameter $\phi$. These patterns are consistent with our results.\(^{19}\)

Presently, we establish the optimal liquidation strategy in the case that the parameters are continuous time functions. To construct the continuous time functions we make a spline interpolation of the hourly estimates. The optimal liquidation strategy is obtained following the methodology described in Section 2.3. Figure 10 displays the optimal liquidation strategy for Orange. We observe that the optimal solution is to liquidate 6400 units at 10:25 a.m., 545 units at 11:50 a.m., 20 units at 1:15 p.m., and 1 unit at 2:40 p.m. From that point on all units will have been sold. The effect of the last two sells is to decrease the time of occurrence of the first 4 liquidations. This feature is reflected in the duration, which decreases by 41 minutes with respect to the benchmark, being now 1 hour and 32 minutes.

\(^{19}\)It is worth clarifying that our model for prices, different from the one of Madhavan, Richardson, and Roomans (1997), includes the impact of the order flow. Therefore, the comparison between the parameters can be done only at the intuitive level. The values of the parameters are not comparable.
4 Conclusion

In this paper, using actual data, we implement an optimal liquidation strategy that takes into account several microstructure elements of the price dynamics. The price dynamics includes (possibly time-varying) permanent as well as transitory impacts of both the order flow and the direction of the order flow. The optimal liquidation strategy minimizes a combination of the expected value and the variance of the total income of liquidation, and endogenizes the quantities sold at each period and the number of liquidation periods over which to liquidate the position.

When the parameters describing the price dynamics are assumed to be constant, the optimal liquidation strategy consists in selling the largest part of the position in the morning: 95% of the initial position should be liquidated before 1 p.m. The duration of this strategy would be quite long, however, around 2 hours and 13 minutes. We also establish that the number of liquidation periods is increasing in the permanent and transitory impact of the order flow, and in the volatility of the transaction price, and is decreasing in the permanent impact of the direction of the order flow, and in the volatility of the news arrival process.

Our empirical evidence suggests that the parameters describing the price dynamics are in fact time varying, and depend on the time-of-the-day. The permanent impact of the direction of the order flow presents a U-shape pattern, whereas the permanent impact of the directed innovation of the order flow increases significantly during the afternoon. In this case, the optimal strategy consists in a much faster liquidation. The duration of this strategy would be only 1 hour and 30 minutes. In selling her position early in the morning, the investor minimizes the impact of her orders on the asset prices.
Appendix 1 - Proof of Proposition 1.

In this appendix, we obtain the solution to the optimization problem described by Equation (1), when the price dynamics is given by Equations (4) and (5).

Replacing \( p_t \) from Equation (5) in the total income of liquidation, \( \sum_{t=1}^{N} n_t p_t \), we have that
\[
\sum_{t=1}^{N} n_t p_t = \sum_{t=1}^{N} n_t m_t + \sum_{t=1}^{N} n_t (-\overline{\psi}_t - \overline{\lambda}_t n_t + \xi_t). \tag{9}
\]

On the other hand, Equation (4) can be written as \( m_t = m_{t-1} + h_t - \lambda_t n_t \), where \( h_t = -\overline{\psi}_t + \sigma_t \tau^{1/2} \omega_t \). Replacing \( m_t \) recursively, we obtain
\[
\sum_{t=1}^{N} n_t m_t = m_0 X + \sum_{t=1}^{N} (h_t - \lambda_t n_t) \left( \sum_{j=t}^{N} n_j \right).
\]
Replacing this result in Equation (9), it follows that
\[
\sum_{t=1}^{N} n_t p_t = m_0 X + \sum_{t=1}^{N} \left( h_t - \lambda_t n_t \right) \left( \sum_{j=t}^{N} n_j \right)
- \sum_{i=1}^{N} \overline{\psi}_i n_t + \sum_{t=1}^{N} n_t (-\overline{\lambda}_t n_t + \xi_t)
\]

Taking the expectation and calculating the variance of \( \sum_{t=1}^{N} n_t p_t \), it holds that
\[
E \left[ \sum_{t=1}^{N} n_t p_t \right] = p_0 X + \sum_{t=1}^{N} E [h_t] \left( \sum_{j=t}^{N} n_j \right) - \sum_{t=1}^{N} \lambda_t n_t \left( \sum_{j=t}^{N} n_j \right)
- \sum_{t=1}^{N} \overline{\psi}_t n_t - \sum_{t=1}^{N} \overline{\lambda}_t n_t^2,
\]
and
\[
V \left[ \sum_{t=1}^{N} n_t p_t \right] = \sum_{t=1}^{N} V [h_t] \left( \sum_{j=t}^{N} n_j \right)^2 + \sum_{t=1}^{N} n_t^2 \sigma_t^2 \xi_t^2.
\]

Replacing \( E \left[ \sum_{t=1}^{N} n_t p_t \right] \) and \( V \left[ \sum_{t=1}^{N} n_t p_t \right] \) in the objective function of Equation (1), grouping common terms, and using the results: \( E [h_t] = -\overline{\psi}_t \) and \( V [h_t] = \sigma_t^2 \tau \), we have that
\[
E \left[ \sum_{t=1}^{N} n_t p_t \right] - \eta V \left[ \sum_{t=1}^{N} n_t p_t \right] = p_0 X - \sum_{t=1}^{N} \psi_t \left( \sum_{j=t}^{N} n_j \right) - \sum_{t=1}^{N} \lambda_t n_t \left( \sum_{j=t}^{N} n_j \right) - \sum_{t=1}^{N} \overline{\psi}_t n_t
- \sum_{t=1}^{N} \overline{\lambda}_t n_t^2 - \eta \sum_{t=1}^{N} \sigma_t^2 \tau \left( \sum_{j=t}^{N} n_j \right)^2 - \eta \sum_{t=1}^{N} \sigma_t^2 \xi_t^2 n_t^2
\]
Taking the derivative with respect to $n_k$, we have that

$$\frac{\partial}{\partial n_k} \left[ -\sum_{t=1}^{N} \psi_t \left( \sum_{j=1}^{N} n_j \right) \right] = -\sum_{t=1}^{k} \psi_t,$$

$$\frac{\partial}{\partial n_k} \left[ -\sum_{t=1}^{N} \lambda_t n_t \left( \sum_{j=1}^{N} n_j \right) \right] = -2\lambda_k n_k - \sum_{t=1}^{k-1} \lambda_t n_t,$$

$$\frac{\partial}{\partial n_k} \left[ -\sum_{t=1}^{N} \overline{\psi}_t n_t \right] = -\overline{\psi}_k,$$

$$\frac{\partial}{\partial n_k} \left[ -\sum_{t=1}^{N} \overline{\lambda}_k n_t^2 \right] = -2\overline{\lambda}_k n_k,$$

$$\frac{\partial}{\partial n_k} \left[ -\eta \sum_{t=1}^{N} \sigma_t^2 \left( \sum_{j=1}^{N} n_j \right)^2 \right] = -\sum_{t=1}^{N} \left( \sum_{j=1}^{\min(t,k)} 2\eta \sigma_j^2 \right) n_t,$$

$$\frac{\partial}{\partial n_k} \left[ -\eta \sum_{t=1}^{N} \sigma_t^2 n_t^2 \right] = -2\eta \sigma_k^2 n_k.$$

Thus, the total derivative of $L = E \left[ \sum_{t=1}^{N} n_t p_t \right] - \eta V \left[ \sum_{t=1}^{N} n_t p_t \right]$ is given by

$$\frac{\partial L}{\partial n_k} = -\sum_{t=1}^{k} \psi_t - 2\lambda_k n_k - \sum_{t=1}^{k-1} \lambda_t n_t - \overline{\psi}_k - 2\overline{\lambda}_k n_k - \sum_{t=1}^{N} \left( \sum_{j=1}^{\min(t,k)} 2\eta \sigma_j^2 \right) n_t - 2\eta \sigma_k^2 n_k.$$

for $k = 1, \cdots, N$. Therefore, the solution to the optimization problem described by Equation (1) is given by the solution to this system of linear equations, subject to the restrictions: $\sum_{t=1}^{N} n_t = X$, and $0 \leq n_k \leq X$ for $k = 1, \cdots, N$. It follows that the optimization problem (1) for the given price dynamics boils down to obtain a solution of the Phase-I of the simplex algorithm. In other words, we are seeking the integers $n_k$ satisfying

$$An = a, \text{ and}$$

$$0 \leq n_i \leq X, \text{ for } i = 1, \cdots, N,$$

where $A$ is an $(N+1, N)$ matrix, $a$ is a vector of dimension $N+1$, and the decision
vector \( \mathbf{n} \) is of dimension \( N \). Specifically, \( A, \mathbf{a}, \) and \( \mathbf{n} \) are defined as

\[
A = \begin{bmatrix}
d_1 - b_1 & -b_1 & -b_1 & \cdots & -b_1 & -b_1 \\
-\lambda_1 - b_1 & d_1 - b_1 & -b_2 & \cdots & -b_2 & -b_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-\lambda_1 & -\lambda_2 - b_2 & -\lambda_3 - b_3 & \cdots & -\lambda_{N-1} - b_{N-1} & -b_N \\
1 & 1 & 1 & \cdots & 1 & 1
\end{bmatrix},
\]

\( \mathbf{a}' = (c_1, \cdots, c_N, X) \), and \( \mathbf{n}' = (n_1, \cdots, n_N) \). There exist very efficient numerical algorithms for this problem.
References


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<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Std. error</th>
<th>t-stat</th>
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<tr>
<td>$\psi$</td>
<td>0.002250</td>
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<td>49.64</td>
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<td>$\bar{\psi}$</td>
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<td>$\lambda$</td>
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<tr>
<td>$\bar{\lambda}$</td>
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<td>$\sigma_y$</td>
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<td>0.000124</td>
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<td>$\sigma_\xi$</td>
<td>0.004061</td>
<td>0.000064</td>
<td>63.99</td>
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Table 1: GMM estimates of the parameters governing the stochastic process of the transaction price the French stock Orange (from January to February 2003).
Figure 1: Hourly estimates of the parameters $\psi$, $\psi$, $\lambda$, $\bar{\lambda}$, $\sigma_y$, and $\sigma_{\xi}$, and their 95 percents confidence intervals for the French stock Orange. On the $x$-axis, 9.5 represents the time interval 9:00–10:00, 10.5 the time interval 10:00–11:00, $\cdots$, 16.5 the time interval 16:00–17:00, and 17.25 the time interval 17:00–17:30. On the $y$-axis are the GMM estimates.
Figure 2: Optimal selling strategy for the benchmark: $X = 6966$, $T = 8.5$ hours, $\eta = 4$, $\psi = 0.0022479$, $\bar{\psi} = 0.0047696$, $\lambda = 0.000045$, $\bar{\lambda} = -0.0000307$, $\sigma_y = 0.0058662$, and $\sigma_\xi = 0.0040590$. Parameters are the GMM estimates for Orange over the period January - February 2003.
Figure 3: Percentage of the initial position, $X = 6966$, that has been sold, against time. The figure represents a comparative static exercise. The benchmark has as parameters $T = 8.5$ hours, $\eta = 4$, $\psi = 0.0022479$, $\overline{\psi} = 0.0047696$, $\lambda = 0.000045$, $\overline{\lambda} = -0.0000307$, $\sigma_y = 0.0058662$, and $\sigma_\xi = 0.0040590$. Deviations are obtained as follows: $\eta$ gets increased to 8, $\psi$ is halved, $\lambda$ doubled, $\overline{\lambda}$ is halved, $\sigma_y$ is halved, and $\sigma_\xi$ is increased up to $\sigma_\xi = 0.006088$. 
Figure 4: Optimal number of liquidation periods for Orange as a function of $\psi$, when $X = 6966$, $T = 8.5$ hours, $\eta = 4$, and the other parameters are the GMM estimates: $\overline{\psi} = 0.0047696$, $\lambda = 0.000045$, $\overline{\lambda} = -0.0000307$, $\sigma_{\psi} = 0.0058662$, and $\sigma_{\xi} = 0.0040590$. 
Figure 5: Optimal number of liquidation periods for Orange as a function of $\lambda$, when $X = 6966$, $T = 8.5$ hours, $\eta = 4$, $\psi = 0.0022479$, $\bar{\psi} = 0.0047696$, $\bar{X} = -0.0000307$, $\sigma_y = 0.0058662$, and $\sigma_\xi = 0.0040590$. 
Figure 6: Optimal number of liquidation periods for Orange as a function of $\bar{\lambda}$, when $X = 6966, T = 8.5$ hours, $\eta = 4, \psi = 0.0022479, \bar{\psi} = 0.0047696, \lambda = 0.000045, \sigma_y = 0.0058662$, and $\sigma_{\xi} = 0.0040590$. 
Figure 7: Optimal number of liquidation periods for Orange as a function of $\sigma_y$, when $X = 6966$, $T = 8.5$ hours, $\eta = 4$, $\psi = 0.0022479$, $\tilde{\psi} = 0.0047696$, $\lambda = 0.000045$, $\bar{\lambda} = -0.0000307$, and $\sigma_z = 0.0040590$. 
Figure 8: Optimal number of liquidation periods for Orange as a function of $\sigma_\xi$, when $X = 6966, T = 8.5$ hours, $\eta = 4, \psi = 0.0022479, \overline{\psi} = 0.0047696, \lambda = 0.000045$, $\overline{\lambda} = -0.0000307$, and $\sigma_y = 0.0058662$. 
Figure 9: Top panel: Functional form of the price-impact function. It corresponds to the specification $\lambda_t = (257.59 - 50.62t + 99.80t^2) \cdot 10^{-6}$. Bottom panel: Optimal liquidation strategy when the parameter $\lambda_t$ follows the pattern displayed in the top panel.
Figure 10: Optimal liquidation strategy for $X = 6966$, $T = 8.5$ hours, $\eta = 4$, and the parameters $\psi_t$, $\overline{\psi}_t$, $\lambda_t$, $\overline{\lambda}_t$, $\sigma_{yt}$, and $\sigma_{\xi t}$, are the continuous time functions that correspond to the spline interpolation of the hourly estimates displayed in Figure 1.