How Higher Moments Affect the Allocation of Assets

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Abstract

We evaluate how deviations from normality may affect the allocation of assets. A Taylor expansion of expected utility allows us to focus on certain moments and to compute numerically the optimal portfolio allocation. We obtain that for small values of the risk-aversion parameter, non-normality does not alter significantly the optimal allocation. In contrast, when the investor is strongly risk averse, and restricted to invest in risky assets only, we also obtain significant changes in portfolio weights.

Key words: Asset allocation, Stock returns, Non-normality, Utility function.
JEL classification: C22, C51, G12.

1. INTRODUCTION

The non-normality of asset returns is a well-established empirical regularity. For stocks, this regularity may be due to time-varying parameters or rare, yet extreme realisations. For other assets, such as hedge funds, non-normality may be due to complex allocation strategies. This leaves the question open how utility maximising individuals should allocate their wealth among a set of assets that exhibit non-normality. We attempt to answer this question by assuming that investors care about the first four moments. So in addition to mean and variance, investors are supposed to like positive skewness and to dislike fat-tailedness as measured by kurtosis.

Our model is related to the existing literature in the following way. First, there are contributions that focus on the implications of higher moments within an equilibrium context. Rubinstein (1973) constructs the first Capital Asset Pricing Model involving up to the third moment. Kraus and Litzenberger (1976) provide an empirical implementation. Other work, also containing further references, is by Harvey and Siddique (2000), Jurczenko and Maillet (2001) and Adcock (2002). Athayde and Flôres (2001) focus on the computation of the efficient frontier when several moments matter.

Second, there is a set of contributions that investigate how higher moments affect portfolio allocation. For instance, Ang and Bekaert (2002) show how regime switches of returns may affect portfolio allocation. Chunachinda, Dandapani, Hamid, and Prakash (1997) proposed a numerical solution to solve the asset-allocation problem when skewness is taken into account.

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2. THE OPTIMAL PORTFOLIO ALLOCATION

We consider an investor who allocates her portfolio to maximise the expected utility $U(W)$ of the next period wealth W. There are n risky assets with return vector $R = (R_1,...,R_n)'$ and a riskless asset with return r_f. The initial wealth is arbitrarily set equal to one. Next period wealth is given by $W = 1 + (1-\alpha' e)\, r_f + \alpha' R$. The vector $\alpha = (\alpha_1,\ldots, \alpha_n)'$ represents the fractions of wealth allocated to the risky assets and e is a column vector of n ones. If $\alpha' e = 1$, then the investor can only invest in risky assets. If the α_is are forced to be positive, it is not possible to short sell assets. Our general problem is to solve

$$\max_{\{\alpha\}} E\left(U\left(1+(1-\alpha' e)r_f + \alpha' R\right)\right) \tag{1}$$

To assess the importance of higher moments on the asset allocation, we now approximate the expected utility by a Taylor series expansion around the expected wealth.[1] We define $\mu_p = E(r_p)$, $\sigma_p^2 = E\left((r_p-\mu_p)^2\right)$, $s_p^3 = E\left((r_p-\mu_p)^3\right)$, and $\kappa_p^4 = E\left((r_p-\mu_p)^4\right)$. Focusing on the first four moments only, for an investor with constant relative risk aversion (CRRA) utility function with γ as parameter of risk aversion, the first-order conditions yielding the optimal portfolio weights, α_i, may be written as

$$\begin{aligned}(\mu - r_f)&\left(1+\frac{\gamma(\gamma+1)}{2(1+\mu_p)^2}\sigma_p^2 - \frac{\gamma(\gamma+1)(\gamma+2)}{6(1+\mu_p)^3}s_p^3 + \frac{\gamma(\gamma+1)(\gamma+2)(\gamma+3)}{24(1+\mu_p)^4}\kappa_p^4\right)\\ &= \frac{\gamma}{2(1+\mu_p)}\frac{\partial\sigma_p^2}{\partial\alpha} - \frac{\gamma(\gamma+1)}{6(1+\mu_p)^2}\frac{\partial s_p^3}{\partial\alpha} + \frac{\gamma(\gamma+1)(\gamma+2)}{24(1+\mu_p)^3}\frac{\partial\kappa_p^4}{\partial\alpha}.\end{aligned} \tag{2}$$

This equation may be solved numerically in a very efficient manner.

3. DATA AND THE PORTFOLIO ALLOCATION

We use weekly returns (from Friday to Friday) for dollar-denominated stock indices of the main geographical areas: North America, Europe excluding UK, UK, Asia excluding Japan, and Japan. The data set consists of total return indices from Morgan Stanley Capital International (MSCI) and covers the period from January 1976 to December 2001, for a total of 1345 observations.

In order to provide evidence on how the CRRA utility function incorporates information on higher moments, we perform two kinds of asset allocation exercises. In the first one, reported in Table 1, short sales and borrowing are allowed, so that no constraints are put on the portfolio weights. In the second one, reported in Table 2, we assume that short sales are not possible and that the entire wealth must be invested in risky assets.

Table 1. Optimal portfolio characteristics. Investment in risk free asset is possible

	Portfolio weights					Portfolio-return moments			
γ	α_1	α_2	α_3	α_4	α_5	μ_p	σ_p^2	s_p^3	κ_p^4
Panel A: Taylor expansion up to order 2									
2	0.881	0.257	0.635	0.037	-0.209	0.339	10.069	-0.857	8.094
10	0.176	0.051	0.127	0.007	-0.042	0.179	0.401	-0.857	8.092
20	0.088	0.026	0.063	0.004	-0.021	0.159	0.100	-0.857	8.093
50	0.035	0.010	0.026	0.001	-0.008	0.147	0.016	-0.852	8.078
Panel B: Taylor expansion up to order 3									
2	0.848	0.243	0.624	0.018	-0.199	0.332	9.333	-0.838	7.969
10	0.171	0.049	0.125	0.005	-0.040	0.178	0.379	-0.844	8.002
20	0.086	0.024	0.063	0.003	-0.020	0.159	0.095	-0.845	8.020
50	0.034	0.010	0.025	0.001	-0.008	0.147	0.015	-0.843	8.004

[1] It has been put forward by Jurczenko and Maillet (2001) that the Taylor series expansion is exact, i.e., the remainder is equal to zero, for some return distributions such as the Gaussian one. The same holds for some utility functions such as polynomial utility.

				Panel C: Taylor expansion up to order 4					
2	0.837	0.242	0.617	0.015	-0.198	0.329	9.089	-0.835	7.940
10	0.169	0.049	0.124	0.005	-0.040	0.178	0.372	-0.843	7.990
20	0.085	0.025	0.062	0.002	-0.020	0.159	0.094	-0.843	7.994
50	0.034	0.010	0.025	0.001	-0.008	0.147	0.015	-0.843	7.996

Table 2. Optimal portfolio characteristics. Investment in risk free asset is not possible

	Portfolio weights					**Portfolio-return moments**			
γ	α_1	α_2	α_3	α_4	α_5	μ_p	σ_p^2	s_p^3	κ_p^4
				Panel A: Taylor expansion up to order 2					
2	0.513	0.000	0.487	0.000	0.000	0.261	3.878	-0.736	8.014
10	0.438	0.281	0.176	0.054	0.051	0.244	3.068	-0.921	8.642
20	0.412	0.309	0.125	0.061	0.093	0.239	2.994	-0.908	8.474
50	0.398	0.324	0.099	0.065	0.115	0.236	2.973	-0.896	8.334
				Panel B: Taylor expansion up to order 3					
2	0.523	0.000	0.478	0.000	0.000	0.261	3.861	-0.741	8.044
10	0.436	0.287	0.172	0.037	0.068	0.243	3.052	-0.889	8.411
20	0.412	0.316	0.121	0.031	0.120	0.237	2.990	-0.847	8.044
50	0.401	0.332	0.095	0.000	0.173	0.234	3.008	-0.757	7.410
				Panel C: Taylor expansion up to order 4					
2	0.509	0.009	0.483	0.000	0.000	0.261	3.858	-0.739	8.017
10	0.433	0.296	0.166	0.032	0.073	0.242	3.044	-0.878	8.316
20	0.405	0.340	0.107	0.017	0.132	0.236	2.991	-0.816	7.773
50	0.367	0.393	0.049	0.000	0.191	0.230	3.024	-0.732	7.061

In our investigation, the parameter γ ranges between 2 and 50. This covers most values found in the literature. Figure 1 reports portfolio weights for four of the areas, when the risk-aversion parameter varies.

Several points are worth noting:

1. As Table 1 displays, when the investor may invest in all the assets including the risk free one, asset weights are essentially unaffected by the introduction of a concern for skewness or kurtosis. This indicates that the asset allocation is dominated by the concern for minimising variance. When the risk-aversion parameter increases, the weight of risky assets decreases markedly, so that, for $\gamma = 30$, the weight of the riskless asset is as high as 90%. Therefore, portfolio skewness and kurtosis are not changed as the investor considers higher and higher moments.

2. Table 2 shows that when the investor is restricted to invest in the risky assets only, skewness and kurtosis may play a role for high γ. This comes from the fact that the weight on skewness becomes rather big. We also notice that for $\gamma \leq 10$, the effect on asset weights α_i does not exceed 1.5 percentage points. For instance, as we shift from Panel A to Panel C in Table 2, and keep γ at 10, the α's change from (0.438, 0.281, 0.176, 0.054, 0.051) to (0.433, 0.296, 0.166, 0.032, 0.073), a small change indeed.

3. When γ becomes larger, we obtain changes in portfolio weights. Comparison of Panel A and B of Table 2 shows for instance, for $\gamma = 50$, that the weight on Asia (α_4) decreases from 6.5% to 0% when skewness is introduced in the optimisation process. This may be explained by the very large negative skewness of returns sampled from this geographic area. In contrast, Japan (α_4) is characterised by a positively skewed return, so that the weight of this asset markedly increases (from 11.5% to 17.3%) when a concern for skewness is introduced.

4. Comparing how portfolio moments are modified, when higher moments are introduced, also provides interesting results. When the Taylor expansion is performed up to order 2, portfolio variance decreases when γ increases. However, when a concern for skewness is allowed, this is not necessarily the case. For large values of γ (say 50), a very slight increase in the variance is admitted to obtain a significant increase in skewness.

4. CONCLUSION

Our results complement the ones reported by Aït-Sahalia and Brandt (2002) who report that for small values of the parameter of risk aversion, higher moments do not matter. These authors also use the CRRA utility function. They consider rather low risk-aversion parameters and only two assets, characterised by a slight departure from normality.

Here, we used very aggregated data. We expect stronger effects for less aggregated data and for conditional models where skewness and kurtosis may assume very high values. Conditional higher moments may be obtained in the setting of Hansen (1994) and extended by Jondeau and Rockinger (2003).

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Figure 1a. Weight on North-America

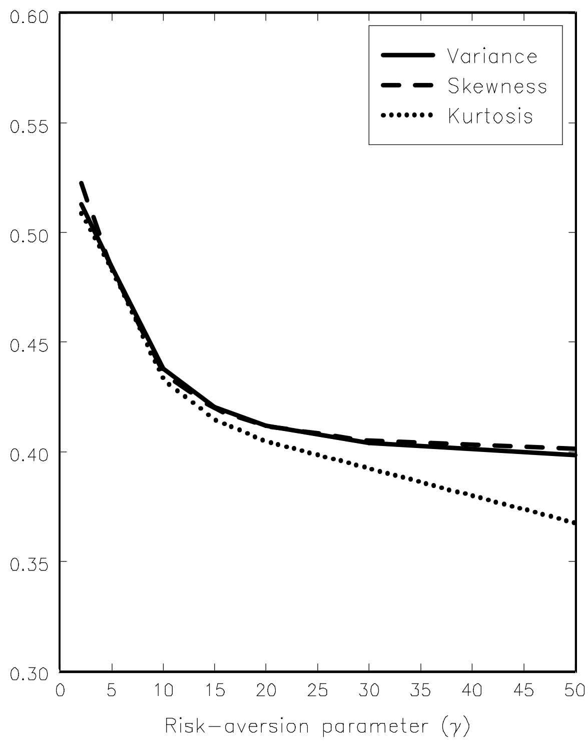

Figure 1b. Weight on Europe ex. the UK

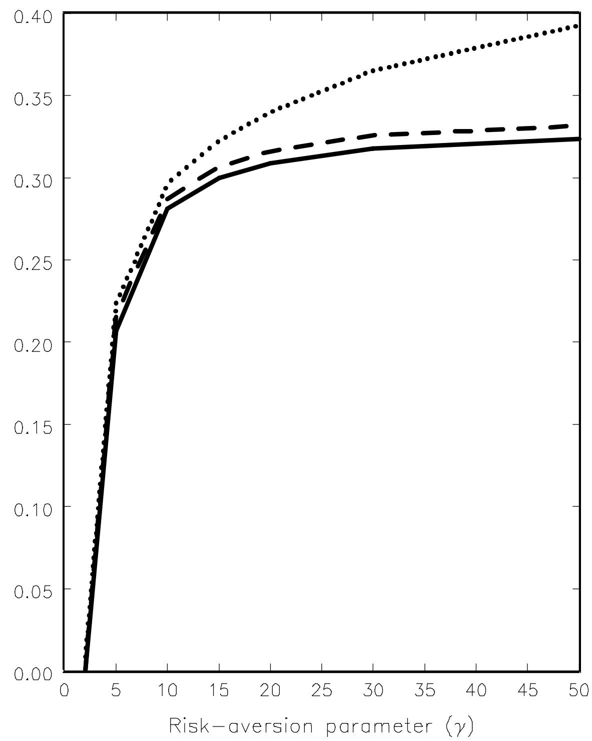

Figure 1c. Weight on the UK

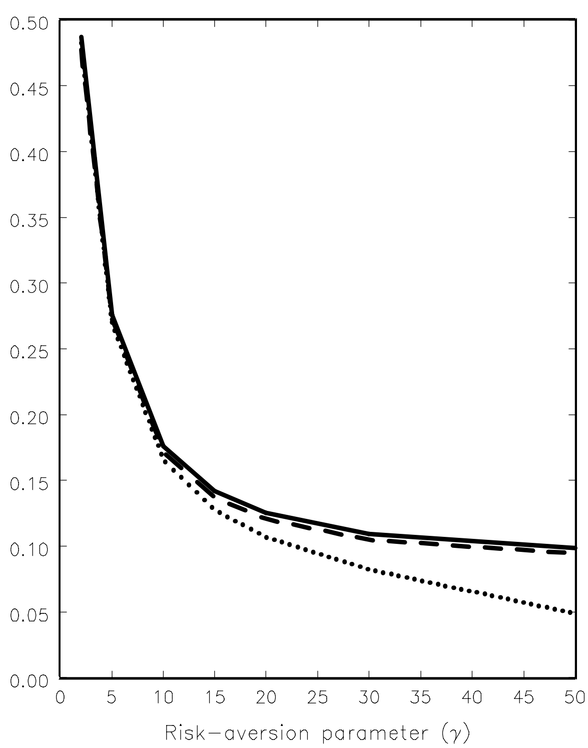

Figure 1d. Weight on Japan