Market Selection and Welfare in a Multi-Asset Economy

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August, 2010

Abstract

We analyze the performance of irrational investors, who mistake expected returns of assets, in a multi-asset economy. Under general conditions, irrational investors are severely punished compared with rational investors, in that their fractions of consumption and wealth decrease quickly. Further, the welfare cost of such underperformance is often high. Our results contrast with previous studies of single-asset economies, which find modest underperformance by irrational investors. In a calibration, we find that an irrational investor, who mistakes expected returns by 20%, loses almost 95% of his wealth in about 25 years. The welfare cost of this underperformance is significant, about 40% of the total wealth in the economy.

JEL: G0, G11.

Keywords: Portfolio choice, market selection, heterogeneous investors, asset pricing.

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*We thank Ulf Axelson, Kerry Back, Jonathan Berk, Tomas Björk, Greg Duffee, Phil Dybvig, Terry Hendershott, Christopher Hennessy, David Hirshleifer, Pete Kyle, Hayne Leland, Mark Loewenstein, Christine Parlour, Mark Rubinstein, Jacob Sagi, Mark Seasholes, Costis Skiadas, Richard Stanton, Mark Westerfield and Hongjun Yan for valuable comments and suggestions. We also thank seminar participants at UC Berkeley, The Stockholm School of Economics, Arizona State University and the 2007 meetings of the Western Finance Association. Finally, we thank Sebastien Betermier for research assistance. Some of the ideas in this paper previously occurred in an unpublished manuscript “High-speed natural selection in financial markets with large state spaces.”

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1 Introduction

The recent financial crisis and its associated welfare costs have been blamed on the innovation of complex derivatives and other financial instruments that allowed unsophisticated investors to take on large risks they did not understand.\(^1\) This view stands in contrast to the rational neoclassical view that such financial innovation increases efficiency by completing markets. At the individual investor level, suboptimal investment behavior by unsophisticated investors has been documented by several studies. For example, Barber, Lee, Liu, and Odean (2009) find an underperformance of 2.1% per year for individual investors relative to institutional investors—a 50% underperformance over a 30-year time horizon. Moreover, as shown in Calvet, Campbell, and Sodini (2007, 2009), the welfare costs of suboptimal investments by unsophisticated investors are also significant. A solid theoretical understanding of the equilibrium effects of investor irrationality on consumption, wealth, and welfare is therefore important.

The equilibrium effects of investor irrationality can be understood by using investor survival analysis, developed in Sandroni (2000, 2005) and Blume and Easley (1992, 2006). Specifically, the survival index, introduced in Blume and Easley (2006), provides an asymptotic result (for large time periods) for the rate at which irrational investors underperform rational ones. Using this approach, and building on the general equilibrium literature with heterogeneous investors (see Detemple and Murthy (1994a), Basak (2000) and David (2009), Yan (2008) calibrates a standard exchange economy with a representative firm, and shows that it may take several hundred years before rational investors significantly outperform irrational investors. Similar results are derived in Dumas, Kurshev, and Uppal (2009), under slightly different assumptions, and used in Branger, Schlag, and Wu (2006).\(^2\)

\(^1\) An early warning was issued by Mr. Warren Buffet, calling these instruments “financial weapons of mass destruction” in his 2003 newsletter to Berkshire Hathaway’s shareholders. In the letter, Mr. Buffet argues that the range of derivatives is only limited by the imagination of madmen, and that investors’ biased forecasts together with fraudulent accounting impose severe systemic risk on the economy.

\(^2\) Of course, as shown in many studies, irrational investors do not always underperform. For example, over-optimistic investors may invest a larger share of their wealth in risky assets and ultimately dominate the market when prices are set exogenously (DeLong, Shleifer, Summers, and Waldman (1991)). Similarly, irrational investors with a lower consumption-to-savings ratio than rational investors may come to dominate the market. Moreover, even when rational investors eventually dominate the market measured by fraction of wealth, irrational investors may still have nonnegligible impact on prices (Kogan, Ross, Wang, and Westerfield (2006)). However, when rational and irrational investors have identical utilities, irrational investors will lose out compared with rational ones except under special conditions. In general equilibrium with complete markets, Sandroni (2000) shows that rational investors will eventually dominate the market under general conditions if agents have identical intertemporal discount factors (Blume and Easley (2006) have showed that in incomplete markets, this result may not hold in general, although Sandroni (2005) shows that the result can be extended to incomplete markets in some cases). Loewenstein and Willard (2006) point out that models of the type of DeLong, Shleifer, Summers, and Waldman (1990, 1991) implicitly allow for real transfers of production (between risk-less storage and risky technology), due to sentiment, and for changes in aggregate consumption. We study a general equilibrium in a complete market, so in line with Sandroni (2000) irrational investors will eventually lose out. The literature also relates to the original literature on market selection, see Alchian (1950) and Friedman (1953), Cootner (1964) and...
One caveat with these previous theoretical studies, which find only modest underperformance by irrational investors, is that they are based on representative firm economies, i.e., on economies with a single traded risky asset. Although this is a harmless assumption when all investors are identical, since aggregate production is sufficient for all purposes in this case, it is unclear whether the assumption is harmless when some investors are irrational. In fact, one may suspect a source of underperformance to be that irrational investors hold very different portfolios compared with rational ones. With only one firm to invest in, however, the heterogeneity in portfolio holdings is severely limited. For example, all investors will hold portfolios with the same Sharpe ratios when there is only one risky asset.

In this paper, we study a general exchange economy with many risky assets and show that irrational investors severely underperform rational ones in such an economy. Our results stand in stark contrast to previous studies. For example, over a 25-year horizon, a moderately irrational investor is expected to lose about 93% of consumption and wealth to a rational investor. Somewhat surprising, the welfare costs of irrationality are also severe, even in this standard exchange economy setting, in which irrational investors neither affect the total output, nor generate negative externalities by creating systemic risk. Instead, the welfare costs arise because the consumption allocation between the rational and irrational investor is severely suboptimal. When the initial wealth of the rational and irrational investor is the same, the welfare cost as a fraction of total wealth is about 40% in a 25-year horizon, highlighting the importance of financial education in markets with a large fraction of unsophisticated investors, and suggesting that it may be welfare increasing to restrict the asset span in such markets.

The underperformance of irrational investors is proportional to the number of risky assets. For example, if it takes 1,500 years to reach a prescribed loss in a market with one risky asset, it takes three years in a market with the same aggregate dynamics, but with 500 risky assets. Thus, although the model with one representative firm qualitatively gives the same result as the multi-firm model (the eventual extinction of irrational traders), the quantitative difference is striking. Our results are robust to various assumptions about the sentiments of irrational investors and the risk structure of the stock market. Indeed, the only situations in which irrational investors do not severely underperform rational ones is when there is no spread of investor sentiment across assets, or when asset risks are uncorrelated. In these cases, our results are the same as in the representative firm setting.

The irrational investors in our model consistently mistake the growth rates of firms. Such behavior could, e.g., arise if investors receive noisy signals about growth rates and are overconfident about the quality of these signals. The assumption can more generally be viewed as a

Fama (1965). Other recent contributions to the literature include Cvitanic and Malamud (2010a) and Cvitanic and Malamud (2010b).
reduced form representation of the behavior of investors who are unsophisticated in how they treat probabilities. Our results are so strong that it is clear that they would be present under various model extensions. For example, there is no structural uncertainty in our model, but if unknown structural breaks were added, the underperformance of irrational investors would still be severe. Rational investors would quickly learn enough about the new structure of the economy after a structural break and then take advantage of irrational investors. Also, we do not consider frictions. Transaction costs, for example, may be another source of underperformance by unsophisticated investors, as shown by Barber and Odean (2000). A more realistic model with frictions, although clearly of interest, is outside of the scope of this paper. Finally, we focus on investors who are irrational in the way that they update their probabilistic beliefs. Irrationality in the form of deviations from the expected utility framework is also outside of the scope of our analysis.

The paper is organized as follows. In the next section we introduce the model. In section 3 we provide the main results on the consumption, wealth and welfare of rational and irrational investors, and in section 4 we discuss robustness under variations and generalizations of the base model. Finally, in section 5, we make some concluding remarks. Details and proofs are left to the appendix.

2 Model

We assume a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}), 0 \leq t \leq T\), and \(N\)-dimensional \(\mathcal{F}_t\)-adapted standard Brownian motions \(B_t = (B_{1,t}, ..., B_{N,t})'\), (where ' denotes transpose), satisfying the usual assumptions. Here, \(T\) could be finite or infinite, although we mainly focus on the infinite horizon case. We use the notation \(\mathbf{a} = [a_i]_i\) to “build” vectors from scalars, and \(a_i = (\mathbf{a})_i\) to extract scalars from vectors. Similarly, we define the matrix \(\mathbf{A} = [a_{ij}]_{ij}\) and the scalars \(a_{ij} = (\mathbf{A})_{ij}\).

There is an \(N\)-dimensional state vector, \(\omega_t \in \mathbb{R}^N\), which evolves according to

\[
d\omega_t = g dt + \sigma_\omega dB_t. \tag{1}
\]

Here, \(g \in \mathbb{R}^N\) and \(\sigma_\omega \in \mathbb{R}^{N \times N}\) may be (smooth) functions of \(\omega_t\) and \(t\), so that \(\omega_t\) is a general diffusion process. We sometimes suppress the time dependence of variables when this can be done without causing confusion, e.g., writing \(\omega\) instead of \(\omega_t\). The variance-covariance matrix of \(\omega\) is \(\Sigma = \sigma_\omega \sigma_\omega'\). We assume that \(\Sigma\) is invertible at all points in time. There is a dynamically complete competitive market of contingent claims, such that claims on each realization of \(\omega\) are traded, i.e., Arrow-Debreu securities exist for each state of the world.
The first $M$ elements (where $1 \leq M \leq N$) of the state vector are associated with firms that produce consumption goods. The latter $M - N$ elements are associated with claims on zero net supply assets. The latter elements could, for example, represent derivatives markets on unspanned risk, but also other claims on idiosyncratic risk, e.g., insurance contracts. Specifically, firm $i$ instantaneously produces $D_{i,t} dt$ of a perishable consumption good, where

$$D_{i,t} = D_{i,0} e^{\omega_{i,t}}, \quad D_{i,0} > 0,$$

and where $\omega_{i,t} = (\omega_t)_i$ is the $i$th element of the vector $\omega$ at time $t$. We define $D = (D_{1,0}, D_{2,0}, \ldots, D_{N,0})'$, where $D_{M+1,0}, \ldots, D_{N,0} = 0$. The aggregate consumption is given by

$$C_t = \sum_{i=1}^M D_{i,t}.$$

Our results do not depend on the type of securities used to implement the complete market. A standard implementation, however, is to assume that there are $N$ securities, where security $i$ represents a claim to the consumption stream $e^{\omega_{i,t}}$, and where the net supply of asset $i$ is $D_{i,0}$ for $1 \leq i \leq M$ and zero for $M + 1 \leq i \leq N$, and that there is also a risk-free bond available in zero net supply.\(^3\)

The set-up is very general, and contains several interesting sub-cases. If $M = N$, and $g$ and $\sigma_{\omega}$ are constants, then the economy corresponds to a standard $N$-tree Lucas economy, see Cochrane, Longstaff, and Santa-Clara (2008); Martin (2008); Parlour, Stanton, and Walden (2010).\(^4\) Also, the models in Anderson and Raimondo (2008), Santos and Veronesi (2005), Campbell and Cochrane (1999) and Bansal and Yaron (2004) fall within this setting, with $M = N$. The case when $M = 1$, i.e., when there is one stock in positive net supply and many zero net supply claims, covers the models in Dumas, Kurshev, and Uppal (2009) (with $N = 2$) and Buraschi and Jiltsov (2006).\(^5\) Further generalizations are also possible. For example, it is neither crucial that aggregate consumption is of the form (3), nor that the underlying process $\omega$ is a diffusion process. For notational simplicity, we stick with the — already very general — set-up. Intuitively, it may be easiest to think of the situation when $M = N$, in which case each risky asset represents a stock, i.e., the claim on the dividend process of a firm.

There are two price taking investors, $k \in \{1, 2\}$. Investor 1 is a rational von Neumann-
Morgenstern expected utility optimizer, who has a complete understanding of parameters and dynamics in the economy. It has been repeatedly documented that many investors deviate from the rational expected utility framework. Broadly speaking, there are two types of deviations (see Barberis and Thaler (2003)). First, investor behavior is not consistent with agents having a (subjective) expected utility function. Second, investors do not update their beliefs in consistency with Bayes rule. Following Yan (2008) (see also Kogan, Ross, Wang, and Westerfield (2006, 2009), Cvitanic and Malamud (2010a) and Chen, Joslin, and Tran (2010)), we make a parsimonious assumption in line with the second type of deviation. Specifically, we assume that investor 2 mistakes the drift term for $g^2 = (g_1^2, ..., g_N^2) = g + \delta$, $\delta \neq 0$. We call $\delta$ the irrational investor’s sentiment vector. Of course, if there is uncertainty about structural parameters, even a rational investor may initially be mistaken about parameters. We would, however, expect a rational investor who knows about the uncertainty to update his beliefs over time. Investor 2, being irrational, does not update his belief. This type of irrationality could, for example, arise if the irrational investor is overconfident about a noisy signal of $g$. We will briefly discuss the robustness of our results to other types of deviations from rationality in Section 4.

We further assume that investors $k \in \{1, 2\}$ have initial wealths $W_k$ and CRRA preferences with time discount factors $\rho_k$, and common relative risk aversion parameter, $\gamma$. For expositional reasons, we mainly focus on the case when $\gamma \neq 1$, although our results also hold under logarithmic utility. In the next section we will generalize to agent specific risk aversion coefficients. Thus, investor $k$ optimizes

$$U_k = E_k \left[ \int_0^T e^{-\rho_k t} \frac{c_{k,t}^{1-\gamma}}{1-\gamma} dt \right],$$

subject to his budget constraint, where $c_{k,t}$ is the instantaneous consumption at $t$ of investor $k$. Here, since the two investors have different expectations, the $k$-subscript of the expectation operator is motivated. The total initial wealth is $W = W_1 + W_2$. The economic environment can be summarized by the quadruplet $E = (\delta, g, \Sigma, D)$, whereas the agents’ preferences are summarized by the triplet $(\gamma, \rho_1, \rho_2)$.

The expected realized utilities of consumption of the two agents are calculated using objective probabilities, i.e., using the rational agent’s probability estimates:

$$U_k^{OBJ} = E_1 \left[ \int_0^T e^{-\rho_k t} \frac{c_{k,t}^{1-\gamma}}{1-\gamma} dt \right].$$

We call $U_1^{OBJ}$ and $U_2^{OBJ}$ the objective expected utilities of the rational and irrational agent.

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6A potential extension would be to endow the two agents with idiosyncratic endowment shocks, providing a motive for zero-net supply “insurance” assets (i.e., for $N - M > 0$).
respectively. The objective expected utility of the rational agent coincides with his “personal” expected utility. For the irrational agent, however, the personal and objective expected utilities differ, because of his incorrect beliefs. We note that welfare analysis under this specific form of agent irrationality is quite straightforward, since the true welfare is derived from the objective expected utilities, i.e., from the two agents’ true expectations of realized utility of consumption. Thus, although agent 2 at \( t = 0 \) may believe that he is going to be very well off, it is the true utility he gets from consumption that matters for the welfare in the economy.

It follows immediately that the perceived shocks to the state variable \( \omega \) by agent \( k = 1, 2 \) are

\[
d\omega_t - g_t^k dt = \sigma_\omega dB_t^k,
\]

where \( B_t^k \) is a standard \( N \)-dimensional Brownian motion under agent \( k \)’s probability measure. The relation between \( B_t^1 = B_t \) and \( B_t^2 \) is

\[
dB_t^2 = dB_t - \Delta_t dt,
\]

where \( \Delta = \sigma_\omega^{-1}\delta \).

To solve for the Walrasian complete market equilibrium, we construct the social planner’s problem with a representative agent (see Dumas (1989), Cuoco and He (1994), Detemple and Murthy (1994b), Wang (1996), Basak (2000) and Gallmeyer and Hollifield (2008)) state by state and time by time from

\[
u(C_t, \lambda_t, t) = \max_{c_{1,t}, c_{2,t}} \left\{ e^{-\rho_1 t} c_{1,t}^{1-\gamma} + \lambda_t e^{-\rho_2 t} c_{2,t}^{1-\gamma} \right\}
\]

s.t.

\[
c_{1,t} + c_{2,t} = C_t.
\]

Here, \( \lambda_t = \lambda_0 \eta_t \), \( \eta_t = \exp \left( -\frac{1}{2} \int_0^t \Delta'_s \Delta_s ds + \int_0^t \Delta'_s dB_s \right) \), and \( \lambda_0 \in \mathbb{R}_+ \) determines agent 2’s weight in the social planner’s problem. Using martingale methods (Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987)), agent specific stochastic discount factors (SDFs), \( \xi_{k,t} \), can be constructed. The SDFs are agent specific, since the two agents disagree on the shocks, see Basak (2000). The SDF of agent \( k \) follows (see Duffie (2001))

\[
\frac{d\xi_{k,t}}{\xi_{k,t}} = - \left( r_t dt + \kappa'_{k,t} dB_t^k \right).
\]

\(^7\)Formally, \( \eta_t \) is the Radon-Nikodym derivative of the irrational agent’s probability measure with respect to the rational agent’s probability measure.
Here, $\kappa_{k,t}$ is the vector of market prices of risk for the Brownian shocks, as perceived by agent $k$. As agent 1 is rational, his perceived market prices of risk coincide with the market prices of risk under the objective probability measure. The difference in the market prices of risk of agents 1 and 2 is

$$\kappa_2 - \kappa_1 = \Delta.$$  \hspace{1cm} (10)

Our objective is to study the consumption, wealth and welfare of the two agents. We therefore make the following definitions:

**Definition 1**

1. The consumption share (of agent 1) is $f_t \overset{\text{def}}{=} \frac{c_{1,t}}{C_t}$.

2. The wealth share (of agent 1) is $f_{W,t} \overset{\text{def}}{=} \frac{W_{1,t}}{W_t}$.

3. The log-consumption ratio is $h_t \overset{\text{def}}{=} \log \left[ \frac{c_{1,t}}{c_{2,t}} \right] = \log \left[ \frac{f_t}{1-f_t} \right]$.

4. The log-wealth ratio is $h_{W,t} \overset{\text{def}}{=} \log \left[ \frac{W_{1,t}}{W_{2,t}} \right] = \log \left[ \frac{f_{W,t}}{1-f_{W,t}} \right]$.

We note that the consumption share is obtained by a simple transformation of the log-consumption ratio, $f_t = \frac{e^{h_t}}{1+e^{h_t}}$. We will also work with the random stopping time

$$\tau_f = \inf_t \{ t : f_t \geq f \},$$  \hspace{1cm} (11)

that is, $\tau_f$ defines the first point in time at which the consumption share of agent 1 reaches $f$.

The following proposition, which follows directly from the agents’ Euler conditions, summarizes the dynamics of consumption, wealth and the stochastic discount factor in the complete market Walrasian equilibrium:

**Proposition 1** Agent 1’s consumption share at time $t$ is

$$f_t = \frac{1}{1 + e^{(\rho_1 - \rho_2) t/\gamma} \lambda_t^\gamma},$$  \hspace{1cm} (12)

which determines the agents’ consumption:

$$c_{1,t} = f_tC_t,$$
$$c_{2,t} = (1 - f_t)C_t.$$  \hspace{1cm} (13)

The state price density at time $t$ under the objective probability measure is

$$\xi_t \overset{\text{def}}{=} \xi_{1,t} = e^{-\rho_1 t} \left( \frac{c_{1,t}}{c_{1,0}} \right)^{-\gamma} = e^{-\rho_1 t} \left( \frac{f_t}{f_0} \right)^{-\gamma} \left( \frac{C_t}{C_0} \right)^{-\gamma},$$  \hspace{1cm} (14)
which determines the agents’ wealths:

\[
W_{1,t} = E_t \left[ \int_t^T \frac{\xi_s}{\xi_t} c_{1,s} ds \right] = E_t \left[ \int_t^T \frac{\xi_s}{\xi_t} f_s C_s ds \right],
\]

\[
W_{2,t} = E_t \left[ \int_t^T \frac{\xi_s}{\xi_t} c_{2,s} ds \right] = E_t \left[ \int_t^T \frac{\xi_s}{\xi_t} (1 - f_s) C_s ds \right].
\]

We see that the consumption share, \( f_t \), together with aggregate consumption, \( C_t \), completely determine the dynamics of the economy.

3 Results

We study the dynamics of consumption and wealth of the two agents in equilibrium, and the welfare consequences of irrationality. The following proposition provides the key result for our analysis:

**Proposition 2** Define the instantaneous transfer index

\[
K = \delta' \Sigma^{-1} \delta.
\]

Here, \( \delta \in \mathbb{R}^N \) is the irrational agent’s sentiment vector and \( \Sigma \) is the instantaneous variance covariance matrix of \( \omega \). Then, the instantaneous dynamics of the log-consumption ratio, \( h_t \), is

\[
dh_t = \left( \frac{1}{2\gamma} K + \frac{\rho_2 - \rho_1}{\gamma} \right) dt + \frac{1}{\gamma} \sqrt{K} dB_t,
\]

where \( \tilde{B} \) is a standardized Brownian motion.

Proposition 2 is valid under the most general assumptions of our model, with time and state dependent diffusion coefficients, in which case \( K \) will also be time and state dependent. The transfer index is related to the survival index (see Blume and Easley (2006) and Yan (2008)), but it is preference independent, i.e., it only depends on the economic environment. As we shall see, in multi-asset markets, the transfer index typically dominates preferences in determining if and how market selection takes place, i.e., a large \( K \) typically implies a significant consumption and wealth transfer from agent 2 to agent 1.

From (17) it follows that \( E[\Delta h_t] = \left( \frac{1}{2\gamma} K + \frac{\rho_2 - \rho_1}{\gamma} \right) dt \), which, if \( K \) is constant, immediately implies that

\[
E[h_t - h_0] = \left( \frac{1}{2\gamma} K + \frac{\rho_2 - \rho_1}{\gamma} \right) t.
\]
Therefore, if $\rho_2 = \rho_1$, since (16) implies that $K > 0$ (as the covariance matrix, $\Sigma$, is positive definite), the log-consumption ratio is expected to increase at each point in time. More generally, if the agents have different time preference parameters, the log-consumption ratio is expected to increase if $K > 2(\rho_1 - \rho_2)$.

If $\rho_1 = \rho_2$, it is straightforward to show that the consumption share is also expected to increase: $E[dt] = \frac{e^{\theta t(\gamma + 1 + e^{\theta t(\gamma - 1)})}}{2(1 + e^{\theta t})^2} K dt$. Thus, regardless of the sentiment of the irrational investor, he is expected to underperform (in consumption growth terms) the rational investor at each point in time and the larger the transfer index is, the more severe the underperformance.

To quantify the underperformance of the irrational investor, we focus on the case when model parameters are constant, so that $K$ is constant. In this case we get closed form expressions for most variables of interest.\footnote{The theory of differential inequalities can be used to derive theoretical bounds on the underperformance when coefficients are state and/or time dependent, although the analysis becomes more complex. Intuitively, a lower bound on the transfer index leads to a lower bound on the underperformance through (17).} We study the expected stopping time, $E[\tau_f]$, i.e., the time it is expected to take for the rational investor to reach a consumption share of $f$. We have:

**Proposition 3** Assume that the irrational agent’s sentiment, $\delta$, and the covariance matrix, $\Sigma$, are constants, and that the rational agent’s (agent 1’s) initial consumption share is $f_0$. The expected time for the rational agent to reach the consumption share $f$, where $f > f_0$, is

$$E(\tau_f) = \frac{2\gamma \nu}{K + 2(\rho_2 - \rho_1)}$$

and the variance is

$$Var(\tau_f) = \frac{4\gamma K \nu}{(K + 2(\rho_2 - \rho_1))^3}$$

where $\nu = \log \left( \frac{f}{1-f} \right) - \log \left( \frac{f_0}{1-f_0} \right)$.

We note that if $K$ is large, any difference in the investors’ discount factors will be swamped by $K$. We will show that $K$ is indeed large, so going forward, we simply assume that the investors’ discount factors are the same, $\rho_1 = \rho_2$.

### 3.1 An example

We calibrate the model to a symmetric example. We consider a textbook style economy, in which all risky assets are affected by a market-wide shock and also by independent, idiosyncratic shocks. Specifically, we assume that $[g_i, g, i = 1, \ldots, N]$ and that $[\Sigma]_{i,i} = 2kN \sigma^2$, $i = 1, \ldots, N$, $[\Sigma]_{i,j\neq i} = kN \sigma^2$, $i, j = 1, \ldots, N$, where $k_N = \frac{N}{N+1}$ is a scaling factor introduced to make the total risk independent of $N$ (without it, the aggregate consumption volatility would be higher
for small $N$, since the benefit of diversification is lower for small $N$). For the time being, we allow $M$ to be any number between 1 and $N$. The economy thus has one systematic risk-factor and, for large $N$, one half of the risk is idiosyncratic whereas the other half is systematic.

We choose a growth rate of $g = 0.02$, in line with standard practice, and a growth volatility of $\sigma = 0.03218$, as found in Campbell (2003). From (17) and (19), we see that a high risk-aversion coefficient will lead to a lower degree of underperformance by the irrational investor. We therefore choose a somewhat high risk-aversion coefficient, $\gamma = 6$, to show that our results are robust. We use the common personal discount rate, $\rho_1 = \rho_2 = 1\%$. Finally, we assume that the irrational investor is slightly bullish about the first half of the stocks, $[\delta]_i = q \times g$, $i = 1, \ldots, N/2$, and slightly bearish about the other half, $[\delta]_i = -q \times g$, $i = N/2 + 1, \ldots, N$, where we for simplicity assume that $N$ is even. We assume that $q = 0.2$. Thus, the irrational investor overestimates the real dividend growth rate by 20% for half of the stocks (believing that it is 0.024) and underestimates it by the same amount for the other half (believing that it is 0.016). Although not necessary, it is natural to think of the case when $M = N$, and the $N$ risky assets represent the real growth processes of $N$ firms.

It follows immediately from (16) that the transfer index in this economy is:

$$K = \frac{g^2}{\sigma^2} q^2 \left( N + 1 \right), \quad N \text{ even.} \quad (21)$$

We see that $K$ is increasing in the real growth rate and decreasing in the real growth volatility. Moreover, not surprisingly, it is increasing in the irrational investor’s sentiment. What is crucial for our analysis, however, is that it is increasing in $N$. Thus, all else equal, the more risky assets there are in the economy, the more severe is the underperformance of the irrational investor. The intuition behind this result is that it is difficult for the rational investor to take advantage of the irrational one in a market with a restricted state space. For example, when there is only one stock, the only difference between the rational and irrational investors’ portfolios is the amount they invest in the market, which does not allow much separation. When there are many stocks, more portfolio separation is possible, the rational investor can better take advantage of the irrational one, and the irrational investors’ underperformance therefore becomes more severe.

Equation (21) holds for even $N$. It is easy to check that for odd $N$, the formula becomes

$$K = \frac{g^2}{\sigma^2} q^2 \left( N + 1 - \frac{1}{N} \right), \quad N \text{ odd.} \quad (22)$$

Therefore, $K$ triples when the number of stocks increases from one to two. This drastic increase in the transfer index occurs because with two stocks, the irrational agent will push down the price of the stock he is pessimistic about. Similarly, he will push the price of the other stock up.
The rational agent can then take on a relatively large position in the stock with the deflated price, financed with a relatively smaller position in the stock with an inflated price. This, in turn, leads to a higher Sharpe ratio for the rational agent than for the irrational agent. In contrast, when there is only one stock, differences in the two agent’s portfolio holdings are financed by different holdings of the risk-free asset and the agents therefore hold portfolios with the same Sharpe ratio, leading to less underperformance by the irrational agent. Thus, the representative firm setting (i.e., the setting with \( N = 1 \)) severely limits the underperformance by the irrational agent, and even with a modest number of assets his underperformance may be much more severe. Such a situation could, for example, arise if the irrational agent invests in a small number of mutual funds.

In Figure 1, we show the expected time to reach different consumption shares for \( N = 1, 10, 50 \) and 100 stocks, when the rational and irrational investor have the same initial consumption share \((f_0 = 1/2)\). We see that the difference between the representative firm \((N = 1)\) and the multi-firm \((N >> 1)\) settings is indeed drastic. With one stock, it is expected to take 1,706 years for the rational investor’s consumption share to reach 90%, whereas it takes 569 years with 2 stocks, 17 years with 100 stocks, and only 3.4 years with 500 stocks. With \( N = 100 \), the consumption share is expected to be 3.7% after 25 years, so in this case the irrational agent is expected to lose \((0.5-0.037)/0.5=93\)% of his initial consumption share of 50% to the rational agent.

The distribution of \( \tau_f \) is thin-tailed, so the time it takes to reach \( f \) will, with high probability, be close to \( E[\tau_f] \). In fact, as shown in the proof of Proposition 3, \( \tau_f \) has the first passage time distribution of a Brownian motion with drift \( K^2 \gamma \) and variance \( K^2 \gamma^2 \) per unit time, with a probability distribution function that decreases faster than that of a normal distribution for any fixed \( f \).

In Figure 2, we show the distribution of the time it takes to reach \( f = 90\% \), with \( f_0 = 0.5 \), for \( N = 1, N = 10, N = 50, \) and \( N = 100 \) stocks. For example, for \( N = 100 \), in which case the expected time is 17 years, the probability that it takes more than twice the expected time (34 years) is negligible.

The severe underperformance by the irrational investor is driven by the transfer index, \( K \). In this example we assumed that the personal discount rates of the two agents were the same. If the rational agent has a higher discount rate than the irrational agent, the irrational agent’s underperformance will decrease since the rational agent consumes at a higher rate. However, from (19) it follows that the differences in discount rates have to be very large to offset the underperformance generated by the transfer index. For example, in the previous example with \( N = 100 \), the rational agent’s discount rate has to be 39% per year to increase the expected time to reach a 90% consumption share by a factor of two, from 17 years to 34 years.
Figure 1: Expected time in years, $E[\tau_f]$, for the rational investor to reach the consumption share, $f$, for $N = 1$, $N = 10$, $N = 50$ and $N = 100$, when the irrational investor’s sentiment is $q = 20\%$ and the initial consumption shares are the same.

Figure 2: Probability distribution of the time it takes for rational investors to reach 90\% of the consumption share for $N = 1$, $N = 10$, $N = 50$ and $N = 100$, when the irrational investor’s sentiment is $q = 20\%$ and the initial consumption shares are the same.
Thus, in consumption terms, the irrational agent is severely punished in the multi-asset economy. We next study the effects on wealth and welfare.

3.2 Wealth and Welfare

It can be shown that the dynamics of the wealth share, \( f_{W,t} \), are very similar to those of the consumption share, \( f_t \). For some special cases, we have closed form solutions and for other cases we verify numerically that the dynamics of the consumption share and the wealth share are very similar. For simplicity, we restrict ourselves to the case when the agents have the same personal discount rates \( (\rho_1 = \rho_2) \) and the relative risk aversion coefficient is an integer; generalizations are straightforward.\(^9\)

There are two special cases in which closed form solutions for the wealth share are obtainable. First, when the investors have log-utility, \( \gamma = 1 \), it trivially follows that the dynamics of the consumption and wealth shares are identical, since both investors consume constant fractions of their wealths at all times.

**Proposition 4** When the investors have logarithmic utility, \( \gamma = 1 \), \( f_t \equiv f_{W,t} \) for all \( t \), and the results for the consumption share therefore also hold for the wealth share.

Second, for the case when aggregate consumption follows a constant coefficient geometric Brownian motion, e.g., when \( M = 1 \) in the model in the previous section, we have:

**Proposition 5** When \( M = 1 \) and \( g \) and \( \Sigma \) are constants, then the wealths of agents 1 and 2 are

\[
W_{1,t} = \frac{C_t}{1 + \lambda_t^\frac{1}{\gamma}} \sum_{k=0}^{\gamma-1} \left( \frac{\gamma - 1}{k} \right) \frac{1}{\alpha_k} \left( 1 - e^{-\alpha_k(T-t)} \right) \tag{23}
\]

\[
W_{2,t} = \frac{C_t}{1 + \lambda_t^\frac{1}{\gamma}} \sum_{k=0}^{\gamma-1} \left( \frac{\gamma - 1}{k} \right) \frac{1}{\alpha_{k+1}} \left( 1 - e^{-\alpha_{k+1}(T-t)} \right), \tag{24}
\]

where \( \alpha_k = \rho + \frac{k}{2\gamma} \left( 1 - \frac{k}{\gamma} \right) \Delta' \Delta + (\gamma - 1) \left( g_1 + \frac{1-\gamma}{2} \sigma_{\omega_1} \sigma_{\omega_1} + \frac{k}{\gamma} \sigma_{\omega_1} \Delta \right) \), and where we have defined the vector \( \sigma_{\omega_1} = ((\sigma_{\omega})_{11}, ..., (\sigma_{\omega})_{1N})' \), i.e., \( \sigma_{\omega_1} \) is the transpose of the first row of \( \sigma_\omega \).

It is straightforward to use Proposition 5 to see that when \( M = 1 \), the wealth and consumption shares are similar and that the severe underperformance of the irrational investor in terms of consumption therefore carries over to wealth. For example, in Figure 3, the wealth fraction

\[^9\]The proof in the appendix covers the case with different personal discount rates. The approach in Bhamra and Uppal (2009) can be used to further extend the results to noninteger \( \gamma \)'s.
divided by the consumption fraction, \( Z = \frac{W_1}{W_2} c_1/c_2 \), is shown as a function of the consumption fraction, for \( N = 1, 10, 50, 100 \) assets, and \( M = 1 \). The fraction is close to one, and it is greater than one when \( c_1 > c_2 \), implying that \( \frac{W_1}{W_2} > \frac{c_1}{c_2} \) when \( c_1 > c_2 \). The dynamics of the wealth share is therefore very similar to that of the consumption share in this case, and since \( c_1 \) will quickly become larger than \( c_2 \) when \( N \) is large, the wealth share of the irrational agent is typically even higher than the consumption share. We have also verified, using simulations, that the dynamics of the consumption share and wealth share when \( M = N \) (i.e., for the \( N \)-tree Lucas economy) are almost identical to the case when \( M = 1 \). The results are reported in the Appendix. Continuing with the example of the previous section, with \( N = 100 \) trees the wealth share of the rational agent is expected to be 3.24% after 25 years, i.e., it is even lower than the expected consumption share, which is 3.7%. The irrational agent is therefore expected to lose \((0.5-0.0324)/0.5=93.5\)% of his wealth in 25 years.

Figure 3: Wealth fraction divided by consumption fraction, \( Z = \frac{W_1}{W_2} \), as a function of consumption fraction, for \( N = 1, 10, 50, 100 \) and \( M = 1 \). \( Z \) is close to one, implying that the wealth fraction is very similar to the consumption fraction. Parameters: \( q = 20\% \), \( \sigma = 0.0328 \), \( g = 0.02 \), \( \rho_1 = \rho_2 = 1\% \), \( \gamma = 6 \), \( T = \infty \).

It is a priori unclear whether the underperformance of the irrational investor is associated with large welfare costs. In fact, since we are studying an exchange economy, the irrational investor does not influence production in any way, so one might suspect that the main effect of the underperformance is a transfer from the irrational to the rational investor. This intuition is incorrect and the welfare costs of agent 2’s irrationality may actually be high. The reason is
that he consumes in the wrong states of the world and at the wrong points in time. Especially, he tends to consume too early compared to what is objectively optimal.

In analyzing the welfare costs of agent 2’s irrationality, we use the objective expected utilities defined in Section 2, i.e., the expected realized utility of consumption of the two agents. We compare the objective expected utilities of agents 1 and 2 with the expected utilities they could realize in an economy with the same aggregate consumption process and utility weights, but with Pareto efficient consumption allocations in all states. We focus on the case when $T < \infty$ and $\gamma > 1$.10 Formally, we define the utility weight, $y = \frac{U_{OBJ1}}{U_{OBJ2}}$. A Pareto efficient allocation with the same utility weights would be achieved by allowing agent 1 to consume $\zeta C_t$ in all states of the world at all times, where $\zeta = \frac{y}{1+y^{1-\gamma}}$. Agent 2 would then consume $(1-\zeta)C_t$. We define the expected utility of a rational representative investor,

$$\hat{U}^{OBJ} = E_t \left[ \int_{0}^{T} e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt \right].$$ (25)

The expected utilities of the two agents with this Pareto efficient allocation would then be $\hat{U}_{OBJ1} = \zeta^{1-\gamma} \hat{U}_{OBJ}$, and $\hat{U}_{OBJ2} = (1-\zeta)^{1-\gamma} \hat{U}_{OBJ}$. Therefore, the relative expected utility improvements of the two agents, under the efficient allocation, are the same,

$$\zeta^{1-\gamma} \frac{\hat{U}_{OBJ}}{U_{OBJ1}} = \frac{y}{1+y^{1-\gamma}} \frac{\hat{U}_{OBJ}}{yU_{OBJ2}} = \frac{1}{1+y^{1-\gamma}} \frac{\hat{U}_{OBJ}}{U_{OBJ2}} = (1-\zeta)^{1-\gamma} \frac{\hat{U}_{OBJ}}{U_{OBJ2}}.$$ (25)

It follows almost immediately that the welfare cost of agent 2’s irrationality is $\theta = 1 - \zeta^{-1} \left( \frac{U_{OBJ1}}{U_{OBJ2}} \right)^{1-\gamma}$.11

That is, given that the two agents have wealths $W_1$ and $W_2$ in the original economy and objective expected utilities $U_{OBJ1}$ and $U_{OBJ2}$, the objective expected utilities they would realize in the Pareto efficient implementation correspond to what they would realize with wealths $\frac{1}{1-\theta}W_1$ and $\frac{1}{1-\theta}W_2$ in the original economy. We rewrite and summarize this in:

**Proposition 6** Given that the objective expected utilities of agents 1 and 2 are $U_{OBJ1}$ and $U_{OBJ2}$ and that the objective expected utility of a representative agent is $\hat{U}_{OBJ}$, the welfare cost of agent

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10If $T = \infty$, the objective utility of the irrational agent may not even be defined, since it may be negative infinity. The analysis of the case when $\gamma = 1$ is straightforward, although the formulas differ.

11Since expected utility is homogeneous of degree $1-\gamma$ in wealth for both agents, the corresponding relative wealth increase for both agents under a proportional sharing rule is $\beta = \zeta \left( \frac{U_{OBJ1}}{U_{OBJ2}} \right)^{-\gamma}$. The relative welfare of the economy with irrationality is therefore a fraction $\frac{1}{\beta}$ of the optimal economy, so the welfare cost is $1 - \frac{1}{\beta}$, which takes the given form.
2’s irrationality is
\[
\theta = 1 - \frac{|U_1^{OBJ}|^{\frac{1}{1-\gamma}} + |U_2^{OBJ}|^{\frac{1}{1-\gamma}}}{|\hat{U}^{OBJ}|^{\frac{1}{1-\gamma}}}. \tag{26}
\]

For special cases, we have closed form solutions for the welfare cost, just as was the case for the wealth dynamics:

**Proposition 7** When \( M = 1 \), and \( g \) and \( \Sigma \) are constants, the relative welfare cost in an economy with horizon \( T \) is

\[
\theta = 1 - \frac{\left( \sum_{k=0}^{\gamma-1} \left( \frac{\gamma - 1}{k} \right) \frac{\lambda_0}{\alpha_k} \left( 1 - e^{-\alpha_k T} \right) \right)^{\frac{1}{1-\gamma}} + \left( \sum_{k=0}^{\gamma-1} \left( \frac{\gamma - 1}{k} \right) \frac{\lambda_0}{\alpha_k} \left( \frac{1 - e^{-\alpha_k (1+T) \gamma}}{\alpha_k} \right) \right)^{\frac{1}{1-\gamma}}}{\left( \frac{\lambda_0}{\alpha_0} \left( 1 - e^{-\alpha_0 T} \right) \right)^{\frac{1}{1-\gamma}}}. \tag{27}
\]

Here, \( \alpha_k = \rho + \frac{k}{2\gamma} \left( 1 - \frac{k}{\gamma} \right) \Delta' \Delta + (\gamma - 1) \left( g_1 + \frac{1-\gamma}{2} \sigma_{\omega_1}' \sigma_{\omega_1} + \frac{k}{\gamma} \sigma_{\omega_1}' \Delta \right) \), \( \sigma_{\omega_1} = ((\sigma_{\omega})_{11}, \ldots, (\sigma_{\omega})_{1N})' \) and \( \lambda_0 \) is the social planner’s weight coefficient, defined in Section 2.

In Figure 4, we show the welfare cost of agent 2’s irrationality for economies with \( N = 1, 10, 50, 100 \) risky assets and \( M = 1 \), as a function of the horizon of the economy, \( T \), given that the two agents have the same initial wealth, \( W_{1,0} = W_{2,0} \). The welfare cost can indeed be high. For example, for the economy with \( N = 100 \) risky assets, and a 25-year investment horizon, the welfare cost is about 40% of the total wealth in the economy. For \( N = 50 \) it is even higher in longer horizons, about 43%.

Interestingly, the welfare cost is neither monotone in the number of risky assets, \( N \), nor in the time horizon, \( T \): A higher \( N \) may lead to a lower welfare cost, given \( T \), as may a longer horizon, given \( N \). The intuition is as follows. The initial wealths of the two agents are the same, so the values of the consumption paths of agent 1 and agent 2 are the same under the objective probability measure. The first-order condition therefore implies that the rational agent, at the margin, is willing to switch a dollar’s worth of \( c_{1,t} \) consumption for a dollar’s worth of \( c_{2,t} \) consumption. For large \( N \), the irrational agent’s expected realized utility is extremely low. It is therefore easy to make him much better off (objectively measured), by giving him a very small piece of total consumption, \( C_t \), at all times. The rational agent will hence consume almost the whole of \( C_t \), i.e., he almost increases his consumption by \( c_{2,t} \) in all states. However, although a dollar’s worth of \( c_{2,t} \) is worth a dollar in certainty equivalent for agent 1, the certainty equivalent of the whole \( c_{2,t} \) consumption stream is worth less for agent 1 than its dollar value. Indeed, the consumption increase is “lumped” in such a way so that it provides a lot more consumption for low \( t \), but almost no increase in consumption for large \( t \). Given the agent’s concave utility
function, the certainty equivalent of this additional consumption stream to agent 1 is therefore lower than the dollar value of the $c_{2,t}$ consumption stream. Thus, the higher $N$ is, the easier it is to make agent 2 better off, which increases the welfare cost. However, the higher $N$ is, the more difficult it is to make agent 1 better off, since a higher $N$ leads to more lumped consumption in time (because market selection is so fast). The two effects are offsetting, and the total effect is that the welfare cost may initially be increasing in $N$, but then decreasing, for a fixed $T > 0$.

A similar argument can be made for why, for a fixed high $N$, the welfare cost may be nonmonotone as a function of the investment horizon, $T$: Initially $\theta$ is increasing in $T$, since the higher $T$ is, the easier it is to make agent 2 better off. For larger $T$ however, it is very easy to make agent 2 better off, since his expected realized utility is so low, but it is also difficult to make agent 1 much better off, since even if he is given the whole of $C_t$, the consumption increase is so lumped towards early time periods that his utility does not increase that much. Of course, the larger $T$ is, the more “lumped” is the consumption increase given to agent 1. Therefore the welfare cost may be decreasing in $T$ for large $T$.

The closed form solution is valid when $M = 1$. In the $N$-tree Lucas economy (i.e., in the economy with $M = N$), we have verified with simulations that the welfare costs are very similar as when $M = 1$. In Table 1 we compare the two cases and show that the results are indeed almost identical.
<table>
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<th>Welfare cost, $\theta$</th>
<th>$T = 1$</th>
<th>$T = 10$</th>
<th>$T = 25$</th>
<th>$T = 50$</th>
</tr>
</thead>
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<td>$N = 10$ $M = 1$</td>
<td>0.0018</td>
<td>0.0168</td>
<td>0.0390</td>
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<tr>
<td>$M = N$</td>
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<td>0.0171</td>
<td>0.0413</td>
<td>0.0672</td>
</tr>
<tr>
<td>$N = 50$ $M = 1$</td>
<td>0.0085</td>
<td>0.1196</td>
<td>0.3389</td>
<td>0.4290</td>
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<tr>
<td>$M = N$</td>
<td>0.0091</td>
<td>0.1163</td>
<td>0.3275</td>
<td>0.4227</td>
</tr>
<tr>
<td>$N = 100$ $M = 1$</td>
<td>0.0178</td>
<td>0.2839</td>
<td>0.4003</td>
<td>0.3978</td>
</tr>
<tr>
<td>$M = N$</td>
<td>0.0182</td>
<td>0.2797</td>
<td>0.3977</td>
<td>0.3946</td>
</tr>
</tbody>
</table>

Table 1: Comparison of welfare costs, $\theta$, when $M = 1$ and $M = N$, for different number of risky assets, $N$, and investments horizons, $T$. For a given $N$ and $T$, the welfare costs are almost the same when $M = 1$ and when $M = N$. Parameters: $q = 20\%$, $\sigma = 0.0328$, $g = 0.02$, $\rho_1 = \rho_2 = 1\%$, $\gamma = 6$.

3.3 Potential policy implications

As we have seen, in the exchange economy setting, the welfare costs of agent irrationality can be severe. In a production economy, in which the irrational agents would also affect output, the costs may be even higher, as may also be the case if the irrational agents’ actions create systemic risk, potentially leading to economy-wide negative externalities.

The root of the costs is, of course, the suboptimal behavior by the unsophisticated investors in our model. The most straightforward approach would therefore seem to be to educate these investors. The importance of financial education has been emphasized in the household finance literature, see Campbell (2006). However, Campbell also stresses that financial education alone may not be enough and that regulation may also be important. He writes: “As a financial educator, I am tempted to call for an expansion of financial education. However, academic finance may have more to offer by influencing consumer regulation, disclosure rules and the provision of investment default options…”

So what is the appropriate regulatory response to the type of irrational behavior studied in this paper? In our model, it is the market completeness that allows the irrational agents to invest in severely suboptimal portfolios. In fact, if the asset span was restricted to include only claims on aggregate consumption, together with a risk-free asset, i.e., if the economy would have $N = 1$ risky assets, the welfare costs would be negligible, see Figure 4.\footnote{In other settings, finding an efficient regulatory policy when irrational agents are present is often difficult, since the agents’ responses to a well-intended regulation may actually lead to a worse outcome, see, Salanie and Treich (2009) for an example.} Thus, when irrational agents are present, the neoclassical view that financial innovation allows efficient risk sharing by completing markets — and thereby is welfare increasing — does not always tell the whole story, suggesting that a regulator may wish to restrict the asset span for unsophisticated investors.
Of course, in absence of other sources of agent heterogeneity than differences in degree of rationality, only claims on aggregate consumption are needed and a severely restricted asset span is optimal. In practice, additional sources of heterogeneity exist, because of agents’ hedging motives for idiosyncratic risk and heterogeneous preferences. A regulator must therefore weigh the benefits of increased risk sharing against the costs of unsophisticated investors’ suboptimal investments when determining the optimal asset span, the general rule being that unsophisticated investors may be restricted from trading in financial instruments on (disaggregated) idiosyncratic risks that offer no or few hedging benefits. An example of such a policy may be found in U.S. hedge fund regulations. Only accredited investors, i.e., investors with a minimum net worth of USD 1 million are allowed to invest in hedge funds, keeping poorer (less sophisticated) investors away from hedge funds’ volatile returns, dynamic trading strategies and investments in nontraditional asset classes.

Another regulatory policy is to enforce delegated management. The pension systems in several countries may serve as examples of such an approach. For example, between 1998-2001, Sweden moved away from a pay-as-you-go defined benefit system with government managed pension plans to a defined contribution system, in which a part of a worker’s pension contribution is invested through an individual retirement account, managed by the worker (SSIA, 2008). However, only about 15% of the capital is entrusted with the worker, whereas the remaining 85% is invested by a Swedish pension agency in government run pension funds. Moreover, the 15% managed by the worker can only be invested in funds that have been approved by the pension agency. Thus, the system has multiple mechanisms in place that restrict the feasible investment strategies.

4 Extensions

Our results are robust to several generalizations and variations.

4.1 Different risk aversion coefficients

Although the analysis becomes less tractable, the irrational agent also underperforms severely when the investors have different risk-aversion coefficients, $\gamma_1$ and $\gamma_2$, respectively. For example, it is possible to derive the following bound on the expected change in the log-consumption ratio, $E[h_t - h_0]$:

**Proposition 8** Given that the agents have risk aversion coefficients $\gamma_1$ and $\gamma_2$, and personal discount rates $\rho_1$ and $\rho_2$ respectively in the constant coefficient economy $\mathcal{E} = (\delta, g, \Sigma, D)$, define $\bar{g} = \max_i(g)_i$ and $\underline{g} = \min_i(g)_i$. The following inequalities hold for the expectation of the
log-consumption ratio, \( h_t = \log(c_{1t}/c_{2t}) \):

\[
E[h_t - h_0] \geq \begin{cases} 
\left( \frac{K}{2\gamma_2} + \frac{\gamma_2 - \gamma_1}{\gamma_2} \gamma \right) t + \frac{\gamma_2 - \gamma_1}{\gamma_2} \log(1 + e^{-h_0}), & \gamma_1 > \gamma_2, \\
\left( \frac{K}{2\gamma_2} + \frac{\gamma_2 - \gamma_1}{\gamma_2} \gamma \right) t - \frac{\gamma_2 - \gamma_1}{\gamma_2} \log(2) - O(e^{-qKt}), & \gamma_2 > \gamma_1.
\end{cases}
\]

Here \( z = O(f(t)) \) means that, for large \( t \), \( z \leq af(t) \), for some positive constant, \( a \). The term \( O(e^{-qKt}) \) thus quickly becomes small as \( t \) grows, for large \( K \).

Compared with equation (18), which is valid when \( \gamma_1 = \gamma_2 \), when \( \gamma_1 > \gamma_2 \) there are two additional terms in this bound. The first additional term depends on the maximum growth rate, \( \bar{g} \). This term appears because the difference in relative risk aversions of the two agents provides an additional motive for a transfer of consumption, beyond the heterogeneous probability measures. Specifically, since agent 1’s marginal utility decreases at a faster rate than agent 2’s when \( \gamma_1 > \gamma_2 \), a higher consumption is worth relatively more for agent 2 than for agent 1. Agent 2 is therefore willing to give up more of today’s consumption for future consumption, so his current consumption is lower and his future consumption is higher compared with what they would be if his risk aversion coefficient was \( \gamma_1 \), i.e., his consumption is expected to grow at a higher rate. This offsets the underperformance that is due to irrationality. The second additional term does not depend on \( t \). In total, since the transfer index is large for large \( N \), it will dominate the other terms and the irrational agent severely underperforms also when \( \gamma_1 > \gamma_2 \). A similar argument holds when \( \gamma_2 > \gamma_1 \). We note that (28) generalizes equation (18) in Yan (2008) in that it shows the rate at which market selection occurs, and in that it is valid for multi-asset economies, \( N > 1 \).

Similar to what was shown with the personal discount factor, when \( N \) is large, it follows immediately from (28) that the differences in risk aversion need to be very large to offset the market selection generated by the transfer index. In our example in Section 3.1, with \( N = 100 \), agent 1 needs a risk aversion coefficient of \( \gamma_1 = 25.5 \) to double the expected time it takes for him to reach 90% of the consumption share (from 17 to 34 years), when \( \gamma_2 = 6 \).

These results can not be generalized to arbitrary utility functions outside of the CRRA class, as shown in Kogan, Ross, Wang, and Westerfield (2009). Specifically, Kogan, Ross, Wang, and Westerfield (2009) show that the irrational investor may survive in the long term when the investors have unbounded relative risk aversion. It is intuitively clear that the irrational investors may survive within our setting too. This is seen in (28): If the economy reaches a state where \( \gamma_2 - \gamma_1 << 0 \), then market selection may not occur even if \( K \) is large. So, in an economy with varying risk-aversion coefficients, in which \( K \) is large, we initially expect the irrational investor to severely underperform the rational one, since \( K \) will initially dominate \( \gamma_2 - \gamma_1 \). However, if the utilities are such that eventually \( \gamma_2 - \gamma_1 << 0 \), then eventually the irrational investor’s
underperformance slows down and he survives—although with a very small consumption share.

### 4.2 Several investors

The generalization to economies with \( L > 2 \) investors is also straightforward. The following proposition generalizes the results in Proposition 3 to economies with multiple investor groups.

**Proposition 9** In the economy with \( L \) investors, define the consumption share of investor \( i \) with respect to investor \( j \),

\[
f_t \overset{\text{def}}{=} \frac{c_{it}}{c_{it} + c_{jt}},
\]

the stopping time \( \tau_f \overset{\text{def}}{=} \inf \{ t : f_t \geq f \} \), and the constants

\[
K_i \overset{\text{def}}{=} \delta_i^{-1} \Sigma^{-1} \delta_i, \quad K_j \overset{\text{def}}{=} \delta_j^{-1} \Sigma^{-1} \delta_j, \quad K_{ij} \overset{\text{def}}{=} \delta_i^{-1} \Sigma^{-1} \delta_j.
\]

If \( K_j > K_i \), then

\[
E(\tau_f) = \frac{2\gamma \nu}{K_j - K_i + 2(\rho_j - \rho_i)},
\]

and

\[
\text{Var}(\tau_f) = \frac{4\gamma \nu (K_j + K_i - 2K_{ij})}{(K_j - K_i + 2(\rho_j - \rho_i))^3},
\]

where \( \nu = \log\left(\frac{f_i}{1-f_i}\right) - \log\left(\frac{f_0}{1-f_0}\right) \).

Thus, the severity of the underperformance of group \( j \) relative to \( i \) is decided by the difference of their two transfer indices, \( K_j - K_i \), in the case with multiple investors.

### 4.3 General risk structures

So far we have studied an economy with one systematic source of risk that affects all firms equally, symmetric idiosyncratic risk, and an irrational investor with symmetric sentiments. Our results are much more general, however. For example, the irrational investor needs not to be symmetric in his sentiments. To see this, consider the economy with the same growth rate and risk as in Section 3.1, but with asymmetric sentiments. Specifically, the irrational investor believes that the drift is \( g(1 + q_1a) \) for a fraction, \( a \), of the stocks and \( g(1 + q_2) \) for the remaining stocks, where we assume that \( aN \) and \( (1 - a)N \) are both integers. Without loss of generality, we assume that \( 0 < a < 1 \) (as the cases \( a = 0 \) and \( a = 1 \) are covered by taking \( q_1 = q_2 \)).

From the definition of the transfer index, \( K \), in Proposition 2, it is straightforward to show that

\[
K = \frac{g^2}{\sigma^2} \left( q_1^2 a + q_2^2 (1 - a) + (q_1^2 a + q_2^2 (1 - a) - (q_1 a + q_2 (1 - a))^2) \right) \times N.
\]

Thus, by Proposition 3, the underperformance of the irrational investor will be severe for large \( N \), unless the term \( q_1^2 a + q_2^2 (1 - a) - (q_1 a + q_2 (1 - a))^2 \) is equal to zero. It is easy to show that this term is equal to zero if and only if \( q_1 = q_2 \). If \( q_1 = q_2 \), then \( K = \frac{g^2}{\sigma^2} q_1^2 \) so the irrational
investor’s underperformance does not depend on $N$. Effectively, since his sentiment is uniform, he holds the same portfolio as the rational agent, and the model collapses to the representative firm model.

To study the underperformance in an even more general setting, we introduce a sequence of economies $\mathcal{M} = (\mathcal{E}_1, \ldots, \mathcal{E}_N, \ldots)$ where $\mathcal{E}_N = (\delta_N, \mathbf{g}_N, \Sigma_N, \mathbf{D}_N)$. The idea is now to see if the irrational investor’s underperformance becomes severe as $N$ tends to infinity, in the sense that the expected time to reach any prescribed consumption share approaches zero. In this case we say that high-speed market selection occurs for large $N$. Formally, for a sequence of economies, $\mathcal{M}$, we say that

**Definition 2** High-speed market selection occurs if, in market $\mathcal{E}_N$, the expected time to reach the consumption share $f$ when the initial consumption share is $f_0$, satisfies $E(\tau_f) \leq G(f_0, f, N)$ for some function $G : (0,1) \times (0,1) \times \{1, 2, 3, \ldots\} \to \mathbb{R}_+$, which for all $f_0$ and $f > f_0$ satisfies

$$\lim_{N \to \infty} G(f_0, f, N) = 0.$$  

**Definition 3** High-speed market selection of order $\nu$ (where $\nu > 0$ is a constant) occurs if the function, $G$, in Definition 2 can be written in the form $G(f_0, f, N) = H(f_0, f)/N^\nu$.

We let $K_N$ denote the transfer index term in economy $\mathcal{E}_N$. Proposition 3 implies that high-speed market selection of order $\nu$ occurs if and only if

$$k \overset{\text{def}}{=} \liminf_{N \to \infty} \frac{K_N}{N^\nu},$$

where $K_N = \delta_N' \Sigma_N^{-1} \delta_N$, is greater than zero, i.e., if and only if $0 < k \leq \infty$.

We define $\rho(\Sigma)$ to be the spectral radius of the covariance matrix, $\Sigma$.\(^{13}\) We have a couple of immediate results, relating the spectral radius of the covariance matrix, $\rho(\Sigma)$, the sentiment, $\delta$, and the transfer index, $K$. From (21,22), it follows that high-speed market selection of order one occurs in the example in Section 3.1. The following two propositions can be used to show high speed market selection in more general economies:

**Proposition 10** For a sequence of economies, $\mathcal{M}$, high-speed market selection of order $\nu$ occurs if there are positive constants $c$ and $N_0$, such that for all $N > N_0$: $\rho(\Sigma_N) \leq cN^{-\nu} \delta_N'^2 \delta_N$.

**Proposition 11** For an arbitrary vector, $\mathbf{q}$, define $Q_\mathbf{q}$ to be the Euclidean projection operator onto the orthogonal complement of $\mathbf{q}$, and the $N$-vector of ones, $\mathbf{1}_N = (1,1,\ldots,1)^T$.\(^{14}\) For a

\(^{13}\)Since $\Sigma$ symmetric and positive definite, its spectral radius is simply its largest eigenvalue. We use the standard notation $\rho(\Sigma)$, since it should create no confusion with the personal discount rate, $\rho$.

\(^{14}\)That is, in matrix notation, $Q_\mathbf{q} = I - \frac{\mathbf{q}\mathbf{q}^T}{\mathbf{q}^T\mathbf{q}}$, where $I$ is the identity matrix.
sequence of economies, $\mathcal{M}$, high-speed market selection of order $\nu$ occurs if there are positive constants $c$ and $N_0$, such that the following two conditions are satisfied for all $N > N_0$:

- $\mathbf{1}_N$ is an eigenvector of $\Sigma_N$,
- $\rho(Q_{1_N}^\prime \Sigma_N Q_{1_N}) \leq cN^{-\nu} \left( \delta_N^\prime \delta_N - \frac{(\delta_N^\prime \delta_N)^2}{N} \right)$.

It is straightforward to check that both the example in Section 3.1 and the example with asymmetric sentiments in this section satisfy the conditions of Proposition 11 with $\nu = 1$, with the exception of asymmetric sentiments with $q_1 = q_2$.\(^{15}\) We can also use the proposition to study economies with general covariance matrices of the form $\Sigma_N = N^{-1}(aI_N + b\mathbf{1}_N \mathbf{1}_N^\prime)/(a + bN)$, $a > 0$, $b \geq 0$, where $I_N$ is the $N \times N$ identity matrix. For such covariance matrices, the first condition of Proposition 11 is always satisfied. Moreover, the second condition is satisfied with $\nu = 1$, as long as $b > 0$, i.e., as long as the economy has a systematic risk component. If there is no systematic component, there are effectively $N$ separate financial markets, and in each of these markets there is a representative firm. It is easy to show that the market selection process will be slow in this case.

### 4.4 Sequences of random economies

Propositions 10 and 11 can be used to prove high-speed market selection for a specific sequence of markets, but do not say how “often” high-speed market selection occurs. Is high-speed market selection the norm, or are the previous examples just exceptional special cases? To answer this question, we study how often high-speed market selection occurs in randomly generated economies. We look at economies with one systematic risk component affecting all firms, and random parameters. In the appendix we discuss the generalization of the results to $Q > 1$ common risk components.

We make several assumptions about the randomness of the economies, $\mathcal{E}_N = (\delta_N, \mathbf{g}_N, \Sigma_N, \mathbf{D}_N)$. We assume that $(\mathbf{g}_N)_i = \tilde{p}^N_i$, where $\tilde{p}^N_i$ are i.i.d. random variables, $E(\tilde{p}^N_i) = \bar{p} > 0$ and $\text{Var}(\tilde{p}^N_i) = \sigma^2_p > 0$. Similarly, $(\delta_N)_i = \tilde{q}^N_i$, where $\tilde{q}^N_i$ are i.i.d. random variables, $E(\tilde{q}^N_i) = \bar{q}$ and $\text{Var}(\tilde{q}^N_i) = \sigma^2_q > 0$. Furthermore, we assume that the randomness of the $i$th asset, $\sigma_i dB_{it}$, is of the form $\sigma_i dB_{it} = (\beta^N_i d\xi_{0it} + \alpha^N_i d\xi^N_{it})$, where $\xi_{it}$ are i.i.d. standard Brownian motions, and $\alpha^N_i, \beta^N_i$ are i.i.d. random variables: $E(\alpha^N_i) = \bar{\alpha} > 0$, $\text{Var}(\alpha^N_i) = \sigma^2_{\alpha} > 0$, $E(\beta^N_i) = \bar{\beta}$ and $\text{Var}(\beta^N_i) = \sigma^2_{\beta} > 0$. All random variables, $\tilde{p}^N_i, \tilde{q}^N_i, \beta^N_i, \alpha^N_i, \xi^N_{it}$ are jointly independent. For simplicity, we furthermore assume that all random variables are absolutely continuous (with respect to Lebesgue measure) and that the $\beta$’s are (a.s.) bounded below by a strictly positive constant, $\epsilon > 0$.\(^{16}\) We also require the $\tilde{p}$’s to be strictly positive (a.s.).

---

\(^{15}\)If $q_1 = q_2$, the second condition of Proposition 11 fails since the right hand side is identically equal to zero.\(^{16}\)These assumptions can be relaxed in several directions, but at the expense of increased complexity.
For a fixed $N$, the economy $\mathcal{E}_N$ will thus be characterized by $g_N = (\tilde{p}_1^N, \ldots, \tilde{p}_N^N)'$, $\delta_N = (\tilde{q}_1^N, \ldots, \tilde{q}_N^N)'$, $\Sigma_N = (\text{diag}(\alpha_1^N, \ldots, \alpha_N^N)^2 + b_N b_N')$, where $b_N = (\beta_1^N, \ldots, \beta_N^N)'$, and the $D_N$’s are arbitrary weakly positive vectors, with at least one nonzero element. Under these conditions, we have

**Proposition 12** In a sequence of economies, $\mathcal{M} = (\mathcal{E}_1, \mathcal{E}_2, \ldots)$, satisfying the previous assumptions, high-speed market selection of order 1 occurs almost surely.

Thus, high-speed market selection is really the norm in such economies and the exception is when it breaks down.

### 4.5 Other types of irrationality, structural uncertainty and learning

There is no structural uncertainty in our model. We believe that our results would be important even in a set-up with such uncertainty, since the speed of the market selection process is so high that irrational investors may be wiped out even if there are structural breaks, say every decade. In such an economy, we would expect a Bayesian rational investor to quickly learn enough about the new structure of the economy after a structural break to take advantage of irrational investors.

Also, the type of “stubborn” irrationality we have assumed may seem rather extreme: Surely, an irrational investor that is losing a lot of money will learn about the firms’ true growth rates or get out of the market. We do not proceed along these lines, but it is quite clear what happens in a model where irrational investors learn over time. Equation (17) describes the instantaneous transfer and therefore the speed at which an irrational investor needs to learn to avoid extinction. The transfer index therefore induces a “learn or die” bound on how quickly irrational investors must learn. In any case, in a market with a high transfer index the long-term outcome is a market in which rational investors hold the bulk of the wealth, either because irrational investors are wiped out, because they learn to be rational, or because they leave the market.

Our model does not cover deviations from subjective expected utility theory, for example, along the lines of Prospect Theory (Kahneman and Tversky, 1979). A detailed dynamic analysis of such deviations is outside the scope of this paper, but we conjecture that the force of market selection in a market with a large transfer index is unlikely to be offset by less than large deviations in preferences. For example, if the irrational investor has a convex utility function in the loss domain, this implies that his risk-aversion coefficient is negative in that domain. The analysis in Section 4.1 suggests that this will decrease the speed of the market selection process, but also that the convexity must be severe to completely offset a high transfer index.
5 Concluding remarks

We have shown that irrational investors in multi-asset economies, who mistake growth rates of firms, severely underperform rational ones. The severity of the underperformance depends on the economy’s so-called transfer index, which typically swamps differences in personal discount rates and risk aversion coefficients. In a calibration, we show that an irrational investor loses almost 95% of his consumption and wealth shares to a rational investor over a 25-year horizon. Moreover, the realized consumption paths of the two investors are severely suboptimal. The welfare costs are about 40% of the total wealth in the economy in a 25-year horizon. Our results highlight the value of financial education and also suggest that delegated investment management, as well as restrictions on the asset span in the market, may be welfare increasing when unsophisticated investors are present.
Appendix

Proof of Proposition 1: To solve for equilibrium we use the martingale approach (Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987)). Each agent solves the static optimization problem

$$\max_{c_k} E_k \left[ \int_0^T e^{-\rho_k t} \frac{c_{k,t}^{1-\gamma}}{1-\gamma} dt \right]$$

(31)

s.t.

$$E_k \left[ \int_0^T \xi_{k,t} c_{k,t} dt \right] \leq f_{W_k,0} E_k \left[ \int_0^T \xi_{k,t} C_t dt \right],$$

(32)

where $f_{W_k,0}$ is the initial wealth fraction of agent $k$ and $\xi_{k,t}$, $k \in \{0, 1\}$, are the agent specific SDFs, yet to be defined. Necessary and sufficient conditions for optimality of the consumption streams are

$$c_{k,t} = (y_k e^{\rho_k t} \xi_{k,t})^{1-\gamma},$$

(33)

where $y_k > 0$ is such that the budget constraint holds with equality

$$E_k \left[ \int_0^T \xi_{k,t} (y_k e^{\rho_k t} \xi_{k,t})^{1-\gamma} dt \right] = f_{W_k,0} E_k \left[ \int_0^T \xi_{k,t} C_t dt \right].$$

(34)

To solve for the optimal consumption streams of the two agents, we introduce the central planner’s problem

$$u(C_t, \lambda_t, t) = \max_{c_{1,t}, c_{2,t}} \left\{ e^{-\rho_1 t} \frac{c_{1,t}^{1-\gamma}}{1-\gamma} + \lambda_t e^{-\rho_2 t} \frac{c_{2,t}^{1-\gamma}}{1-\gamma} \right\}$$

(35)

s.t.

$$c_{1,t} + c_{2,t} = C_t,$$

(36)

where

$$\lambda_t = \left( \begin{array}{c} y_1 \xi_{1,t} \\ y_2 \xi_{2,t} \end{array} \right),$$

$$= \lambda_0 \eta_t.$$  

(37)

Here, $\lambda_0 = \frac{y_1}{y_2}$ as $x_{1,0} = x_{2,0} = 1$, and $\eta_t = e^{\int_0^t \Delta s d\Delta s + \int_0^t \Delta s dW_s}$. From the first-order conditions of the central planner’s problem in Equation (35) we have

$$c_{2,t} = e^{(\rho_1 - \rho_2)t/\gamma} \frac{1}{\lambda_t} c_{1,t}. $$

(38)

Using (36) and (38), we get

$$c_{1,t} + e^{(\rho_1 - \rho_2)t/\gamma} \frac{1}{\lambda_t} c_{1,t} = C_t.$$  

(39)

Rearranging, we get

$$c_{1,t} = f_t C_t$$

$$c_{2,t} = (1 - f_t) C_t$$

(40)
where \( f_t = \frac{1}{1 + \epsilon^{(\rho_1 - \rho_2)t/\gamma \lambda_t^\gamma}} \). Finally, solving Equation (33) for \( \xi_{1,t} \) we get

\[
\xi_{1,t} = e^{-\rho_1 t} \left( \frac{c_{1,t}}{c_{1,0}} \right)^{-\gamma} = e^{-\rho_1 t} \left( \frac{f_t}{f_0} \right)^{-\gamma} \left( \frac{C_t}{C_0} \right)^{-\gamma}.
\]

(41)

The expressions for the wealth of the two agents follows from discounting future optimal consumption using the SDF.

Proof of Proposition 2: From the optimal consumption allocations in Proposition 1 we have

\[
h_t = \log \left( \frac{c_{1,t}}{c_{2,t}} \right) = \log \left( e^{-\rho_1^2t/\gamma \lambda_t^\gamma} \right) = -\frac{1}{\gamma} \log \left( e^{\rho_1^2t/\lambda_t^\gamma} \right),
\]

so

\[
dh_t = \left( \frac{1}{2\gamma} \Delta \lambda_t + \frac{1}{\gamma} (\rho_2 - \rho_1) \right) dt - \frac{1}{\gamma} \Delta dB_t
\]

\[
= \left( \frac{1}{2\gamma} K + \frac{1}{\gamma} (\rho_2 - \rho_1) \right) dt + \frac{1}{\gamma} \sqrt{K} \tilde{B}_t
\]

where \( K = \delta' \Sigma^{-1} \delta \) and \( \tilde{B}_t = -\frac{1}{\sqrt{K}} \Delta B_t \) is a standardized Brownian motion.

Proof of Proposition 3: The first passage probability density distribution for the time it takes for \( \log \left( \frac{1-f}{1-f_0} \right) \) to reach \( \log \left( \frac{1-f}{1-f_0} \right) \) with initial condition \( \log \left( \frac{1-f_0}{1-f_0} \right) \) is (Ingersoll, 1987)

\[
p.d.f. (\tau f) = \frac{\log \left( \frac{1-f}{1-f_0} \right) - \log \left( \frac{1-f_0}{1-f_0} \right)}{2\pi \frac{K^3}{2}} \exp \left[ -\log \left( \frac{1-f}{1-f_0} \right) - \log \left( \frac{1-f_0}{1-f_0} \right) - \left( \frac{1}{2\gamma} K + \frac{1}{\gamma} (\rho_2 - \rho_1) \right) \right] \left[ \frac{2\gamma}{K} \right]^\frac{1}{2}.
\]

The expected time is (Ingersoll, 1987)

\[
E (\tau f) = \frac{2\gamma}{K + 2 (\rho_2 - \rho_1)} \left( \frac{1-f}{1-f_0} \right),
\]

(43)

and the variance is

\[
Var (\tau f) = \frac{4\gamma K \left( \frac{1-f}{1-f_0} \right)}{(K + 2 (\rho_2 - \rho_1))^3}.
\]

(44)

Proof of Proposition 4: The result follows from the fact that the wealth-consumption ratio is constant for
investors with logarithmic preferences.

**Proof of Proposition 5:** For generality, we prove the proposition for the case when the personal discount factors are agent specific. From Equation (15)

\[
W_{1,t} = E_t \left[ \int_t^T \frac{\xi_s}{\xi_t} f_t C_t \, ds \right]
\]

\[
= \frac{C_t}{(1 + \lambda_t^+)} \int_t^T e^{-\rho t (s-t)} \left[ \left( 1 + e^{(\rho_1 - \rho_2)s/\gamma} \right)^{(k-1)} \left( \frac{C_s}{C_t} \right)^{1-\gamma} \right] ds
\]

\[
= \frac{C_t}{(1 + \lambda_t^+)} \int_t^T e^{-\rho t (s-t)} \sum_{k=0}^{\gamma-1} \left( \frac{\gamma - 1}{k} \right) e^{(\rho_1 - \rho_2)sk/\gamma} \frac{1}{C_t} \left( \frac{\lambda_t^+}{\lambda_t} \right)^{\frac{1}{\gamma}} E_t \left[ \left( \frac{C_s}{C_t} \right)^{1-\gamma} \right] ds.
\]

(45)

Note that since \( \left( \frac{\lambda_t}{\lambda_t^+} \right)^{1-\gamma} \) and \( \left( \frac{C_t}{C_s} \right)^{1-\gamma} \) are jointly log-normal we get \( E_t \left[ \left( \frac{\lambda_t}{\lambda_t^+} \right)^{1-\gamma} \left( \frac{C_t}{C_s} \right)^{1-\gamma} \right] = e^{A_k(s-t)} \), where

\[
A_k = \frac{\lambda_t^+}{\lambda_t} \left( \frac{1}{\gamma} - 1 \right) \Delta' \Delta + (1 - \gamma) \left( g_1 + \frac{1}{\gamma} \sigma_{\omega_1} \sigma_{\omega_1} + \frac{1}{\gamma} \sigma_{\omega_1} \Delta \right).
\]

Inserting into Equation (45) and solving the integral we get

\[
W_{1,t} = \frac{C_t}{(1 + \lambda_t^+)} \sum_{k=0}^{\gamma-1} \left( \frac{\gamma - 1}{k} \right) e^{(\rho_1 - \rho_2)sk/\gamma} \frac{1}{C_t} \left( \frac{\lambda_t^+}{\lambda_t} \right)^{\frac{1}{\gamma}} \left( 1 - e^{-\alpha_k(s-t)(k+1)/\gamma} \right) / \alpha_k (\rho_1 - \rho_2) (k+1)/\gamma
\]

where \( \alpha_k = \rho_1 - A_k \). Following a similar calculation we get

\[
W_{2,t} = \frac{C_t}{(1 + \lambda_t^+)} \sum_{k=0}^{\gamma-1} \left( \frac{\gamma - 1}{k} \right) e^{(\rho_1 - \rho_2)(k+1)/\gamma} \frac{1}{C_t} \left( \frac{\lambda_t^+}{\lambda_t} \right)^{\frac{1}{\gamma}} \left( 1 - e^{-\alpha_k(s-t)(k+1)/\gamma} \right) / \alpha_k (\rho_1 - \rho_2) (k+1)/\gamma
\]

**Proof of Proposition 7:** Using a similar approach as in the proof of Proposition 5, the objective expected utility of agents 1 and 2 can be written as

\[
U_1^{OBJ} = E \left[ \int_0^T e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt \right]
\]

\[
= \frac{1}{1-\gamma} \int_0^T e^{-\rho t} E \left[ C_t^{1-\gamma} \right] \frac{1}{1-\gamma} \left( 1 + \lambda_t^+ \right)^{\gamma-1} dt
\]

\[
= \frac{C_t^{1-\gamma}}{1-\gamma} \sum_{k=0}^{\gamma-1} \left( \frac{\gamma - 1}{k} \right) \lambda_t^+ / \alpha_k \left( 1 - e^{-\alpha_k s} \right)
\]
and

\[ U^{\text{OBJ}} = \frac{C_0^{1-\gamma} \sum_{k=0}^{\gamma-1} \left( \frac{\gamma-1}{k} \right) \frac{1}{\alpha_{k+1-\gamma}} (1 - e^{-\alpha_{k+1-\gamma} T})}{1 - \gamma} \]

Using the above expressions together with Equation (26) we get

\[ \beta = \frac{\left( \frac{1}{\alpha_0} \right)^{1 - e^{-\alpha_0 T}}}{\left( \sum_{k=0}^{\gamma-1} \left( \frac{\gamma-1}{k} \right) \frac{1}{\alpha_k} (1 - e^{-\alpha_k T}) \right)^{1 - e^{-\alpha_0 T}}} \left( \frac{k+1-\gamma}{\alpha_{k+1-\gamma}} \right)^{1 - e^{-\alpha_{k+1-\gamma} T}} \]

and since \( \theta = 1 - \frac{1}{\gamma} \), (27) follows.

**Proof of Proposition 8:**
Assume that \( \gamma_1 \geq \gamma_2 \). From the agents’ first-order conditions it follows that

\[ \left( \frac{c_{1t}^{1-\gamma}}{c_{2t}^{1-\gamma}} \right)^{1-\gamma_1/\gamma_2} = e^{-\rho_2 - \rho_1} t^{1-\gamma_1/\gamma_2} \lambda_t, \]

and therefore

\[ \frac{c_{1t}^{1-\gamma_1/\gamma_2}}{c_{2t}^{1-\gamma_1/\gamma_2}} = e^{(\rho_2 - \rho_1) t^{1-\gamma_1/\gamma_2}} \lambda_t = q_t. \]

Now, \( c_{1t}^{1-\gamma_1/\gamma_2} = c_{1t}^{1-\gamma_1/\gamma_2} (1 + c_{2t}/c_{1t})^{1-\gamma_1/\gamma_2} \), so we have

\[ q_t = \frac{c_{1t}^{1-\gamma_1/\gamma_2}}{c_{2t}} = \frac{c_{1t}}{c_{2t}} C_{1t}^{1-\gamma_1/\gamma_2} (1 + c_{2t}/c_{1t})^{1-\gamma_1/\gamma_2}, \]

so

\[ \frac{c_{1t}}{c_{2t}} (1 + c_{2t}/c_{1t})^{1-\gamma_1/\gamma_2} = q_t C_{1t}^{1-\gamma_1/\gamma_2}, \quad \text{(46)} \]

and therefore

\[ \left( \frac{c_{1t}/c_{2t}}{c_{10}/c_{20}} \right) \left( \frac{1 + c_{2t}/c_{1t}}{1 + c_{20}/c_{10}} \right)^{1-\gamma_1/\gamma_2} = \left( \frac{q_t}{q_0} \right) \left( \frac{C_t}{C_0} \right)^{1-\gamma_1/\gamma_2}. \]

Since \( \gamma_1 > \gamma_2 \), it follows that \( (1 + c_{2t}/c_{1t})^{1-\gamma_1/\gamma_2} \leq 1 \), and therefore

\[ \left( \frac{c_{1t}/c_{2t}}{c_{10}/c_{20}} \right) \left( \frac{1}{1 + c_{20}/c_{10}} \right)^{1-\gamma_1/\gamma_2} \geq \left( \frac{q_t}{q_0} \right) \left( \frac{C_t}{C_0} \right)^{1-\gamma_1/\gamma_2}. \]

Taking logarithms on both sides leads to

\[ h_t - h_0 - \frac{\gamma_2 - \gamma_1}{\gamma_2} \log(1 + e^{-h_0}) \geq \log \left( \frac{q_t}{q_0} \right) + \frac{\gamma_2 - \gamma_1}{\gamma_2} \log \left( \frac{C_t}{C_0} \right). \]

Taking expectations on both sides and rearranging, using the method in the proof of Proposition 2 for \( q_t \) and
Therefore, function of $E$ and since $(1 + z_i)^{1-\alpha} = e^{h_i}$, so, taking logarithms on both sides,

$$y_t = \alpha h_t + (1 - \alpha) \log(1 + e^{h_t}).$$

It is easily seen that $y$ is a strictly convex function of $h$, such that $y'(-\infty) = \alpha$, $y'(\infty) = 1$, and $y(0) = (1 - \alpha) \log(2)$. Therefore, $h = f(y)$ where $f$ is a strictly concave function, such that $f'(-\infty) = \frac{1}{\alpha}$, $f'(\infty) = 1$ and $f((1 - \alpha) \log(2)) = 0$.

It therefore follows that $h = f(y) \geq y - (1 - \alpha) \log(2) + \left(\frac{1}{\alpha} - 1\right)(y - (1 - \alpha) \log(2))$, where $(x)_-$ defines a function of $x$ that is equal to $x$ when $x < 0$ and to $0$ when $x \geq 0$.

Now, since $y_t = \log(q_t) + (1 - \gamma_1/\gamma_2) \log(C_t)$, using the same approach as in proposition 3, it follows that $(y_t - (1 - \alpha) \log(2))_-$ first order stochastically dominates $(v_t - (1 - \alpha) \log(2))_-$, where

$$v_t \sim N\left(\log(q_0) + (1 - \gamma_1/\gamma_2) \log(C_0) + \left(\frac{K}{2\gamma_2} + \frac{\rho_2 - \rho_1}{\gamma_2}\right) t, \frac{4K^2}{\gamma^2} t\right),$$

so $E[(y_t - (1 - \alpha) \log(2))_-] \geq E[(v_t - (1 - \alpha) \log(2))_-]$.

It is easy to show that for a random variable, $x \sim N(\mu, \sigma^2)$ and constant $\beta < \mu$ it is the case that $E[(x - \beta)_-] = O\left(\frac{\sigma^3}{(\mu - \beta)^2} e^{\frac{(\mu - \beta)^2}{2\sigma^2}}\right)$, from which it follows that $E[(v_t - (1 - \alpha) \log(2))_-] = -O(K^{-1/2} e^{-qKt}) = -O(e^{-qKt})$.

Therefore, $E[h_t] \geq E[y_t] - (1 - \alpha) \log(2) - O(e^{-qKt})$.

Moreover, from (46), it is clear that $h_0 + (1 - \gamma_1/\gamma_2) \log(1 + e^{-h_0}) = y_0$, so $h_0 \leq y_0$. Together these two inequalities imply that

$$E[h_t - h_0] \geq E[y_t - y_0] - (1 - \alpha) \log(2) - O(e^{-qKt}),$$

and since

$$E[y_t - y_0] = -\frac{1}{\gamma_2} E\left[\log\left(\frac{q_t}{q_0}\right)\right] + \frac{\gamma_2 - \gamma_1}{\gamma_2} E\left[\log\left(\frac{C_t}{C_0}\right)\right] \geq \frac{1}{\gamma_2} \left(\frac{K}{2} + \rho_2 - \rho_1\right) t + \frac{\gamma_2 - \gamma_1}{\gamma_2} q_t,$$

the result follows.

**Proof of Proposition 9:** From the agents’ first-order conditions it follows that

$$\frac{c_{i,t}}{c_{j,t}} = \left(\frac{\lambda_{i,t}}{\lambda_{j,t}}\right)^{1/\gamma} e^{(\rho_j - \rho_i) t/\gamma}.$$

Consequently we have that

$$\log\left(\frac{c_{i,t}}{c_{j,t}}\right) = \left(\frac{1}{2\gamma} (K_j - K_i) + \frac{1}{\gamma} (\rho_j - \rho_i)\right) t + \frac{1}{\gamma} (\Delta_i - \Delta_j)' B_t$$

$$= \left(\frac{1}{2\gamma} (K_j - K_i) + \frac{1}{\gamma} (\rho_j - \rho_i)\right) t + \frac{1}{\gamma} \sqrt{(K_i + K_j - 2K_{ij})} B_t,$$

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where $\tilde{B}_t = \frac{1}{\sqrt{(K_j + K_j - 2K_j)}} (\Delta_i - \Delta_j)' B_t$ is a standard Brownian motion. Using the same approach as in the proof of Proposition 3 we get

$$E(\tau_f) = \frac{2\gamma\nu}{K_j - K_i + 2 (\rho_j - \rho_i)},$$

and

$$Var(\tau_f) = \frac{4\gamma\nu (K_j + K_i - 2K_j)j}{(K_j - K_i + 2 (\rho_j - \rho_i))^2},$$

where $\nu = \log \left(\frac{\rho_j}{1 - f_{\rho_j}}\right) - \log \left(\frac{\rho_i}{1 - f_{\rho_i}}\right)$.

**Proof of Proposition 10:** The proof is a straightforward application of spectral decomposition. The spectral theorem ensures that for each $N$, there is a real orthogonal transformation of $\Sigma_N$ into a diagonal matrix with strictly positive elements, $\Sigma_N = R'_N \Lambda_N R_N$, $\Lambda_N = diag(\rho_1, \ldots, \rho_N)$ and $R_N^{-1} = R'_N$. W.l.o.g., we can assume that the $\rho$’s are ordered increasingly, so the spectral radius of $\Sigma_N$ is $\rho_N$. Standard matrix-norm theory implies that

$$\min_{\delta_N \neq 0_N} \frac{\delta_N \Sigma_N^{-1} \delta_N}{\delta_N \delta_N} = \frac{1}{\rho_N},$$

so $\delta_N \Sigma_N^{-1} \delta_N \geq \frac{\delta_N \delta_N}{\rho_N}$, and by our assumptions $\frac{\delta_N \delta_N}{\rho_N} \geq c^{-1} N^\nu$, so $K_N = \delta_N \Sigma_N^{-1} \delta_N \geq c^{-1} N^\nu$.

**Proof of Proposition 11:** As in the proof of the previous proposition, the spectral theorem ensures that for each $N$, there is a real orthogonal transformation of $\Sigma_N$ into a diagonal matrix with strictly positive elements, $\Sigma_N = R'_N \Lambda_N R_N$, $\Lambda = diag(\rho_1, \ldots, \rho_N)$ and $R_N^{-1} = R'_N$. Moreover, the first assumption ensures that there is an eigenvalue, $\rho_1$, with corresponding eigenvector $1_N$. We define $\rho^* = \rho(Q_1 N, \Sigma_N^{1/2} Q_1 N)$. Also, let us denote by $P_N$, the projection operator onto the one-dimensional subspace spanned by $1_N$, so $Q_1 N \perp P_N$. Clearly $P_N \delta_N = \frac{\Sigma_N^{1/2} \delta_N}{\rho^*} 1_N$. We can decompose

$$\delta_N \Sigma_N^{-1} \delta_N = (\delta_N - P_N \delta_N + P_N \delta_N)'\Sigma^{-1} (\delta_N - P_N \delta_N + P_N \delta_N) =

(\delta_N - P_N \delta_N)' Q_1 N^{1/2} \Sigma_1 N^{1/2} Q_1 N (\delta_N - P_N \delta_N) + \frac{(P_N \delta_N)'(P_N \delta_N)}{\rho_1} \geq

(\delta_N - P_N \delta_N)' Q_1 N^{1/2} \Sigma_1 N^{1/2} Q_1 N (\delta_N - P_N \delta_N) \geq

(\delta_N - P_N \delta_N)'(\delta_N - P_N \delta_N) =

\rho^* \frac{\delta_N \delta_N - \frac{(Q_1 N^{1/2} \delta_N)^2}{\rho^*}}{\rho^*}.$$

By the assumptions of the Proposition, we therefore have (for large enough $N$):

$$K_N = \delta_N \Sigma_N^{-1} \delta_N \geq c^{-1} N^\nu.$$

**Proof of Proposition 12:** Define $\Lambda_{n,N} = diag(\alpha_1^N, \ldots, \alpha_N^N)$ and $\lambda_N = \delta_N$. We use the inversion formula
Thus, $K$ grows like $kN$ a.s. as $N$ becomes large. This completes the proof. If $E[(\alpha_1)^{-2}] < \infty$ (which is not guaranteed by our assumptions) then $k < \infty$, so in this case the order of the natural selection process is exactly one. Otherwise it can be faster.

We note that the argument is easy to generalize to more general random structures. For example, a similar result can be derived for $Q$-factor models, $Q > 1$, using the same argument as above, but with the inversion rule $(I_N + XX')^{-1} = I_N - X(I_Q + X'X)^{-1}X'$. Here, $X$ is an $N \times Q$ random matrix, representing the factor loadings of the $N$ stocks on $Q$ factors, $I_N$ is the $N \times N$ identity matrix and $I_Q$ is the $Q \times Q$ identity matrix.

**Wealth fraction over consumption fraction when $M = N$**

The wealth analysis in Section 3.2 is carried out for $M = 1$, i.e. in the case when there is only one asset in positive net supply. The results, however, are very similar for the case when $M = N$, i.e., in the $N$-tree economy. In Figure 5, we show the simulated fractions in the $N$-tree economy. The results are very similar to the case when $M = 1$. 

$$ (I + xx')^{-1} = I - \frac{1}{1 + x'x}xx' $$ for an arbitrary vector $x$ to get

$$ K_N = \frac{1}{N}N\delta^\top\Sigma^{-1}\delta = \frac{1}{N}\lambda_N^2 N^\top\Lambda_N^{-1} + b_N b_N' \lambda_N^{-1} \lambda_N $$

$$ = \frac{1}{N} \left( \Lambda_N^{-1} \lambda_N \right) \left( I_N + \Lambda_N^{-1} (b_N b_N') \Lambda_N^{-1} \lambda_N \right) $$

$$ = \frac{1}{N} \left( \Lambda_N^{-1} \lambda_N \right) \left( I_N - \frac{1}{N} + \frac{b_N b_N^\top}{N} \Lambda_N^{-1} (b_N b_N') \Lambda_N^{-1} \right) \left( \Lambda_N^{-1} \lambda_N \right) $$

$$ = \frac{\lambda_N \Lambda_N^{-2} \lambda_N}{N} - \frac{1}{N} \frac{1}{N} \left( \frac{\lambda_N \Lambda_N^{-2} b_N}{N} \right)^2. $$

The independence of these variables, together with the strong law of large numbers, implies that

$$ \frac{K_N}{N} \rightarrow_{a.s.} E[(\tilde{q}_1)^2] E[(\alpha_1)^{-2}] - \frac{1}{E[(\beta_1^2)] E[(\alpha_1^2)]} \left( E[\tilde{q}_1^2] E[\beta_1^2] E[(\alpha_1^2)] \right)^2 $$

$$ = E[(\beta_1^2)] \left( E[\tilde{q}_1^2] - \frac{E[\tilde{q}_1^2] E[\beta_1^2]}{E[\beta_1^2]} \right) $$

$$ = E[(\beta_1^2)] \left( E[\tilde{q}_1^2] - E[\tilde{q}_1^2] E[\beta_1^2] \right) $$

$$ = E[(\beta_1^2)] \left( E[\tilde{q}_1^2] - \frac{E[\tilde{q}_1^2] E[\beta_1^2]}{E[\beta_1^2]} \right) $$

$$ = E[(\beta_1^2)] \left( \sigma_\beta^2 \sigma_\tilde{\beta}^2 - \sigma_\tilde{\beta}^2 \beta_\tilde{\beta}^2 \right) $$

$$ = E[(\beta_1^2)] \left( \sigma_\beta^2 \sigma_\tilde{\beta}^2 + \sigma_\tilde{\beta}^2 \beta_\tilde{\beta}^2 \right) = k \in (0, \infty). $$

The strict positivity of $k$ is ensured, as $\sigma_\beta > 0, \sigma_\tilde{\beta} > 0$, and Jensen’s inequality ensures that $E[(\alpha_1)^{-2}] \geq \frac{1}{\sigma_\alpha^2 + \alpha^2}$. Thus, $K_N$ grows like $kN$ a.s. as $N$ becomes large. This completes the proof. If $E[(\alpha_1)^{-2}] < \infty$ (which is not guaranteed by our assumptions) then $k < \infty$, so in this case the order of the natural selection process is exactly one. Otherwise it can be faster.

We note that the argument is easy to generalize to more general random structures. For example, a similar result can be derived for $Q$-factor models, $Q > 1$, using the same argument as above, but with the inversion rule $(I_N + XX')^{-1} = I_N - X(I_Q + X'X)^{-1}X'$. Here, $X$ is an $N \times Q$ random matrix, representing the factor loadings of the $N$ stocks on $Q$ factors, $I_N$ is the $N \times N$ identity matrix and $I_Q$ is the $Q \times Q$ identity matrix.
Figure 5: Wealth fraction over consumption fraction, $Z = \frac{W_1/W_2}{c_1/c_2}$, as a function of consumption fraction, for $N = 1, 10, 50, 100$ and $M = N$. $Z$ is close to one, implying that the dynamics of the wealth fraction is similar to that of the consumption fraction. Parameters: $q = 20\%$, $\sigma = 0.0328$, $g = 0.02$, $\rho_1 = \rho_2 = 1\%$, $\gamma = 6$. 
References


