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Optimal Portfolio Allocation under Higher Moments

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Abstract

We evaluate how departure from normality may affect the allocation of assets. A Taylor series expansion of the expected utility allows to focus on certain moments and to compute the optimal portfolio allocation numerically. A decisive advantage of this approach is that it remains operational even for a large number of assets. While the mean-variance criterion provides a good approximation of the expected utility maximisation under moderate non-normality, it may be ineffective under large departure from normality. In such cases, the three-moment or four-moment optimisation strategies may provide a good approximation of the expected utility.

Keywords: *asset allocation*; *stock returns*; *non-normality*; *utility function.*

JEL classification: *C*22, *C*51, *G*12

1. Introduction

It has long been recognised that financial asset returns are non-normal. Strong empirical evidence suggests that returns are driven by asymmetric and/or fat-tailed distributions. On one hand, several authors argued that extreme returns occur too often to be consistent with normality (Mandelbrot, 1963; Fama, 1963; Blattberg and Gonedes, 1974; Kon, 1984; Longin, 1996). On the other hand, crashes are found to occur more often than booms (Fama, 1965; Simkowitz and Beedles, 1978; Singleton and Wingender, 1986; Peiro, 1999).

Subsequently, an abundant literature emerged, questioning the adequacy of the mean-variance criterion proposed by Markowitz (1952) for allocating wealth. Several authors considered how the expected utility function may be approximated by a

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function of higher moments (Arditti, 1967; Levy, 1969; Samuelson, 1970).[1] However, early evidence on the usefulness of additional moments in the allocation process is rather mixed. A number of studies have compared the expected utility (obtained from direct optimisation) with the approximated utility obtained from the mean-variance criterion (Levy and Markowitz, 1979; Pulley, 1981; Kroll *et al.*, 1984; Simaan, 1993b). In most cases, the authors obtained very small differences between the two allocation strategies. Simaan also suggested that the opportunity cost of the mean-variance investment strategy is empirically irrelevant when the opportunity set includes a riskless asset, and small for usual levels of risk aversion when the riskless asset is denied. An explanation of the good performance of the mean-variance criterion in these papers may be that, although returns are non-normal, they are driven by an elliptical distribution. For such a distribution (which includes the normal, Student-t, and Levy distributions), Chamberlain (1983) has shown that the mean-variance approximation of the expected utility is exact for all utility functions.

More recently, techniques have been developed to solve the allocation problem when concern for higher moments is included. Lai (1991), Chunhachinda *et al.* (1997), and Prakash *et al.* (2003) applied the polynomial goal programming (PGP) approach to the portfolio selection with skewness. All these studies provided evidence that incorporating skewness into the portfolio decision causes major changes in the optimal portfolio. A shortcoming of this approach, however, is that the allocation problem solved in the PGP approach cannot be precisely related to an approximation of the expected utility. In particular, the choice of the parameters used to weigh moment objectives is not related to the parameters of the utility function. Consequently, no measure of the quality of the approximation can be inferred from such an exercise. Another drawback is that there is no natural extension to an optimisation problem including moments beyond the third one.

An alternative way of dealing with higher moments in the asset allocation is the use of the Taylor series expansion to derive an approximation of the expected utility function. While this approach has long been used in empirical applications to test the CAPM with higher moments, very few studies have considered the asset allocation problem using Taylor series expansions. Recent contributions are by Harvey *et al.* (2002) and Guidolin and Timmermann (2003). The first study proposed using Bayesian techniques to determine the optimal asset allocation when returns are driven by a skew normal distribution. The second paper investigated how the approximation of the expected utility by a Taylor series expansion can be implemented in the context of returns driven by a Markov-switching model with conditionally normal innovations.

There are several criticisms to the use of Taylor series expansions in the asset allocation context, however. As put forward, for instance, by Lhabitant (1998), the Taylor series expansion may converge to the expected utility under restrictive conditions only. For some utility functions (such as the exponential one), the expansion converges for all possible levels of wealth, while for others (e.g., the power function),

[1]Within a Capital Asset Pricing Model, interest in higher moments goes back to the theoretical work of Rubinstein (1973) and its first empirical implementation by Kraus and Litzenberger (1976). Further work in that area is by Friend and Westerfield (1980), Barone-Adesi (1985), and more recently by Fang and Lai (1997), Harvey and Siddique (2000), Jurczenko and Maillet (2001, 2003).

convergence is obtained only over a given range of final wealth. In addition, the truncation of the Taylor series raises several difficulties. In particular, there is no rule, in general, for selecting the order of the truncation. Worse, the inclusion of an additional moment does not necessarily improve the quality of the approximation (see Brockett and Garven, 1998; Lhabitant, 1998; and Berényi, 2001).

In this paper, we investigate how non-normality of returns may affect the allocation of wealth for utility-maximising investors. To address this issue, we first consider the case of an investor with an exponential utility function. This utility function has been widely used in the literature, because it captures investor's risk aversion in a very simple way. Then, we approximate the expected utility by a Taylor series expansion with two or more moments. Thus, we compare the allocation obtained when the expected utility is directly maximised with the allocation obtained using an expansion up to a given order. We use several criteria for gauging the various optimisation strategies. In addition to a distance measure between portfolio weights, we follow Simaan (1993b), Kandel and Stambaugh (1996), as well as Ang and Bekaert (2002) and estimate the opportunity cost of using a sub-optimal strategy, i.e., an optimisation based on moments, rather than on a direct numerical optimisation of the expected utility function.

In the empirical part of the paper, we pay a particular attention to the statistical properties of the returns investigated. First, we carefully test several hypotheses on returns: the univariate as well as multivariate normality, and the serial correlation of returns and squared returns. In addition, we consider three alternative data sets, which are characterised by different data frequencies and higher-moment features. These data sets contain (1) weekly returns for broad geographical areas; (2) weekly returns from stocks of the S&P100 index; (3) monthly returns for emerging markets. While all these data display departure from normality, they turn out to have very different implications for the quality of the approximation of the expected utility by a function of moments. On one hand, in case of slight non-normality of returns, the different allocation strategies provide basically the same allocation, suggesting that the mean-variance criterion correctly approximates the expected utility. On the other hand, under more severe departure from normality, the difference between the maximisation of the expected utility and the mean-variance criterion may be very large. We provide evidence that in such cases the extension to a three-moment or a four-moment criterion results in a good approximation of the expected utility. The goodness of the approximation is highlighted by the opportunity cost which is found to be very small in all instances for the four-moment criterion.

The remainder of the paper is organised as follows. In Section 2, we describe the asset allocation problem, its approximation by a Taylor series expansion, and the practical implementation of this approximation. In Section 3, we present the various data sets investigated and discuss the statistical properties of returns. Section 4 relies on the asset allocations obtained under direct maximisation of the expected utility and under Taylor series expansion of different orders. The quality of the different approximations is also investigated using several measures. In Section 5, we provide concluding remarks.

2. The Optimal Portfolio Allocation

In this section, we begin with the investor's asset allocation problem, that in general cannot be solved analytically. We then describe how the Taylor series expansion can be used to approximate the allocation problem. Conditions for the expansion to be convergent are detailed for the utility function under study. Last, we indicate how portfolio moments are computed from asset return moments.

2.1. *The investment decision in general*

We consider an investor who allocates her portfolio in order to maximise the expected utility $U(W)$ over her end-of-period wealth W. The initial wealth is arbitrarily set equal to one. There are n risky assets with return vector $R = (R_1, \ldots, R_n)'$ and joint cumulative distribution function $F(R_1, \ldots, R_n)$. End-of-period wealth is given by $W = (1 + r_p)$, with $r_p = \alpha' R$, where the vector $\alpha = (\alpha_1, \ldots, \alpha_n)'$ represents the fractions of wealth allocated to the various risky assets. We assume that the investor does not have access to a riskless asset, implying that the portfolio weights sum to one $\left(\sum_{i=1}^{n} \alpha_i = 1\right)$.[2] In addition, portfolio weights are constrained to be positive, so that short-selling is not allowed.

Formally, the optimal asset allocation is obtained by solving the following problem:

$$\begin{cases} \max\limits_{\{\alpha\}} E[U(W)] = E[U(1+\alpha' R)] = \int \ldots \int U\left(1 + \sum_{i=1}^{n} \alpha_i R_i\right) dF(R_1, \ldots, R_n) \\ \text{s.t.} \sum_{i=1}^{n} \alpha_i = 1 \quad \text{and } \alpha_i \geq 0, \text{ for } i = 1, \cdots n. \end{cases}$$

The n first-order conditions (FOCs) of the optimisation problem are

$$\frac{\partial E[U(W)]}{\partial \alpha} = E\left[R \cdot U^{(1)}(W)\right] = 0 \tag{1}$$

where $U^{(j)}(W)$ denotes the jth derivative of U. We assume that the utility function satisfies the usual properties so that a solution exists and is unique. On one hand, when the empirical distribution of returns is used, the solution to this problem can be easily obtained (see, e.g., Levy and Markowitz, 1979; Pulley, 1981; Kroll *et al.*, 1984). On the other hand, when a parametric joint distribution for returns is used, the FOCs in equation (1) do not have a closed-form solution in general.[3] In such cases, Tauchen and Hussey (1991) provided a numerical solution using quadrature rules. This approach has been applied to normal iid returns by Campbell and Viceira (1999) or to a regime-switching multivariate normal distribution by Ang and Bekaert (2002). The difficulty for non-normal distributions and in particular for distributions that involve asymmetry and fat tails, is that the required number of quadrature points is

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[2]As in Simaan (1993b), we found in early experiments that, when the investor is allowed to invest in a riskless asset, there is virtually no difference between maximising the expected utility and using an optimisation based on moments. The proposed explanation is that, since the weight affected to the riskless asset increases sharply with the degree of risk aversion, the various optimisation strategies cannot display large differences in portfolio weights that would be based on higher-moment properties of returns. For low degrees of risk aversion, the investor puts the emphasis on the expected return of the portfolio, implying that second and higher moments are not taken into account, regardless the optimisation strategy.

[3]The case of normal returns is a trivial exception. When returns are driven by a Markov-switching model with conditionally normal innovations, an analytical solution also exists. Simaan (1993b) also proposes an example with an exponential utility function and a Pearson III distribution in which the expected utility is derived analytically.

likely to increase exponentially with the number of assets. Therefore, solving the optimisation problem using numerical integration becomes tricky for more than two or three assets. For more general distributions of returns, Monte-Carlo simulations may be necessary to approximate the expected utility function.

Since we are primarily interested in measuring the effect of higher moments on the asset allocation, we now approximate the expected utility by a Taylor series expansion around the expected wealth. For this purpose, the utility function is expressed in terms of the wealth distribution, so that

$$E[U(W)] = \int U(w)f(w)dw$$

where $f(w)$ is the probability distribution function of end-of-period wealth, that depends on the multivariate distribution of returns and on the vector of weights α. Hence, the infinite-order Taylor series expansion of the utility function is

$$U(W) = \sum_{k=0}^{\infty} \frac{U^{(k)}(\overline{W})(W-\overline{W})^k}{k!}$$

where $\overline{W} = E[W] = 1 + \alpha'\mu$ denotes the expected end-of-period wealth with $\mu = E[R]$ the expected return vector. Under rather mild conditions (see Lhabitant, 1998, and below), the expected utility is given by

$$E[U(W)] = E\left[\sum_{k=0}^{\infty} \frac{U^{(k)}(\overline{W})(W-\overline{W})^k}{k!}\right] = \sum_{k=0}^{\infty} \frac{U^{(k)}(\overline{W})}{k!} E\left[(W-\overline{W})^k\right]$$

Therefore, the expected utility depends on all central moments of the distribution of the end-of-period wealth.

Necessary conditions for the infinite Taylor series expansion to converge to the expected utility have been explored by Loistl (1976) and Lhabitant (1998). The region of convergence of the series depends on the utility function considered. In particular, the exponential or polynomial utility functions do not put any restriction on the wealth range, while the power utility function converges for wealth levels in the range $[0,2\overline{W}]$. It is worth emphasising that such a range is likely to be large enough for bonds and stocks when short-selling is not authorised. In contrast, it may be too small for options, due to their leverage effect. These results hold for arbitrary return distributions.

Now, since the infinite Taylor series expansion is not suitable for numerical implementation, a solution is to approximate the expected utility by truncating the infinite expansion at a given value $\bar{k}$. For instance, the standard mean-variance criterion proposed by Markowitz (1952) corresponds to the case $\bar{k} = 2$. More generally, an expansion truncated at $\bar{k}$ provides an exact solution to the expected utility when utility is described by a polynomial function of order $\bar{k}$. This result holds because such a utility function depends only on the first $\bar{k}$ moments of the return distribution. This avenue has been followed for instance by Levy (1969), Hanoch and Levy (1970), or Jurczenko and Maillet (2001) for $\bar{k} = 3$ (cubic utility function) and by Benishay (1992) and Jurczenko and Maillet (2003) for $\bar{k} = 4$.

However, it is not clear a priori at what level the Taylor series expansion should be truncated. For instance, Hlawitschka (1994) provides examples in which, even if the infinite expansion converges, adding more terms may worsen the approximation at a given truncation level. In contrast, Lhabitant (1998) describes an example in which omitted terms are of importance. Some arguments put forward by Ederington (1986) as well as Berényi (2001) suggest that introducing the fourth moment will generally improve the approximation of the expected utility.

It should be noticed, at this point, that the approximation of the expected utility by a Taylor series expansion is related to the investor's preference (or aversion) towards all moments of the distribution, that are directly given by derivatives of the utility function. Scott and Horvath (1980) show that, under the assumptions of positive marginal utility, decreasing absolute risk aversion at all wealth levels together with strict consistency for moment preference,[4] the following inequalities hold:

$$U^{(k)}(W) > 0 \quad \forall W \text{ if } k \text{ is odd and}$$
$$U^{(k)}(W) < 0 \quad \forall W \text{ if } k \text{ is even.}$$

Further discussion on the conditions that yield such moment preferences or aversions may be found in Pratt and Zeckhauser (1987), Kimball (1993) and Dittmar (2002). Brockett and Garven (1998) provide examples indicating that expected utility preferences do not necessarily translate into moment preferences. Under rather mild assumptions, a general condition for the smoothness of the convergence of the Taylor series expansion, so that the inclusion of an additional moment will improve the quality of the approximation, is that preference-weighted odd central moments are not dominated by their consecutive preference-weighted even central moments, so that

$$\frac{U^{(2n+1)}(\overline{W})}{2n+1} E[W-\overline{W}]^{2n+1} < \frac{U^{(2n+2)}(\overline{W})}{2n+2} E[W-\overline{W}]^{2n+2,}$$

with n integer. In this case, including skewness and kurtosis always leads to a better approximation of the expected utility.

Focusing on terms up to the fourth one, we obtain

$$E[U(W)] = U(\overline{W}) + U^{(1)}(\overline{W})E[W-\overline{W}] + \frac{1}{2}U^{(2)}(\overline{W})E\left[(W-\overline{W})^2\right] + \frac{1}{3}U^{(3)}(\overline{W})E\left[(W-\overline{W})^3\right] + \frac{1}{4}U^{(4)}(\overline{W})E\left[(W-\overline{W})^4\right] + O(W^4),$$

where $O(W^4)$ is the Taylor remainder. We define the expected return, variance, skewness, and kurtosis of the end-of-period return as[5]

$$\mu_p = E[r_p] = \alpha'\mu,$$
$$\sigma_p^2 = E\left[(r_p-\mu_p)^2\right] = E\left[(W-\overline{W})^2\right],$$
$$s_p^3 = E\left[(r_p-\mu_p)^3\right] = E\left[(W-\overline{W})^3\right],$$
$$\kappa_p^4 = E\left[(r_p-\mu_p)^4\right] = E\left[(W-\overline{W})^4\right].$$

[4]An investor exhibits strict consistency for moment preference if a given moment is always associated with the same preference direction regardless of the wealth level.

[5]These definitions of skewness and kurtosis, as central higher moments, differ from the statistical definitions as standardised central higher moments $E[((r_p-\mu_p)/\sigma_p)^j]$ for $j = 3, 4$.

Hence, the expected utility is simply approximated by the following preference function

$$E\Big[U(W)\Big] \approx U(\overline{W}) + \frac{1}{2}U^{(2)}(\overline{W})\sigma_p^2 + \frac{1}{3!}U^{(3)}(\overline{W})s_p^3 + \frac{1}{4!}U^{(4)}(\overline{W})\kappa_p^4. \qquad (2)$$

Under conditions established by Scott and Horvath (1980), the expected utility depends positively on expected return and skewness and negatively on variance and kurtosis.

2.2. *The case of the CARA utility function*

We consider now the CARA (for Constant Absolute Risk Aversion) utility function. The CARA utility function is defined by:

$$U(W) = -\exp(-\lambda W) \qquad (3)$$

where λ measures the investor's constant absolute risk aversion. This specification has been widely used in the literature because of the appealing interpretation of the associated parameter.[6] The approximation for the expected utility is given by

$$E[U(W)] \approx -\exp(-\lambda\mu_p)\left[1 + \frac{\lambda^2}{2}\sigma_p^2 + \frac{\lambda^3}{3!}s_p^3 + \frac{\lambda^4}{4!}\kappa_p^4\right]. \qquad (4)$$

After some obvious simplifications, the FOCs can be defined respectively as:

$$\mu\left(1 + \frac{\lambda^2}{2}\sigma_p^2 + \frac{\lambda^3}{3!}s_p^3 + \frac{\lambda^4}{4!}\kappa_p^4\right) = \frac{\lambda}{2}\frac{\partial\sigma_p^2}{\partial\alpha} + \frac{\lambda^2}{3!}\frac{\partial s_p^3}{\partial\alpha} + \frac{\lambda^3}{4!}\frac{\partial\kappa_p^4}{\partial\alpha}. \qquad (5)$$

Optimal portfolio weights can be obtained alternatively by maximising expression (4) or by solving equalities (5). Inspection of relation (5) reveals that computing this expression would be rather simple if the variance, skewness, and kurtosis of the portfolio return and the derivatives thereof were known.

2.3. *Solving the asset-allocation problem*

Now we briefly describe how the moments of a portfolio return can be expressed in a very convenient way and how their derivatives may be obtained. First, we define the (n,n^2) co-skewness matrix

$$M_3 = E[(R-\mu)(R-\mu)'\otimes(R-\mu)'] = \{s_{ijk}\}$$

and (n,n^3) co-kurtosis matrix

$$M_4 = E[(R-\mu)(R-\mu)'\otimes(R-\mu)'\otimes(R-\mu)'] = \{\kappa_{ijkl}\}$$

with elements

[6] As an alternative, we also considered the CRRA (for Constant Relative Risk Aversion) utility function, that has also been widely studied in the literature. The Taylor series expansion converges toward the expected utility for levels of wealth ranging between 0 and $2\overline{W}$, an interval that may appear rather restrictive in some applications. Yet, it should be noticed that we found basically the same results with both utility functions, even when we considered very non-normal returns, so that we only report results obtained with the CARA utility function.

$$s_{ijk} = E[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)] \quad i,j,k = 1,\ldots,n$$
$$\kappa_{ijkl} = E[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)(R_l - \mu_l)] \quad i,j,k,l = 1,\ldots,n.$$

where $\otimes$ stands for the Kronecker product. This notation extends the definition for the covariance matrix, which is denoted M_2. Such notation has been used by Harvey *et al.* (2002), Prakash *et al.* (2003), and Athayde and Flôres (2004). For instance, in the case of $n = 3$ assets, the resulting (3,9) co-skewness matrix is

$$M_3 = \left[\begin{array}{ccc|ccc|ccc} s_{111} & s_{112} & s_{113} & s_{211} & s_{212} & s_{213} & s_{311} & s_{312} & s_{313} \\ s_{121} & s_{122} & s_{123} & s_{221} & s_{222} & s_{223} & s_{321} & s_{322} & s_{323} \\ s_{131} & s_{132} & s_{133} & s_{231} & s_{232} & s_{233} & s_{331} & s_{332} & s_{333} \end{array}\right] = [S_{1jk} \quad S_{2jk} \quad S_{3jk}]$$

where S_{1jk} is a short notation for the (n,n) matrix $\{s_{1jk}\}_{j,k=1,2,3}$. Similarly, the (3,27) co-kurtosis matrix is

$$M_4 = \left[K_{11kl} \quad K_{12kl} \quad K_{13kl} | K_{21kl} \quad K_{22kl} \quad K_{23kl} | K_{31kl} \quad K_{32kl} \quad K_{33kl} |\right]$$

where K_{11kp} denotes the (n,n) matrix $\{\kappa_{11kl}\}_{k,l=1,2,3}$. It should be noticed that, because of certain symmetries, not all the elements of these matrices need to be computed. The dimension of the covariance matrix is (n,n), but only $n(n + 1)/2$ elements have to be computed. Similarly, the co-skewness and co-kurtosis matrices have dimensions (n,n^2) and (n,n^3), but involve only $n(n + 1)(n + 2)/6$ elements and $n(n + 1)(n + 2)\,(n + 3)/24$ different elements respectively.[7,8]

Now, using these notations, moments of the portfolio return can be computed in a very tractable way. For a given portfolio weight vector α, expected return, variance, skewness, and kurtosis of the portfolio return are, respectively:

$$\mu_p = \alpha'\mu$$
$$\sigma_p^2 = \alpha' M_2 \alpha$$
$$s_p^3 = \alpha' M_3 (\alpha \otimes \alpha)$$
$$\kappa_p^4 = \alpha' M_4 (\alpha \otimes \alpha \otimes \alpha).$$

The moments of the portfolio return may also be expressed as follows:

[7] For $n = 3$, one has to compute 6 different elements for the covariance matrix, 10 elements for the co-skewness matrix and 15 elements for the co-kurtosis matrix (whereas these matrices have 9, 27, and 81 elements, respectively).

[8] It should be noticed that there exist alternative ways of measuring co-moments. In particular, it may be possible to improve the measure of co-moments by specifying the joint distribution of asset returns. For instance, Simaan (1993a) resorts to a model of asset returns in which all returns depend on a random factor with a non-spherical distribution. Co-skewness between asset returns are then defined as a function of the factor third moment. See also Jurczenko and Maillet (2001).

$$\sigma_p^2 = E\left[\sum_{i=1}^{n} \alpha_i (R_i - \mu_i)(r_p - \mu_p)\right] = \alpha' \Sigma_p$$
$$s_p^3 = E\left[\sum_{i=1}^{n} \alpha_i (R_i - \mu_i)(r_p - \mu_p)^2\right] = \alpha' S_p$$
$$\kappa_p^4 = E\left[\sum_{i=1}^{n} \alpha_i (R_i - \mu_i)(r_p - \mu_p)^3\right] = \alpha' K_p$$

where

$$\Sigma_p = E[(R_i - \mu_i)(r_p - \mu_p)] = M_2 \alpha$$
$$S_p = E[(R_i - \mu_i)(r_p - \mu_p)^2] = M_3(\alpha \otimes \alpha)$$
$$K_p = E[(R_i - \mu_i)(r_p - \mu_p)^3] = M_4(\alpha \otimes \alpha \otimes \alpha)$$

are the $(n,1)$ vectors of covariances, co-skewness, and co-kurtosis between the asset returns and the portfolio return, respectively. These notations are obviously equivalent to the previous ones and they offer the advantage of requiring only small-dimensional vectors.[9]

Notations above allow a straightforward computation of the derivatives with respect to the weight vector, that is:

$$\frac{\partial \mu_p}{\partial \alpha} = \mu$$
$$\frac{\partial \sigma_p^2}{\partial \alpha} = 2 M_2 \alpha$$
$$\frac{\partial s_p^3}{\partial \alpha} = 3 M_3 (\alpha \otimes \alpha)$$
$$\frac{\partial \kappa_p^4}{\partial \alpha} = 4 M_4 (\alpha \otimes \alpha \otimes \alpha).$$

Equations (5) can thus be rewritten as

$$\mu - \delta_1(\alpha)[M_2\alpha] + \delta_2(\alpha)[M_3(\alpha \otimes \alpha)] - \delta_3(\alpha)[M_4(\alpha \otimes \alpha \otimes \alpha)] = 0 \qquad (6)$$

where δ_1, δ_2, and δ_3 are non-linear functions of α, such that $\delta_i(\alpha) = \lambda_i/(i!A)$, $i = 1, 2, 3$, with

$$A = 1 + \frac{\lambda^2}{2}\sigma_p^2 + \frac{\lambda^3}{3!}s_p^3 + \frac{\lambda^4}{4!}\kappa_p^4$$

.

The n equations (6) can be easily solved numerically, using a standard optimisation package. The difficulty to solve this problem is not of the same order as compared to

[9]The notations Σ_p, S_p, and K_p are directly related to the notions of systematic risk, that are widely-used in the literature on higher-moment CAPM, see Kraus and Litzenberger (1976), Hwang and Satchell (1999), Berényi (2001), Jurczenko and Maillet (2001, 2003).

problems involving numerical integration of the utility function. This approach provides an alternative way of solving the asset allocation problem to the PGP approach developed by Lai (1991) and Chunhachinda *et al.* (1997). The main advantage of the approach proposed here is that weights attributed to the various portfolio moments in equation (6) are selected on the basis of the utility function, while they are arbitrarily chosen in the PGP approach. Solving equation (6) also provides an alternative to the rather time-consuming approach based on maximising the expected utility numerically. Here, a very accurate solution is obtained in just a few seconds, even in the case of a large number of assets. The price to pay is that the focus is put on a finite number of moments only.

3. Data

We explore three data sets with very different characteristics of returns. The first data set contains weekly returns for dollar-denominated stock indices for the main geographical areas (North America, Europe, and Asia). It consists of total return indices from Morgan Stanley Capital International (MSCI), from January 1976 through December 2001.[10] These data are very aggregated returns, at the geographical level, their statistical behaviour may, therefore, be expected to be close to a normal distribution. The second data set contains weekly returns for stocks included in the S&P100, from January 1973 through January 2003. The selected stocks were not chosen randomly as in some previous studies, but among those with a moderate to large departure from normality. As a matter of fact, several studies used randomly selected US stocks to illustrate that the mean-variance criterion may be relevant in approximating the expected utility (Levy and Markowitz, 1979, Kroll *et al.*, 1984, Simaan, 1993b). Here, our aim is to show that, in some instances, the widely-used mean-variance criterion may be inappropriate in selecting the optimal portfolio weights. The last data set contains monthly returns for three dollar-denominated emerging-market indices (Hong Kong, South Korea, and Thailand), from February 1975 through June 2002. Emerging markets have been shown to display very non-normal behaviour (Harvey, 1995; Bekaert and Harvey, 1997; Hwang and Satchell, 1999; Jondeau and Rockinger, 2003b).

The data sets are described in Table 1.[11] Let R_{it}, $t = 1, \ldots, T$, denote the log-return of market or asset i at date t. As a preliminary investigation of the data, Table 2 reports univariate and multivariate summary statistics on returns. For each data set, we begin with an estimate of the first four moments and a test of the null hypothesis of normality of the univariate distributions. Since the normality hypothesis is crucial to our analysis, we paid a particular attention to this test. Although a large number of tests have been proposed in the literature, we focus on three well-known procedures, that have proved to be relevant in similar contexts: (1) the statistic (JB) proposed by Jarque and Bera (1980) tests whether skewness and excess kurtosis are jointly zero, using the asymptotic distribution of the estimators. This test is known to be suitable for large samples only, because sample skewness and kurtosis approach normality only very slowly. (2) The omnibus statistic (Omnibus) proposed by Doornik and

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[10] At the end of 1999, the North American, European, and Asian markets represented 47.2%, 30.3%, and 19.4% of total market capitalisation in the world MSCI index.

[11] It is worth emphasising that the frequency of data is often claimed to affect both departure from normality and the serial correlation pattern of returns and volatilities. The latter effect is not an issue in our context, since we focus on the unconditional sample distribution of returns.

Table 1
Description of data sets under investigation.

Data	Description of variables	Period	Frequency	Number of observations
DS1	Returns for dollar-denominated MSCI stock-market indices for the main geographical areas (North America, Europe, and Asia)	From 1/1976 to 12/2001	Weekly	1354
DS2	Returns for stocks included in the S&P100 (Delta Air Lines, Gillette, and Southern)	From 1/1973 to 1/2003	Weekly	1568
DS3	Returns for emerging stock-market indices (Hang Seng index for Hong Kong, KOSPI index for South Korea, and S.E.T. index for Thailand)	From 2/1975 to 6/2002	Monthly	336

Note: This table describes the data sets used to investigate the effect of higher moments on the optimal asset allocation.

Hansen (1994) is based on the approximated finite-sample distribution of skewness and kurtosis. (3) The Kolmogorov-Smirnov statistic (KS) consists in the comparison between the theoretical and the empirical cumulative distribution functions. Since these statistics are likely to have different finite-sample properties (see, e.g., Doornik and Hansen (1994) or Bekaert and Harvey (1997)), we performed Monte-Carlo simulations to evaluate the critical values corresponding to the sample size of each data.[12] As Table 2 (Panel A) confirms, all test statistics consistently reject the normality hypothesis for all return series at the 1% significance level (using both theoretical and size-adjusted critical values).

We also consider a test for serial correlation of returns. Given the high level of serial correlation of squared returns, we use a version of the Ljung-Box statistic (QW), corrected for heteroskedasticity (as suggested by Diebold (1987)), to test for the nullity of the first four serial correlations of returns. This statistics is distributed as a $\chi^2(4)$ under the null of no serial correlation. Inspection of the QW statistics does not reveal significant linear dependencies of returns. Next, we test for heteroskedasticity by regressing squared returns on once lagged squared returns. The standard test statistics proposed by Engle (1982), (LM), is distributed as a $\chi^2(1)$ under the null hypothesis of homoskedasticity and normality. We also consider a test for heteroskedasticity developed by Lee and King (1993), (LK), that does not require normality. Under the null, the LK statistic is distributed as a one-sided $N(0,1)$. The table provides evidence of second-moment dependencies for all data sets, confirming that there is a large amount of heteroskedasticity in the data.[13]

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[12] Monte-Carlo experiments were based on 10,000 replications of samples of size 150, 350, and 1500 of an iid $N(0,1)$ variable. Rejection rates as well as size-adjusted critical values of the various tests used in this paper are available from the authors.

[13] We obtained the same conclusions using different lags (from 2 to 10) for the QW, LM, and LK test statistics.

Table 2a

Statistics on weekly MSCI returns (DS1).

$T = 1355$	North America		Europe		Asia	
Panel A: Univariate statistics						
Moments	stat.	s.e.	stat.	s.e.	stat.	s.e.
Mean	0.251	(0.055)	0.246	(0.058)	0.195	(0.083)
Std	2.094	(0.099)	2.031	(0.086)	2.595	(0.100)
SK	−0.660	(0.342)	−0.604	(0.316)	−0.038	(0.123)
XKu	3.873	(2.137)	2.984	(1.819)	1.488	(0.292)
Normality tests	stat.	p-val.	stat.	p-val.	stat.	p-val.
JB	945.239	(0.000)	585.198	(0.000)	125.404	(0.000)
Omnibus	258.956	(0.000)	181.404	(0.000)	89.080	(0.000)
KS	1.529	[a]	1.759	[a]	1.710	[a]
Serial correlation						
QW(4)	1.854	(0.763)	2.320	(0.677)	10.412	(0.034)
LM(1)	104.460	(0.000)	135.093	(0.000)	20.620	(0.000)
LK(1)	10.215	(0.000)	11.627	(0.000)	4.542	(0.000)

Panel B: Multivariate statistics

Correlation matrix	stat.	s.e.	stat.	s.e.	stat.	s.e.	stat.	s.e.
	x_1		x_2		x_3			
x_1	1		0.526	(0.036)	0.327	(0.035)		
x_2	0.526	(0.036)	1		0.478	(0.025)		
x_3	0.327	(0.035)	0.478	(0.025)	1			
Co-skewness matrix								
	x_1^2		x_2^2		x_3^2		x_1x_2	
x_1	−0.660	(0.342)	−0.523	(0.329)	−0.264	(0.128)		
x_2	−0.585	(0.338)	−0.604	(0.316)	−0.231	(0.117)		
x_3	−0.231	(0.207)	−0.348	(0.168)	−0.038	(0.123)	−0.265	(0.189)
Co-kurtosis matrix								
	x_1^3		x_2^3		x_3^3			
x_1	6.873	(2.137)	4.376	(2.008)	1.716	(0.482)		
x_2	4.712	(2.164)	5.984	(1.819)	2.217	(0.393)		
x_3	2.956	(1.326)	3.027	(0.957)	4.488	(0.292)		
	$x_1x_2^2$		$x_1x_3^2$		$x_2x_3^2$		$x_1^2x_2$	
x_1	4.449	(2.075)	2.645	(0.717)				
x_2			1.960	(0.668)	2.516	(0.561)		
x_3	2.404	(1.101)					2.540	(1.199)

Multivariate Normality test	stat.	s.e.
Omnibus	309.812	(0.000)
Small	287.877	(0.000)
Mardia	1229.963	(0.000)

Table 2b

Statistics on weekly S&P100 stock returns (DS2).

$T = 1568$	Delta Air Lines		Gillette		Southern	
Panel A: Univariate statistics						
Moments	stat.	s.e.	stat.	s.e.	stat.	s.e.
Mean	−0.021	(0.123)	0.175	(0.098)	0.097	(0.061)
Std	5.032	(0.272)	4.188	(0.263)	2.776	(0.116)
SK	−0.888	(0.506)	−1.347	(0.925)	0.197	(0.227)
XKu	8.240	(4.234)	16.875	(10.519)	4.264	(0.904)
Normality tests	stat.	p-val.	stat.	p-val.	stat.	p-val.
JB	4641.44	(0.000)	19077.67	(0.000)	1198.07	(0.000)
Omnibus	801.032	(0.000)	1674.067	(0.000)	503.176	(0.000)
KS	2.019	[a]	2.386	[a]	2.622	[a]
Serial correlation						
QW(4)	4.554	(0.336)	5.574	(0.233)	4.425	(0.352)
LM(1)	13.340	(0.000)	0.152	(0.697)	21.910	(0.000)
LK(1)	3.645	(0.000)	0.391	(0.348)	4.685	(0.000)

Panel B: Multivariate statistics

Correlation matrix	stat.	s.e.	stat.	s.e.	stat.	s.e.	stat.	s.e.
	x_1		x_2		x_3			
x_1	1		0.147	(0.033)	0.283	(0.035)		
x_2	0.282	(0.034)	1		0.234	(0.036)		
x_3	0.147	(0.033)	0.234	(0.036)	1			
Co-skewness matrix								
	x_1^2		x_2^2		x_3^2		x_1x_2	
x_1	−0.888	(0.506)	−0.472	(0.407)	−0.130	(0.109)		
x_2	−0.237	(0.167)	−1.347	(0.925)	−0.179	(0.200)		
x_3	0.002	(0.100)	−0.457	(0.446)	0.197	(0.227)	−0.217	(0.185)
Co-kurtosis matrix								
	x_1^3		x_2^3		x_3^3			
x_1	11.240	(4.234)	6.687	(4.788)	0.984	(0.587)		
x_2	1.980	(0.809)	19.875	(10.519)	1.947	(0.946)		
x_3	0.256	(0.644)	6.459	(5.054)	7.264	(0.904)		
	$x_1x_2^2$		$x_1x_3^2$		$x_2x_3^2$		$x_1^2x_2$	
x_1	3.320	(1.970)	1.696	(0.397)				
x_2			1.361	(0.928)	3.673	(2.122)		
x_3	2.721	(2.119)					1.231	(0.899)

Multivariate Normality test	stat.	s.e.
Omnibus	2900.84	(0.000)
Small	2876.56	(0.000)
Mardia	14997.99	(0.000)

Table 2c

Statistics on monthly emerging stock-market returns (DS3).

$T = 336$	Hong Kong		South Korea		Thailand	
Panel A: Univariate statistics						
Moments	stat.	s.e.	stat.	s.e.	stat.	s.e.
Mean	1.078	(0.438)	0.737	(0.497)	0.898	(0.602)
Std	8.698	(0.657)	7.861	(0.756)	9.834	(0.872)
SK	−0.684	(0.304)	0.052	(0.380)	0.339	(0.194)
XKu	2.856	(1.066)	2.522	(1.159)	1.473	(0.391)
Normality tests	stat.	p-val.	stat.	p-val.	stat.	p-val.
JB	140.425	(0.000)	89.217	(0.000)	36.805	(0.000)
Omnibus	44.593	(0.000)	58.333	(0.000)	22.465	(0.000)
KS	1.300	[a]	1.126	[a]	1.214	[a]
Serial correlation						
QW(4)	4.732	(0.316)	0.525	(0.971)	3.341	(0.503)
LM(1)	3.449	(0.063)	32.935	(0.000)	36.903	(0.000)
LK(1)	1.868	(0.031)	5.748	(0.000)	5.956	(0.000)

Panel B: Multivariate statistics

Correlation matrix	stat.	s.e.	stat.	s.e.	stat.	s.e.	stat.	s.e.
	x_1		x_2		x_3			
x_1	1		0.254	(0.064)	0.271	(0.067)		
x_2	0.254	(0.064)	1		0.228	(0.060)		
x_3	0.271	(0.067)	0.228	(0.060)	1			
Co-skewness matrix								
	x_1^2		x_2^2		x_3^2		x_1x_2	
x_1	−0.684	(0.304)	0.088	(0.157)	−0.097	(0.095)		
x_2	0.056	(0.166)	0.052	(0.380)	0.014	(0.079)		
x_3	−0.184	(0.152)	0.017	(0.091)	0.339	(0.194)	−0.008	(0.085)
Co-kurtosis matrix								
	x_1^3		x_2^3		x_3^3			
x_1	5.856	(1.066)	0.776	(0.781)	0.855	(0.303)		
x_2	1.469	(0.427)	5.522	(1.159)	0.495	(0.249)		
x_3	1.826	(0.524)	0.298	(0.532)	4.473	(0.391)		
	$x_1x_2^2$		$x_1x_3^2$		$x_2x_3^2$		${x_1}^2x_2$	
x_1	1.967	(0.360)	1.355	(0.247)				
x_2			0.446	(0.170)	1.227	(0.161)		
x_3	0.867	(0.243)					0.690	(0.196)
Multivariate Normality test			stat.	s.e.				
Omnibus			151.654	(0.000)				
Small			151.643	(0.000)				
Mardia			307.998	(0.000)				

Note: Panel A of this table reports univariate summary statistics on returns. Mean, Std, Sk, and XKu denote the mean, the standard deviation, the skewness, and the excess kurtosis of returns, respectively. Standard errors are computed with the GMM-based procedure proposed by Bekaert and Harvey (1997). JB, Omnibus, and KS stand for the Jarque-Bera statistic (Jarque and Bera, 1980), the omnibus statistic (Doornik and Hansen, 1994), and the Kolmogorov-Smirnov statistic for the test of the null hypothesis of a normal distribution. QW(4) is the Box-Ljung statistic for serial correlation, corrected for heteroskedasticity, computed with 4 lags. Under the null of no serial correlation, it is distributed as a $\chi^2(4)$. LM(1) and LK(1) are the Engle (1982) and Lee and King (1993) statistics for heteroskedasticity. Under the null of no serial correlation of squared returns, the statistics are distributed as a $\chi^2(1)$ and a $N(0,1)$ respectively. Panel B of the table reports multivariate summary statistics on returns. We present the non-redundant elements of correlation, co-skewness, and co-kurtosis matrices. Standard errors are computed with a GMM-based procedure of Bekaert and Harvey (1997). Omnibus, Small, and Mardia stand for the multivariate omnibus statistic (Doornik and Hansen, 1994), and the statistics proposed by Small (1980) and Mardia (1970) respectively. Under the null of multivariate normality, the statistics are distributed as χ^2 with $2n$, $2n$, and $n(n+1)(n+2)/6+1$ degrees of freedom, respectively. [a]means that the statistic is significant at the 1% level.

Then, we turn to the multivariate analysis. In Table 2 (Panel B), we report all non-redundant elements of the correlation, co-skewness, and co-kurtosis matrices, estimated simultaneously using the relations:

$$
\begin{aligned}
&E[R_{it}] = \mu_i \quad i = 1,\ldots,n \\
&E[(R_{it}-\mu_i)(R_{jt}-\mu_j)] = \sigma_{ij} \quad i = 1,\ldots,n, j = i+1,\ldots,n \\
&E[(R_{it}-\mu_i)(R_{jt}-\mu_j)(R_{kt}-\mu_k)] = s_{ijk} \quad i,j,k = 1,\ldots,n \\
&E[(R_{it}-\mu_i)(R_{jt}-\mu_j)(R_{kt}-\mu_k)(R_{lt}-\mu_l)] = \kappa_{ijkl} \quad i,j,k,l = 1,\ldots,n.
\end{aligned}
$$

We also report finite-sample standard errors computed with the GMM-based procedure of Bekaert and Harvey (1997).[14] Finally, we perform several multivariate normality tests. As compared with the univariate tests discussed above, these multivariate tests incorporate hypotheses on the co-skewness and co-kurtosis matrices. We focus on three tests for multivariate normality: (1) the omnibus statistics described by Doornik and Hansen (1994) extents the univariate test discussed above. This test computes higher moments on variates which have been preliminary transformed to approximately independent normals. (2) The statistics proposed by Small (1980) weighs the marginal skewness and kurtosis coefficients of the raw variables by their approximate correlations. (3) The statistics described by Mardia (1970) is based on multivariate measures of skewness and kurtosis. These statistics are described in Doornik and Hansen (1994). The distribution of the first two statistics is known to be approximately a chi-square, while the last statistic is distributed only asymptotically as a chi-square. Monte-Carlo experiments confirmed that these tests are correctly sized even in small sample.

[14] A number of studies suggest to improve the asset allocation by using more robust definitions of empirical moments. These techniques (such as the shrinkage method of Ledoit and Wolf 2004, or the range-based estimator of Brandt and Diebold, 2003) have proved to be very efficient in computing conditional covariance matrices. Their use in a context with higher moments remains an open issue that we leave for further investigation.

In Table 2 (Panel B), we observe very contrasted patterns of skewness in the data sets under study. In data set DS1, all co-skewness between MSCI global indices are found to be negative, most of them being statistically significant. In data set DS2, most co-skewness between S&P stocks are negative, although not significantly. Finally, in data set DS3, no clear pattern emerges between emerging markets, with most co-skewness being insignificant.

The broad picture for co-kurtosis is much clearer, since a number of co-kurtosis are significantly larger than their expected value under multivariate normality.[15] Largest values are found for MSCI global indices (DS1) and S&P stocks (DS2). The three multivariate normality tests reject the null hypothesis for all data sets at any usual significance level.

4. Portfolio Allocation under Non-normality

In this section, we address two related issues how asset allocation is modified when returns are driven by a non-normal distribution. The first issue is how the allocation is altered when the investor is concerned by more than just the first two moments of returns. The second issue is how far the allocation based on a truncated expansion of utility is from the allocation based on a direct maximisation of the expected utility. For this purpose, we follow the approach of Simaan (1993b) and compute, for a given parameter set, the optimal asset allocation for the expected utility function (using direct maximisation) and for the Taylor series expansions with $\bar{k} = 2, 3, 4$ (based on moment computation), corresponding to the cases where we incorporate information on volatility, skewness, and kurtosis respectively.

Table 3 reports results for optimally selected portfolios for several values of the risk aversion parameter λ ranging between 1 and 20. This range covers most values investigated in the literature. Results include the optimal portfolio weights and the absolute distance between portfolio weights obtained with the strategies based on moments $(\hat{\alpha}_i)$ and the strategy based on the expected utility $\left(\alpha_i^*\right)$, that is $norm = \Sigma_{i=1}^n |\hat{\alpha}_i - \alpha_i^*|$. We also report the first four moments of the optimal portfolio returns and the opportunity cost (or optimisation premium) of investment strategies based on moments rather than on expected utility. If we denote r_p^* the optimal portfolio return obtained by direct optimisation of the expected utility, and $\hat{r}_p$ the optimal portfolio return from a given approximation, then the opportunity cost θ is defined as the return that needs to be added to the portfolio return of the approximation, so that the investor becomes indifferent with the direct optimisation

$$E[U(1 + \hat{r}_p + \theta)] = E[U(1 + r_p^*)]. \tag{7}$$

The reported premium θ is obtained by solving equation (7) numerically.

[15] In the normal case, all co-skewness are equal to zero, while co-kurtosis are given by $\kappa_{iiii} = 3$, $\kappa_{iijj} = 1.125$, $\kappa_{iiij} = 0.75$, $\kappa_{iijk} = 0.375$, and $\kappa_{ijkl} = 0$, for $i \neq j \neq k \neq l$, see Kotz *et al.* (2000).

4.1. *MSCI returns on geographical areas (DS1)*

Results for MSCI returns are reported in Table 3a. Very small differences are found in the portfolio weights when the optimisation is based on the first two, three, or four moments. As reported in the table, for a given risk aversion, λ, optimal weights barely change when the concern for an additional moment is introduced. This result, also illustrated in Figure 1, may be explained by the fact that returns for these large geographic areas depart only slightly from normality. Yet, it appears clearly that changes in portfolio weights follow from the values of skewness and kurtosis reported for returns. For instance, we observe that the weight of Asia is slightly higher when concern for skewness is introduced in the optimisation criterion. This is related to the fact that returns in North America and Europe have a relatively large negative skewness while skewness in Asia is close to 0.

Table 3a

Optimal allocation for weekly MSCI returns (DS1).

	Portfolio weights				Moments of portfolio return				Opportunity
λ	α_1	α_2	α_3	Norm	μ_p	${\sigma_p}^2$	${s_p}^3$	${\kappa_p}^4$	cost
Panel A: Direct optimization									
1	0.607	0.393	0.000	–	0.249	1.822	−0.858	8.166	–
2	0.537	0.463	0.000	–	0.249	1.806	−0.863	8.184	–
5	0.489	0.498	0.013	–	0.248	1.794	−0.858	8.158	–
10	0.443	0.440	0.117	–	0.242	1.751	−0.824	8.076	–
15	0.421	0.418	0.161	–	0.240	1.744	−0.800	7.944	–
20	0.401	0.406	0.194	–	0.238	1.742	−0.778	7.802	–
Panel B: Taylor expansion up to order 2									
1	0.608	0.392	0.000	0.002	0.249	1.822	−0.858	8.165	0.000
2	0.538	0.462	0.000	0.003	0.249	1.806	−0.863	8.185	0.000
5	0.495	0.502	0.003	0.020	0.248	1.800	−0.860	8.153	0.002
10	0.455	0.448	0.098	0.039	0.243	1.757	−0.833	8.120	0.002
15	0.441	0.430	0.129	0.064	0.242	1.749	−0.819	8.054	0.002
20	0.435	0.421	0.144	0.099	0.241	1.746	−0.810	8.011	0.002
Panel C: Taylor expansion up to order 3									
1	0.607	0.393	0.000	0.000	0.249	1.822	−0.858	8.166	0.000
2	0.537	0.463	0.000	0.000	0.249	1.806	−0.863	8.184	0.000
5	0.491	0.498	0.011	0.004	0.248	1.795	−0.859	8.158	0.001
10	0.448	0.441	0.111	0.012	0.243	1.753	−0.827	8.093	0.001
15	0.433	0.420	0.148	0.027	0.241	1.745	−0.808	8.000	0.001
20	0.424	0.408	0.168	0.051	0.240	1.743	−0.796	7.932	0.001
Panel D: Taylor expansion up to order 4									
1	0.607	0.393	0.000	0.000	0.249	1.822	−0.858	8.166	0.000
2	0.537	0.463	0.000	0.000	0.249	1.806	−0.863	8.184	0.000
5	0.490	0.498	0.013	0.000	0.248	1.794	−0.858	8.158	0.000
10	0.445	0.440	0.116	0.003	0.242	1.752	−0.825	8.081	0.000
15	0.426	0.419	0.156	0.011	0.240	1.744	−0.803	7.968	0.000
20	0.413	0.407	0.180	0.028	0.239	1.742	−0.788	7.874	0.000

Table 3b

Optimal allocation for monthly S&P100 stock returns (DS2).

	Portfolio weights				Moments of portfolio return				Opportunity
λ	α_1	α_2	α_3	Norm	μ_p	${\sigma_p}^2$	${s_p}^3$	${\kappa_p}^4$	cost
Panel A: Direct optimization									
1	0.000	0.635	0.365	–	0.146	3.059	−1.271	19.121	–
2	0.000	0.434	0.567	–	0.131	2.666	−0.887	14.614	–
5	0.016	0.291	0.694	–	0.118	2.528	−0.488	10.814	–
10	0.081	0.187	0.733	–	0.102	2.466	−0.294	9.034	–
15	0.107	0.120	0.773	–	0.094	2.479	−0.155	7.971	–
20	0.126	0.067	0.808	–	0.087	2.513	−0.058	7.331	–
Panel B: Taylor expansion up to order 2									
1	0.000	0.647	0.354	0.023	0.147	3.088	−1.283	19.274	0.233
2	0.000	0.449	0.551	0.032	0.132	2.688	−0.929	15.070	0.129
5	0.013	0.325	0.661	0.069	0.121	2.548	−0.593	11.718	0.117
10	0.074	0.259	0.667	0.146	0.108	2.473	−0.509	10.716	0.140
15	0.093	0.238	0.669	0.235	0.104	2.459	−0.478	10.307	0.154
20	0.103	0.228	0.670	0.323	0.103	2.454	−0.463	10.103	0.149
Panel C: Taylor expansion up to order 3									
1	0.000	0.636	0.364	0.003	0.147	3.063	−1.272	19.141	0.029
2	0.000	0.436	0.564	0.006	0.131	2.670	−0.894	14.695	0.022
5	0.013	0.302	0.685	0.022	0.119	2.536	−0.516	11.059	0.033
10	0.072	0.221	0.708	0.068	0.106	2.468	−0.381	9.719	0.052
15	0.089	0.185	0.726	0.130	0.101	2.461	−0.305	9.047	0.056
20	0.095	0.162	0.743	0.191	0.098	2.464	−0.249	8.622	0.043
Panel D: Taylor expansion up to order 4									
1	0.000	0.635	0.365	0.000	0.146	3.059	−1.271	19.122	0.002
2	0.000	0.434	0.566	0.001	0.131	2.666	−0.888	14.625	0.003
5	0.016	0.294	0.691	0.007	0.118	2.530	−0.497	10.894	0.010
10	0.079	0.203	0.718	0.034	0.103	2.464	−0.341	9.376	0.024
15	0.102	0.160	0.738	0.079	0.097	2.461	−0.255	8.600	0.029
20	0.117	0.131	0.752	0.130	0.093	2.467	−0.200	8.136	0.021

Table 3c

Optimal allocation for monthly emerging stock-market returns (DS3).

	Portfolio weights				Moments of portfolio return				Opportunity
λ	α_1	α_2	α_3	Norm	μ_p	${\sigma_p}^2$	${s_p}^3$	${\kappa_p}^4$	cost
Panel A: Direct optimization									
1	1.000	0.000	0.000	–	0.416	8.106	−0.652	5.526	–
2	0.757	0.000	0.243	–	0.264	6.792	−0.441	4.802	–
5	0.532	0.024	0.445	–	0.112	6.114	−0.133	3.724	–
10	0.393	0.129	0.479	–	−0.022	5.846	−0.070	3.139	–
15	0.334	0.170	0.497	–	−0.077	5.815	−0.034	2.959	–
20	0.300	0.194	0.506	–	−0.108	5.818	−0.015	2.883	–

Panel B: Taylor expansion up to order 2									
1	1.000	0.000	0.000	0.000	0.416	8.106	−0.652	5.526	0.000
2	0.787	0.000	0.213	0.060	0.282	6.924	−0.478	4.933	0.516
5	0.580	0.026	0.394	0.101	0.141	6.196	−0.217	3.950	0.385
10	0.476	0.114	0.410	0.168	0.037	5.914	−0.179	3.460	0.328
15	0.448	0.138	0.414	0.229	0.009	5.864	−0.164	3.332	0.298
20	0.439	0.146	0.415	0.277	0.000	5.850	−0.158	3.291	0.276
Panel C: Taylor expansion up to order 3									
1	1.000	0.000	0.000	0.000	0.416	8.106	−0.652	5.526	0.000
2	0.763	0.000	0.237	0.013	0.268	6.819	−0.449	4.831	0.107
5	0.554	0.006	0.440	0.044	0.134	6.182	−0.139	3.822	0.150
10	0.434	0.089	0.477	0.083	0.022	5.912	−0.081	3.305	0.122
15	0.393	0.101	0.505	0.136	−0.009	5.888	−0.038	3.171	0.099
20	0.376	0.087	0.537	0.213	−0.014	5.922	0.003	3.151	0.091
Panel D: Taylor expansion up to order 4									
1	1.000	0.000	0.000	0.000	0.416	8.106	−0.652	5.526	0.000
2	0.758	0.000	0.242	0.001	0.264	6.794	−0.442	4.804	0.009
5	0.534	0.024	0.442	0.006	0.114	6.118	−0.138	3.736	0.019
10	0.402	0.128	0.470	0.019	−0.015	5.847	−0.083	3.168	0.025
15	0.353	0.167	0.480	0.038	−0.064	5.808	−0.056	3.001	0.022
20	0.329	0.187	0.484	0.057	−0.088	5.801	−0.043	2.934	0.012

Note: This table reports statistics for optimal portfolios for several values of the risk-aversion parameter λ ranging from 1 to 20. We report the optimal weights (α_i, $i = 1, 2, 3$); the absolute distance between portfolio weights obtained with moment-based strategies ($\hat{\alpha}_i$) and with expected-utility maximization $\left(\alpha_i^*\right)$, defined as $norm = \Sigma_{i=1}^{n} \left|\hat{\alpha}_i - \alpha_i^*\right|$; the first four moments of portfolio returns; and the opportunity cost θ, defined by $E[U(1+\hat{r}_p+\theta)] = E\left[U\left(1+r_p^*\right)\right]$, where r_p^* denotes the optimal portfolio return obtained by direct optimization of the expected utility, and $\hat{r}_p$ the optimal portfolio return from a moment-based strategy.

In addition, we notice that the difference between optimal weights found for the mean-variance criterion and for the expected utility is small for all risk aversion levels. Even for a very large degree of risk aversion, the difference between optimal weights does not exceed 3.5 percentage points. Hence, the norm between portfolio weights is lower than 0.1 for the mean-variance criterion and even lower than 0.03 for the four-moment criterion. Such a closeness between portfolios translates in an approximately zero opportunity cost. It is lower than 0.2 cent per dollar invested, even for the mean-variance criterion and a large risk aversion.

Data set DS1 illustrates that even when returns are found, on statistical grounds, to be non-normal (with a very strong rejection of the null hypothesis of normality), the mean-variance criterion may be relevant for approximating the expected utility criterion. In fact, this finding is consistent with empirical evidence provided by Levy and Markowitz (1979), Kroll *et al.* (1984) and Simaan (1993b) or with the theoretical result of Chamberlain (1983), who shows that the mean-variance criterion provides an exact approximation of the expected utility for the whole elliptical-distribution family. Therefore, even returns driven by a Student-t or Levy distribution would yield such a result.

4.2. *S&P100 stock returns (DS2)*

As reported in Table 2b, the departure from normality of the selected stocks is much more pronounced than for MSCI indices, resulting from both asymmetry and fat tails.

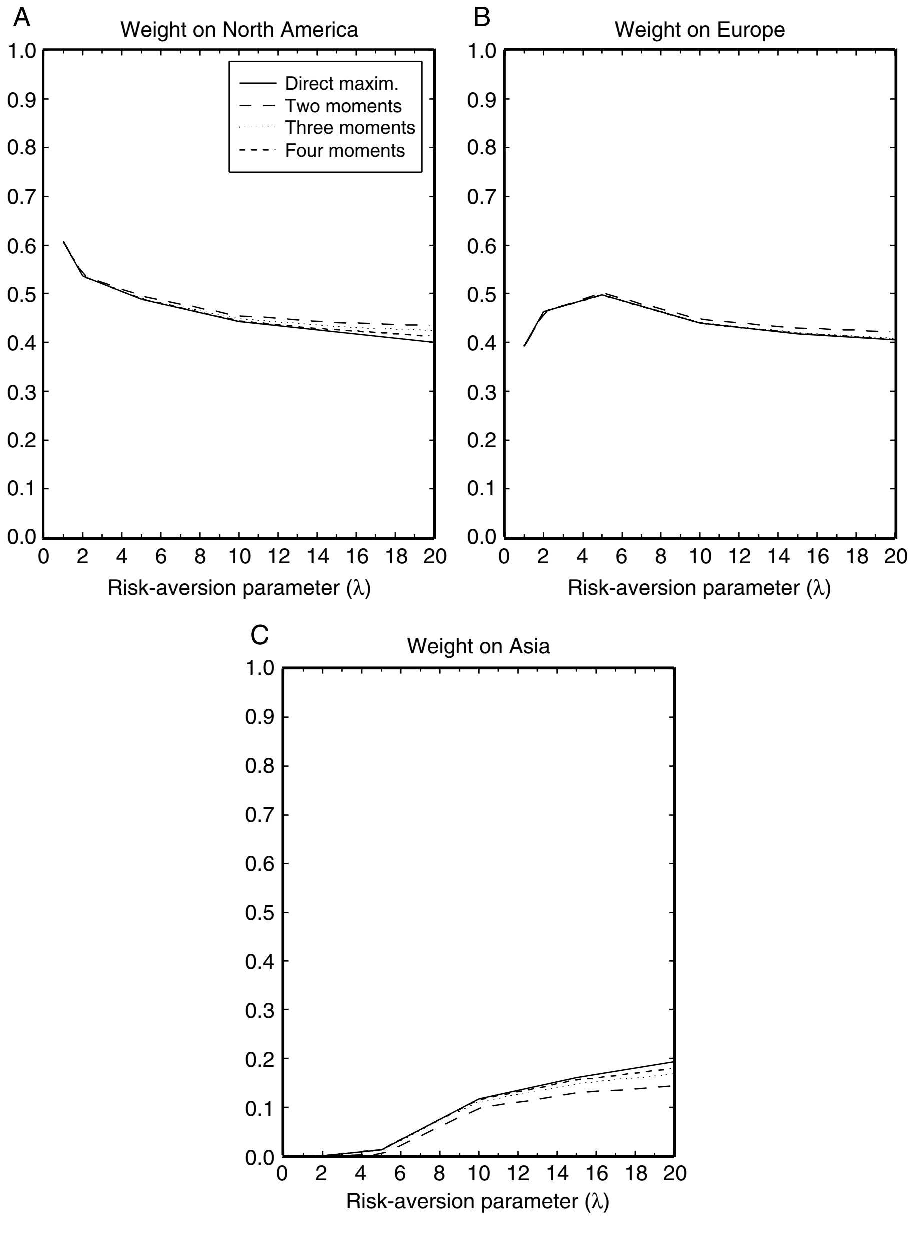

Fig. 1. This figure displays the optimal weights obtained with the different optimisation strategies as a function of the risk-aversion parameter λ. Here the data consists in three MSCI indices.

Consequently, as Table 3b and Figure 2 show, the mean-variance criterion provides a poor approximation of the optimal portfolio weights found by the direct maximisation of the expected utility. For very low levels of risk aversion, the difference of weights is rather moderate, because both optimisation strategies exclude the first stock (Delta Air Lines). Yet, for moderate to large risk aversion levels, the mean-variance criterion puts an excessive weight on the second stock (Gillette) on the basis of its large return and low variance, failing to account for its very negative skewness and large kurtosis.

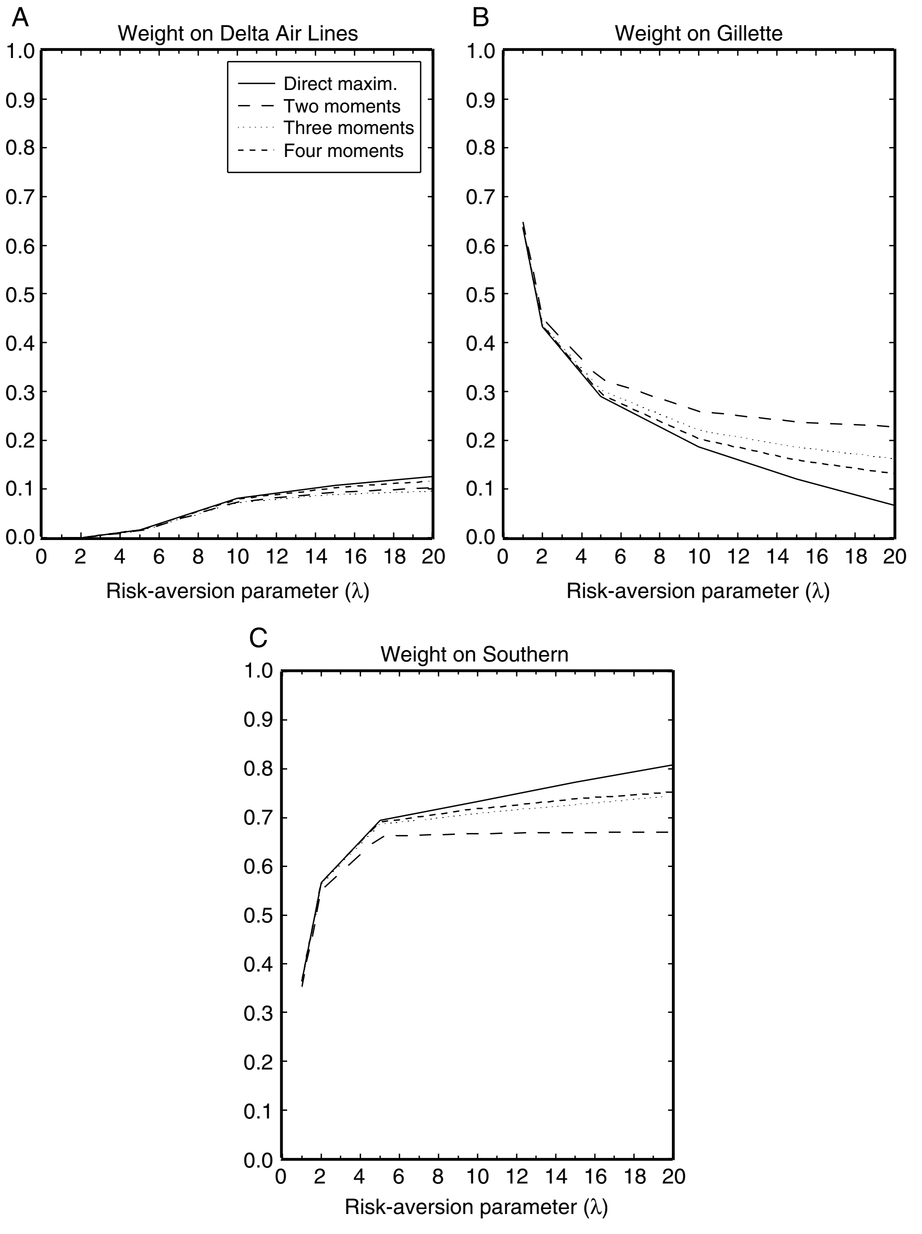

Fig. 2. This figure displays the optimal weights obtained with the different optimisation strategies as a function of the risk-aversion parameter λ. Here the data consists in three components of the S&P100.

Hence, the distance between the portfolio weights obtained with the two optimisation strategies is larger than 0.2 for $\lambda \geq 15$. When concern for skewness and kurtosis is introduced in the optimisation, the gap with the expected utility maximisation significantly reduces. With the four-moment strategy, the distance with expected utility decreases to about 0.15.

We also observe that portfolio moments obtained with an expansion of the utility function are rather distant from those obtained with maximisation of the expected utility. This result can be directly related to the large differences in skewness and kurtosis between the selected stocks. In particular, when only mean and variance are considered, the portfolio skewness is excessively negative while portfolio kurtosis is excessively large. Introducing a concern for higher moments partly fills the gap between portfolio moments obtained with the moment strategy and expected utility maximisation.

Finally, opportunity costs are rather large for the mean-variance strategy, above 10 cents per dollar invested. The optimisation premium lies around 5 cents for the three-moment strategy and only 2 cents for the four-moment strategy. The opportunity cost we obtain for the mean-variance criterion is much larger than the cost reported by Simaan (1993b) for such an optimisation strategy. For similar levels of risk aversion, he found optimisation premia that did not exceed 1 cent. It should be noticed, however, that his data is expected to be much closer to normality than ours.[16]

4.3. *Emerging-market returns (DS3)*

We turn to the case of emerging markets, that are characterised by very large departures from normality. As indicated in Table 3c, in this context, the mean-variance criterion may yield inconsistent portfolio weights, as compared to the expected utility maximisation. Even for moderate risk aversion levels, the difference between optimal weights may exceed 5 percentage points (see also Figure 3). Worse, for $\lambda \geq 15$, the weight allocated to the Hong Kong index (α_1) is larger than 0.43 with the mean-variance criterion, but does not exceed 0.34 with direct maximisation of the expected utility. Even for moderate risk aversion, the norm between the two optimisation strategies is large (more than 0.1 for $\lambda \geq 5$).

Interestingly, introducing skewness in the moment criterion barely improves the allocation, suggesting that asymmetry is not the main source of departure from normality (This result is confirmed in Table 2c, that reports insignificant co-skewness parameters). In contrast, the four-moment criterion provides a very good approximation of the expected utility. Even for large risk aversion, the two optimisation strategies yield close optimal weights, with the norm below 0.06 for all values of λ.

The deficiency of the mean-variance criterion also transpires in the moments of the portfolio return. This strategy is able to yield a slightly larger expected return than the expected utility maximisation, for all risk aversion levels, but at the price of a larger variance, lower skewness, and larger kurtosis. As expected, the three-moment strategy does not succeed in reducing the fat tails of the portfolio return significantly.

Finally, these results translate into a large opportunity cost of adopting sub-optimal investment strategies. The inability of the mean-variance criterion to cope with higher moments is found to cost more than 25 cents per dollar invested for all risk aversion levels and as much as 52 cents for $\lambda = 2$. The opportunity cost of the three-moment strategy is significantly lower, to about 10 cents, while the cost of the four-moment strategy does not exceed 2.5 cents per dollar invested.

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[16] His study uses monthly returns of a random selection of ten stocks chosen among the CRSP Database.

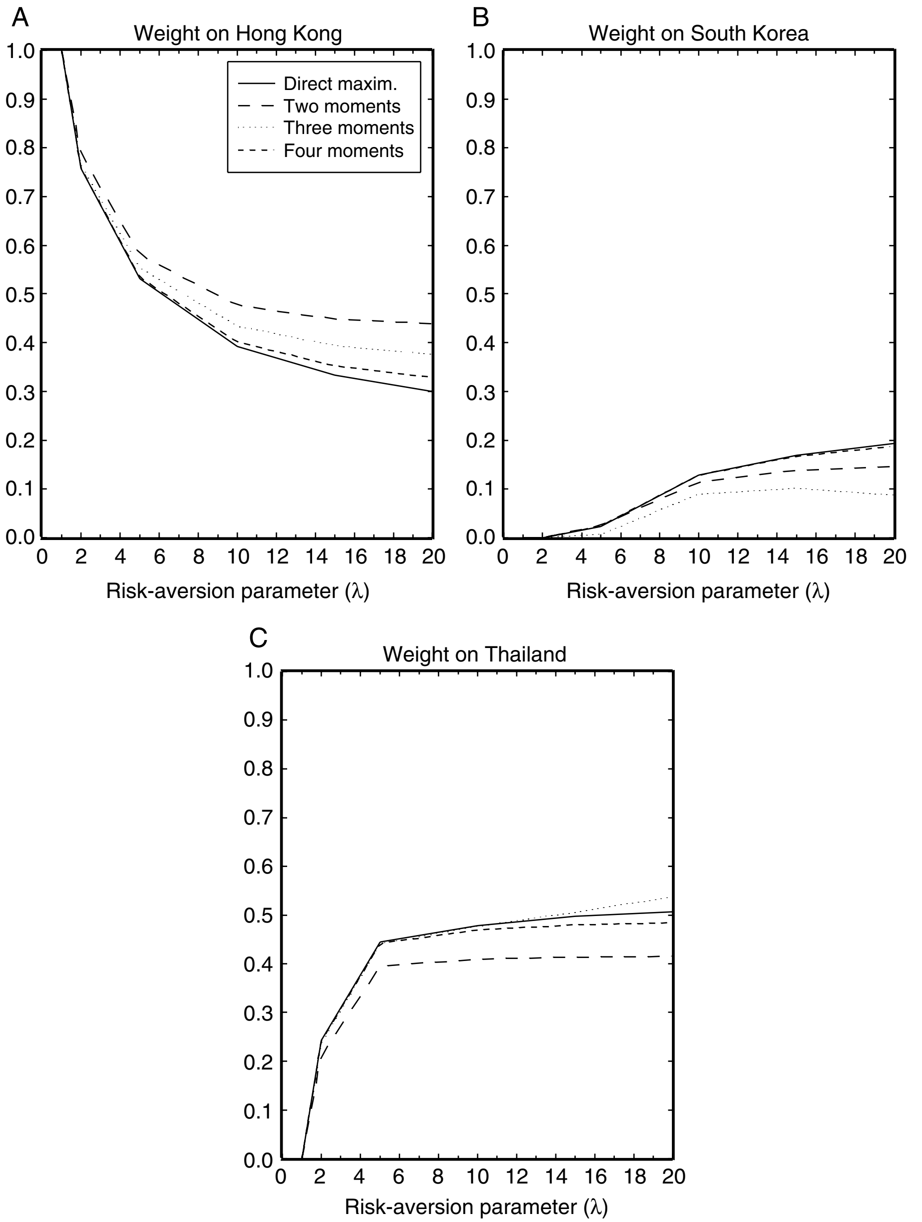

Fig. 3. This figure displays the optimal weights obtained with the different optimisation strategies as a function of the risk-aversion parameter λ. Here the data consists in three emerging markets indices.

5. Conclusion

In this paper, we address two related issues. First, we describe how the impact of non-normality of returns on the allocation of assets may be easily measured in an expected utility framework. In general, maximising the expected utility results in rather cumbersome computations under non-normality. We use a Taylor series expansion to approximate the expected utility as a function of higher moments. Then, we compute the optimal portfolio allocation in a very efficient way. A decisive advantage of this approach is that it can be very easily applied even when the number of assets is large. This results from the numerical stability of the asset-allocation problem regardless of the number of assets.

Second, we consider the extent to which departure from normality is likely to affect the optimal asset allocation when the CARA utility function is used. A number of studies have measured the quality of the approximation of the expected utility by the mean-variance criterion (Levy and Markowitz, 1979; Pulley, 1981; Kroll *et al.*, 1984; Simaan, 1993b). Other studies have described how higher moments may be incorporated in the investor's asset allocation problem using the PGP approach (Lai, 1991; Chunhachinda *et al.*, 1999; Prakash *et al.*, 2003). But no previous study had measured the gain of using a three-moment or a four-moment optimisation strategy for approximating the expected utility. For this purpose, we consider three different data sets, containing returns with both moderate and large departures from normality. We confirm previous empirical evidence (e.g., Kroll *et al.*, 1984) as well as theoretical arguments (Chamberlain, 1983) that, under moderate non-normality, the mean-variance criterion provides a good approximation of the expected utility maximisation. Nevertheless, under large departure from normality (as found in some stocks in mature markets or in some stock indices in emerging markets), the mean-variance criterion may fail to approximate the expected utility correctly. In such cases, the three-moment or four-moment optimisation strategies may provide a good approximation of the expected utility.

An obvious extension of this work is a conditional asset allocation. For a recent contribution arguing for a dynamic portfolio selection see Graflund and Nilsson (2003). One may also consider Malkiel (2003) who argues against. In a dynamic setting, a model for returns with a distribution allowing asymmetry and fat tails should be estimated. This may be done, for instance, in a GARCH framework, with a skewed Student-t distribution for innovations, such as the model proposed by Hansen (1994) and extended by Jondeau and Rockinger (2003a) for time-varying higher moments. In addition, co-moments may be modelled using the multivariate extension developed by Sahu *et al.* (2003) and Bauwens and Laurent (2005). An alternative approach to cope with asymmetry and fat tails would rely on the regime-switching modelling, that provides a very convenient way to incorporate these features (Guidolin and Timmermann, 2003). A definite advantage of these approaches is that moments of the portfolio return can be computed analytically from the multivariate distribution of asset returns.

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