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“A Theory of Repurchase Agreements, Collateral Re-use, and Repo Intermediation”

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Abstract

This paper characterizes repurchase agreements (repos) as equilibrium contracts starting from first principles. We show that a repo allows the borrower to augment its consumption today while hedging both agents against future market price risk. As a result, safer assets will command a lower haircut and a higher liquidity premium relative to riskier assets. Haircuts may also be negative. When lenders can re-use the asset they receive in a repo, we show that there exists a collateral multiplier effect and borrowing increases. In addition, with collateral re-use, lenders might also re-pledge the asset to third parties. In the model, intermediation arises as an equilibrium choice of traders and trustworthy agents play a role as intermediary. These findings are helpful to rationalize chains of trades observed on the repo market.

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A Theory of Repurchase Agreements, 
Collateral Re-use, and Repo Intermediation*

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Abstract

This paper characterizes repurchase agreements (repos) as equilibrium contracts starting from first principles. We show that a repo allows the borrower to augment its consumption today while hedging both agents against future market price risk. As a result, safer assets will command a lower haircut and a higher liquidity premium relative to riskier assets. If collateral is very scarce, haircuts may also be negative. We show that traders benefit from re-using the collateral sold in a repo. First, re-use generates collateral multiplier effect as the economy can sustain more borrowing with a similar quantity of assets. Second, with collateral re-use, lenders might also

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1 Introduction

According to Gorton and Metrick (2012), the financial panic of 2007-08 started with a run on the market for repurchase agreements (repos). Their paper was very influential in shaping our understanding of the crisis. It was quickly followed by many attempts to understand repo markets more deeply, both empirically and theoretically as well as calls to regulate these markets.\footnote{See Acharya (2010) “A Case for Reforming the Repo Market” and (FRBNY 2010)}

A repo is the sale of an asset combined with a forward contract that requires the original seller to repurchase the asset at a given price. Repos are different from simple collateralized loans in (at least) one important way. A repo lender obtains the legal title to the pledged collateral and can thus use the collateral during the length of the forward contract. This practice is known as re-use or re-hypothecation. With standard collateralized loans, borrowers must agree to grant the lender similar rights\footnote{Aghion and Bolton (1992) argue that securities are characterize by cash-flow rights but also control rights. Collateralized loans grant neither cash-flow rights nor control rights over the collateral to the lender unless the counterparties sign an agreement for this purpose. As a sale of the asset, a repo automatically gives the lender full control rights over the security as well as over its cash-flows. Re-use rights follow directly from ownership rights. As Comotto (2014) explains, there is a subtle difference between US and EU law however. Under EU law, a repo is a transfer of the security’s title to the lender. However, a repo in the US falls under New York law which is the predominant jurisdiction in the US. “Under the law of New York, the transfer of title to collateral is not legally robust. In the event of a repo seller becoming insolvent, there is a material risk that the rights of the buyer to liquidate collateral could be successfully challenged in court. Consequently, the transfer of collateral in the US takes the form of the seller giving the buyer (1) a pledge, in which the collateral is transferred into the control of the buyer or his agent, and (2) the right to re-use the collateral at any time during the term of the repo, in other words, a right of re-hypothecation. The right of re-use of the pledged collateral (...) gives US repo the same legal effect as a transfer of title of collateral.” To conlude, although there are legal differences between re-use and rehypothecation, they are economically equivalent (see e.g. Singh, 2011) and we treat them as such in our analysis.}}
Repos are extensively used by market makers and dealer banks as well as other financial institutions as a source of funding, to acquire securities that are on specials, or simply to obtain a safe return on idle cash. As such, they are closely linked to market liquidity and so they are important to understand from the viewpoint of Finance. The Federal Reserve Bank, the central bank in the United States, and other central banks use repos to steer the short term nominal interest rate. The Fed’s newly introduced reverse repos are considered an effective tool to increase the money market rate when there are large excess reserves. Repos thus became essential to the conduct of monetary policy. Finally, firms also rent capital and use collateralized borrowing and some forms of repos to finance their activities or hedge exposures (notably interest rate risk, see BIS, 1999). This affects real activities, and so repos are also an important funding instrument for the macroeconomy.

Most existing research papers study specific aspects of the repo markets, e.g. exemption from automatic stay, fire sales, etc., taking the repo contract and most of its idiosyncrasies as given. These theories leave many fundamental questions unanswered, such as why are repos different from collateralized loans? What is the nature of the economic problem solved by the repo contract? To answer these questions, to understand the repo market and the effect of regulations, one cannot presume the existence or the design of repo contracts. In this paper we characterize a simple economic environment where repos emerge as the funding instrument of choice. More precisely, we borrow techniques from security design to derive the equilibrium collateralized contract. The interpretation as a repo contract is natural since the borrower ultimately sells an asset spot combined with a promise to re-purchase at an agreed price.

The model has three periods and two types of agents, a natural borrower and a natural lender, both risk-averse. The borrower is endowed with an asset that yields an uncertain payoff in the last period. The payoff realization becomes known in the second period and is reflected in the second period price of the asset. To increase his consumption in the first period, the borrower could sell the asset to the lender in the spot market. However this trade will expose both parties to price risk in the second period. Instead, the borrower can obtain resources from the lender by selling the asset combined with a forward contract promising to repurchase the
asset in period 2. Unlike in an outright sale, a constant repurchase price in a repo hedges market price risk. Under limited commitment however, the borrower might not honor his promise. Indeed, he may find it optimal to default if the value of collateral falls below the promised repayment\(^3\). We assume that in addition to the loss of the collateral, a defaulting borrower incurs a cost commensurate with the size of default. To avoid this wasteful default, the repurchase price of the repo contract should thus lie below a multiplier of the asset price proportional to this default cost parameter. In high states of the world or when the asset is abundant, this constraint does not bind and the repurchase price is constant. In low states of the world however, the asset pays very little and the borrower exhausts his borrowing capacity: the repurchase price increases with the spot market price.

Using this equilibrium contract we derive comparative statics for haircuts and liquidity premia. Haircuts increase with counterparty risk as a riskier agent can promise less income per unit of asset pledged. More risky collateral commands a higher haircut and a lower liquidity premium. Compared to a safe asset, a risky security pays less in bad times and more in good times. Since agents are constrained in bad times, this is precisely when collateral is valuable. Hence the liquidity premium is higher for the safe asset. In good times, agents do not exploit the higher value of the riskier collateral since the repurchase price becomes constant. Hence, compared to the safe asset, less of the risky asset’s payoff is pledged and the haircut is larger.

In Section 4, we introduce collateral re-use. In a repo, the lender indeed acquires ownership of the asset used as collateral in the repo transaction. In our model, a lender might re-use a fraction of the asset he receives as collateral. We show that agents strictly prefer to re-use as it increases the borrowing capacity of the repo seller. To fix ideas, suppose the collateral is perfectly safe and pays $100 in the second period. The net interest rate is 0 so that $100 is also the price of the asset in period 1. With the extra cost for default, a borrower can promise to repay more than $100 per unit of the asset in period 2, say $110. The lender can the re-use some of the collateral by selling it back to the borrower. The latter can

\(^3\)In practice, even in the absence of outright default, traders opportunistically delay the settlement of transactions, as documented by Fleming and Garbade (2005). In our model, the repurchase price can be made state-contingent which rules out default in equilibrium. The state-contingency somehow mimics margin adjustments in actual transactions.
now pledge another $110 per unit. With one round of re-use, the borrower netted an extra $10 per unit. These trades can be repeated until no collateral may be re-used. Overall, re-use has a multiplier effect since a borrower can pledge more income with the same quantity of the asset in those states where he is constrained\(^4\). Without the non-pecuniary penalty, this extra borrowing capacity disappears and re-use does not affect the equilibrium allocation, a result in line with Maurin (2015). Overall, the model implies that collateral re-use should be more prevalent for assets that command low haircuts and when the lender’s trading partners have low counterparty risk.

Finally, Section 5 discusses the implications of collateral re-use for the repo market structure. We argue that some participants naturally emerge as intermediaries when they can re-use collateral. In practice, dealer banks indeed make for a significant share of this market by intermediating between natural borrowers (say hedge funds) and lenders (say money market funds or MMF). This might seem puzzling if direct trading platforms are available for both parties to bypass the dealer bank\(^5\). Our model rationalizes intermediation with difference in trustworthiness and ability to re-deploy the collateral. In our example, the hedge fund delegates borrowing to the dealer bank if the later is more trustworthy. Although there are larger gains from trade with the MMF, the hedge fund prefers borrowing from the dealer bank if he is more efficient at re-using collateral. Indeed, through re-use, one unit pledged to the dealer bank can then support more borrowing in the chain of transactions. Our model thus provides an endogenous theory for repo intermediation based on fundamental heterogeneity between traders.

**Relation to the literature**

Gorton and Metrick (2012) argue that the recent crisis started with a run on repo whereby funding dropped dramatically for many financial institutions. Subsequent studies by Krishnamurty et al. (2014) and Copeland et al. (2014) have qualified this finding by showing that the run was specific to the - large - bilateral segment of the repo market. Recent theoretical works indeed highlighted some

\(^4\)Our stripped down example suggested that re-use only works when haircuts are negative. However, borrowers only want to pledge more income in low payoff states where they are constrained. In good states, they might still want to pledge less income than the future value of the asset. The haircut averages over states and might thus be positive.

\(^5\)In the US, Direct Repo\(^\text{TM}\) provides this service
features of repo contracts as sources of funding fragility. As a short-term debt instrument to finance long-term assets, Zhang (2014) and Martin et al. (2014) show that repos are subject to roll-over risk. Antinolfi et al. (2015) emphasize the trade-off from the exemption from automatic stay for repo collateral. Lenders easy access to the borrower’s collateral may be privately optimal but collectively nefarious in the presence of fire sales, a point also made by Infante (2013) and Kuong (2015).

These papers usually take repurchase agreements as given while we want to understand their emergence as a funding instrument. One natural question is to ask why borrowers do not simply sell the collateral to lenders? Most papers including our highlight the role of the commitment to the repurchase price. In Narajabad and Monnet (2012), Tomura (2013) and Parlatore (2015), it allows lenders to avoid search frictions in the spot market when reselling the asset. In contrast, our model is fully competitive but assets’ payoff are risky. As a result, repos are essential because the repurchase price provides hedging against price risk. Bigio (2015) and Madison (2016) emphasize asymmetry of information about the quality of the asset. There, the commitment to repurchase insulates uninformed buyers from the information-sensitive part of the asset cash flow. The repo contract thus resembles a leasing agreement as in Hendel and Lizzeri (2002) or the optimal debt financing arrangement of DeMarzo and Duffie (1999), both of which mitigate adverse selection. Our model has symmetric information but assets payoff are random. With uncertainty, agents may also want to pledge less than the future value of the cash flow when it is expected to be high (the hedging component). Besides the different economic motivation, these works essentially identify repos with standard collateralized loans. We account for the sale of collateral in a repo by considering re-use. In addition, our theory rationalizes haircuts since borrowers choose repos when they could obtain more cash in the spot market.

To derive the repo contract, we follow Geanakoplos (1996), Araújo et al. (2000) and Geanakoplos and Zame (2014) where collateralized promises traded by agents are selected in equilibrium. Our model differs from theirs as we allow for an extra non-pecuniary penalty for default in the spirit of Dubey et al. (2005). While our results on the design of repo contracts carry through without this penalty, it is

\[\text{In particular, we do not need transactions costs as suggested by Duffie (1996).}\]
crucial for the results in Section 4 and 5 related to collateral re-use.

In the second part of the paper, we indeed account for the transfer of the legal title to the collateral to the lender, opening the possibility for re-use. Singh and Aitken (2010) and Singh (2011) argue that collateral re-use or rehypothecation lubricates transactions in the financial system\(^7\). However rehypothecation may entail risks for collateral pledgers as explained by Monnet (2011). While Bottazzi et al. (2012) or Andolfatto et al. (2014) abstract from the limited commitment problem of the collateral receiver, Maurin (2015) shows that re-use risk seriously mitigates the benefits from circulation. In our model indeed, re-use relaxes collateral constraints only thanks to the extra penalty for default for borrowers besides the collateral loss. Asset re-use then plays a role similar to pyramiding (see Gottardi and Kubler, 2015). One difference is that lenders re-use the collateral backing the debt rather than the debt itself as collateral. We stress the role of collateral re-use in explaining repo market intermediation as in Infante (2015) and Muley (2015). Unlike these papers, intermediation arises endogenously in our model as trustworthy agents re-use the collateral from risky counterparties to borrow on their behalf. In an empirical paper, Issa and Jarnecic (2016) indeed suggested that the fee based view of repo intermediation whereby dealers gain from differences in haircuts does not stand in the data.

The structure of the paper is as follows. We present the model and the complete market benchmark in Section 2. We analyze the optimal repo contracts, including properties for haircuts, liquidity premiums, and repo rates in Section 3. In Section 4, we allow for collateral re-use and study intermediation in Section 5. Finally, Section 6 concludes.

2 The Model

2.1 Setting

The economy lasts three dates, \(t = 1, 2, 3\). There are two types of agents \(i = 1, 2\) and only one good each period. Both agents have endowment \(\omega\) in all but the

\(^7\)Fuhrer et al. (2015) estimate an average 5\% re-use rate in the Swiss repo market over 2006-2013.
last period. Agent 1 is also endowed with $a$ units of an asset while agent 2 has none. This asset pays dividend $s$ in date 3. The dividend is distributed according to a cumulative distribution function $F(s)$ with support $S = [\underline{s}, \bar{s}]$ and with mean $E[s] = 1$. In date 2, the realization of $s$ in date 3 is known to all agents. This is an easy way to model price risk at date 2.

Preferences from consumption profile $(c_1, c_2, c_3)$ for agent 1 and 2 are:

$$U^1(c_1, c_2, c_3) = c_1 + v(c_2) + c_3$$
$$U^2(c_1, c_2, c_3) = c_1 + u(c_2) + \beta c_3$$

where $\beta < 1$, $u(.)$ and $v(.)$ are respectively strictly concave and concave functions. We assume $u'(\omega) > v'(\omega)$ and $u'(2\omega) < v'(0)$, so that there are gains from transferring resources from agent 1 to agent 2 in date 2 and the optimal allocation is interior. These preferences contain two important elements. First, as $\beta < 1$, agent 2 values less consumption in date 3 so that agent 1 is the natural holder of the asset in that period. Second, agents with concave utility function dislike consumption variability in period 2.

While they may want to engage in borrowing and lending, agents are not able to fully commit to future promised payments. Borrower can pledge the asset as collateral in a repurchase agreement to alleviate this friction. In Section 2.3, we define contracts and study agents’ incentives to default formally. For clarity, we introduce and discuss the ingredients of our repo model here. When facing a default, a creditor can seize the asset used as collateral, which he can hold or sell in the spot market. In addition, he recovers a fraction $\alpha \in [0, 1]$ of the shortfall, that is the difference between the promised repayment and the market value of the collateral. Finally, a defaulting agent $i$ incurs a non-pecuniary cost equal to a fraction $\pi_i \in [0, 1]$ of the contractual repayment, measured in consumption units.

Our assumptions about default costs match several features of repo contracts. First, in a repo, the lender gets possession of the collateral and may thus sell it when the borrower defaults. Second, repos are recourse-loan. Under the most popular master agreement described in ICMA (2013), an agent may claim the shortfall

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8While a repo is not characterized as a sale in the US, the exemption from automatic stay for repo collateral gives similar rights for the lender. See also footnote 2 on this point.
to a defaulting counterparty in a "close-out" process. Our partial recovery rate \( \alpha \) captures the monetary value of delay or other impediments in recouping this shortfall. Finally, the non-pecuniary component proxies for legal and reputation costs or losses from future market exclusion\(^9\). We allow the parameter \( \pi \) to depend on the identity of the borrower. The functional form will ensure that prices are linear function of trades.

The last building block of our model of repo is the ability for the lender to re-use collateral. Again, this follows naturally from the transfer of ownership of the asset used as collateral. We assume that lender \( i \) can re-use a fraction \( \nu_i \) of the collateral he receives where \( \nu_i \in [0, 1] \). We interpret \( \nu_i \) as a measure of the operational efficiency of a trader in re-deploying collateral for his own trades\(^{10}\).

Our model highlights some key features of repos: they are collateralized loans, with recourse and the lender gets possession of the collateral. The environment is a simple set-up where these features will play out. There are two risk-averse agents and one wishes to borrow funds from the other. Limited commitment implies that the asset must be used to transfer funds across time. Unlike a combination of spot trades, a repo allows to hedge price risk thanks to the repurchase price. However, it exposes traders to default risk. When the asset is scarce, the ability to re-use collateral proves valuable because borrowers can increase leverage. Throughout the paper, markets are competitive and agents are price-takers.

### 2.2 Perfect commitment

As a benchmark, we solve for equilibrium when agents can perfectly commit to future promises. Markets are thus complete and the equilibrium allocation is efficient. As a result, marginal rates of substitution are equalized unless one agent is at a corner. We guess that this is the case between the first and the second

\(^9\)We thus depart from most models of collateralized lending a la Geanakoplos (1996) which assume \( \alpha = \pi = 0 \). As we argued, our assumption that \( \alpha > 0 \) is natural for repos which are recourse loans. In addition, the non-pecuniary punishment (\( \pi > 0 \)) is necessary to explain re-use as we show in Proposition 6 and 9.

\(^{10}\)Singh (2011) discusses the role played by collateral desks at large dealer banks in channeling these assets across different business lines. These desks might not be available for less sophisticated repo market participants such as money market mutual funds or pension funds. In practice, the bulk of traded repos have short maturity, limiting the scope for re-use.
period$^{11}$. Let $c^t_i$ denote agent $i$ consumption in period $t$. We obtain the following equilibrium conditions:

$$\begin{align*}
    u'(c^2_{2,*}) &= v'(2\omega - c^2_{2,*}) \\
    c^2_{3,*} &= 0
\end{align*}$$

(1)

where we used the resource constraint of period 2 to substitute for $c^1_{2,*} = 2\omega - c^2_{2,*}$. Intuitively, since $\beta < 1$, agent 2 does not consume in period 3 because he has a lower marginal utility than agent 1. The implicit prices for period 2 and 3 consumption are respectively $u'(c^2_{2,*})$ and 1. To pin down the equilibrium allocation completely, we use the budget constraint of agent 2 and obtain $c^2_{1,*} = \omega - u'(c^2_{2,*})(c^2_{2,*} - \omega)$. This expression is positive if:

$$\omega \geq u'(c^2_{2,*})(c^2_{2,*} - \omega)$$

(2)

which we assume in the remainder of the text. In equilibrium, agent 1 borrows $c^2_{2,*} - \omega$ at a net interest rate $r^* = 1/u'(c^2_{2,*}) - 1$, using unsecured credit. Observe that agents first best consumption $(c^1_{2,*}, c^2_{2,*})$ in period 2 is deterministic although the asset payoff $s$ is already known. Indeed, risk averse agents prefer a smooth consumption profile.

### 2.3 Incomplete Markets with Limited Commitment

We now turn to the more interesting case where agents face limited commitment. Observe that spot trading is always feasible, independently of the severity of the friction. To gain intuition about the benefit from using repos, we show first that agents cannot achieve the first best allocation by using only spot trades, as it exposes them to price risk.

$^{11}$Conjecturing instead that marginal rates of substitution are equalized between the second and the third period, we find a contradiction since the resulting allocation is not budget feasible at the implied market prices.
2.3.1 Spot Transactions

Suppose agents can only trade the asset in a spot market. The spot market price in period 1 (resp. period 2 and state $s$) is denoted $p_1$ (resp. $p_2(s)$). The price in period 2 indeed reflects the future known payoff $s$ of the asset. Let us denote $a_i^1$ (resp. $a_i^2(s)$) the asset holdings of agent $i$ after trading in period 1 (resp. period 2 and date $s$). The budget constraints of agent 2 in period 1 and 2 write

\[
\begin{align*}
  c_i^1 &= \omega + p_1 a_i^2 \\
  c_i^2(s) &= \omega + p_2(s)(a_i^2 - a_i^2(s))
\end{align*}
\]

Using spot trades, agent 2 can implicitly lend to agent 1 if he buys the asset in period 1, that is $a_2^1 > 0$ and re-sells it in period 2, that is $a_2^2(s) < a_2^1$. We give a formal characterization of the equilibrium in the Appendix. Here, we stress our main point: a combination of spot trades can never finance the first-best allocation (1). Since agent 2 does not want to consume in period 3 ($\beta < 1$), he would resell all the asset bought in period 1 so that $a_2^2(s) = 0$. This implies $c_i^2(s) = \omega + p_2(s)a_i^2$. Agent 2 consumption must then vary with $s$ because of price risk, while the first best consumption level $c_{2,\ast}^2$ is deterministic. Indeed, spot trades are too limited an instrument to transfer wealth across time. In particular, asset price risk generates undesirable consumption variability in period 2. As we will see, the repo allows agents to commit to a repurchase price to hedge against the asset payoff variability.

2.3.2 Trading in Spot and Repo Markets

In this section, we specify the agents’ problem when they have access to both spot and credit markets. We naturally define a repo as the sale of an asset with a forward contract to buy it back. Compared to a combination of spot trades, a repo offers more flexibility as the repurchase price can hedge price risk. However, each agent is now exposed to the default risk of his counterparty. As it will be clear, a repo seller effectively borrows with a loan collateralized by the asset sold. Our model speaks to repos specifically because lenders can re-use a fraction of the asset pledged. This is a natural feature in a repo where the collateral is sold.

**Definition 1.** A repo contract is a price schedule $f = \{f(s)\}_{s \in S}$ whereby the seller
agrees to repurchase each unit sold in period 1 at price $f(s)$ in state $s$ of period 2.

When trading one unit of repo $f$, a seller $i$ transfers one unit of the asset to the buyer $j$. In exchange, he receives $q_{ij}(f)$ which is the price of the repo. We explain below how this price may depend on traders’ type. Repo $f$ is similar to a standard collateralized loan where the seller/borrower obtains $q_{ij}(f)$ per unit of asset pledged and promises to repay $\{f(s)\}_{s \in S}$. However, standard model of collateralized borrowing do not account for collateral re-use, as we do here.

**Borrower and Lender Default**

In a repo, the borrower promises to repay the lender who pledges to return the collateral. Hence, a dual limited commitment problem arises. To explicit each counterparty incentives to default, consider a trade of one unit of repo contract $f$ between borrower $i$ and lender $j$. This comes without loss of generality because penalties for default are linear in the amount traded.

Borrower $i$ prefers to repay rather than default if and only if:

$$f(s) \leq p_2(s) + \alpha(f(s) - p_2(s)) + \pi_i f(s) \quad (3)$$

The left hand side is the repurchase price of the asset. For the borrower to repay, $f(s)$ must not exceed the total default cost. The first component is the loss of the market value $p_2(s)$ of the collateral seized by the lender. The second term $\alpha(f(s) - p_2(s))$ is the fraction of the shortfall recovered by the lender. The third component $\pi_i f(s)$ is the non-pecuniary cost for the borrower.

We now turn to the lender’s incentives to return the asset\(^{12}\). Observe that he can only re-use a fraction $\nu_j$ of the collateral. We assume that he deposits or segregates the non re-usable fraction $1 - \nu_j$ with a collateral custodian. As a result, he may only abscond with the re-usable fraction of the collateral\(^{13}\). When

\(^{12}\)Technically, most Master Agreements characterize as “fails” and not outright defaults the event where the lender does not return the collateral immediately. While our model does not distinguish the two events, lenders also incur penalties when they fail.

\(^{13}\)It is easy to understand why this is optimal for him ex-ante. First, he is less likely to default ex-post. Second, by definition, he would not derive ownership benefits from keeping the non re-usable collateral on his balance sheet. In the tri-party repo market, BNY Mellon and JP Morgan provide these services. Our results extend with some modification to the case where segregation is not available. Essentially, the no-default constraint of the lender might become binding for high values of $s$, while it is not in our baseline specification.
the lender defaults, the borrower gets the $1 - \nu_j$ units of segregated collateral back. He also recovers a fraction $\alpha$ of the shortfall $p_2(s) - f(s) - (1 - \nu_j)p_2(s)$, symmetrically with the case of a borrower's default. Hence, the lender prefers to return the re-usable collateral rather than default if and only if

$$\nu_j p_2(s) \leq f(s) + \alpha(\nu_j p_2(s) - f(s)) + \pi_j f(s)$$  \hfill (4)

The left hand side is the cost of returning the re-usable units of collateral at market value. The right hand side is the total cost of defaulting. The lender then foregoes the payment $f(s)$ from the borrower. He also loses the fraction $\alpha$ of the shortfall $\nu_j p(s) - f(s)$ recovered by the borrower. Finally, he incurs the non-pecuniary cost $\pi_j f(s)$.

Our model has subtle implications for the cost and benefit of default. First, the non-pecuniary punishment generates a deadweight loss. This should encourage agents to trade default-free contracts. However, because loans are recourse, borrowers can indirectly pledge the endowment $\omega$ through the recovery payment when they default. To illustrate this trade-off in a stark way, suppose that the asset is worthless, that is $s = 0$ for all $s$. From (3), a borrower would default on any repo. This does not mean that credit markets shut down however. Indeed, suppose he sells contract $f$ such that $f(s) = 100$ for all $s$. The deadweight cost is $100\pi$. However, the lender would recover $\alpha(100 - p_2(s)) = 100\alpha$. The borrower effectively pledged $100\alpha$ through default. Intuitively, when $\pi$ is high and $\alpha$ is low, agents will avoid contracts with equilibrium default. We show in the Proof of Proposition 3 that focusing on default-free contracts comes without loss of generality when the following condition holds:

$$\pi v'(\omega) \geq \alpha(u'(\omega) - v'(\omega))$$  \hfill (5)

Intuitively, repo contracts inducing defaults are dominated if the marginal cost of default $\pi v'(\omega)$ exceeds the marginal benefits $\alpha(u'(\omega) - v'(\omega))$ through the pe-

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14A lender might re-use collateral and not have it on his balance sheet when he must return it to the lender. However, observe that he can purchase the relevant quantity of the asset in the spot market to satisfy his obligation. When he returns the asset, the lender effectively covers a short position $-\nu_j$. 

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cuniary transfer.

We can now define the set of no-default repo contracts $F_{ij}$ between two agents $i$ and $j$ as a function of the period 2 spot market price $p_2 = \{p_2(s)\}_{s \in S}$. To simplify notation, we let $\theta_i := \pi_i/(1 - \alpha)$. Transforming equations (3) and (4), we obtain the set of no-default repos.

$$F_{ij}(p_2) = \left\{ f \mid \forall s \in [s, \bar{s}], \frac{\nu_j p_2(s)}{1 + \theta_j} \leq f(s) \leq \frac{p_2(s)}{1 - \theta_i} \right\}$$

(6)

Since agent $i$ is less likely to default when $\theta_i$ is high, we interpret this parameter as a measure of creditworthiness. Observe that the set $F_{ij}(p_2)$ is convex. Second, prices and quantities are linear functions of quantity traded. In addition, we normalized all contracts by unit of asset pledged. Hence, for any combination of multiple contracts sold by $i$, there exists an equivalent trade of a single repo contract. In the following, we thus call without ambiguity $f_{12}$ and $f_{21}$ the equilibrium contracts.

Agents optimization problem.

We can now write the agent’s optimization problem. We let $b^{ij}$ (resp $l^{ij}$) denote the amount agent $i$ borrows (resp. lends) with $j$ using equilibrium contract $f_{ij}$ (resp $f_{ji}$). We call $q_{ij}$ denotes the price of the equilibrium contract $f_{ij}$ traded by agents $i$ and $j$. When indexing a contract, the subscript $ij$ reflects the equilibrium choice of repos by agents $i$ and $j$. The subscript $ij$ also indexes the price to the extent that some repo contracts might have different prices when traded by different pairs of agents because of heterogeneous incentives to default. For simplicity, we write
\[ q_{ij} := q_{ij}(f_{ij}). \]

\[
\max_{a_i^1, b_i^j, l_i^j} E \left[ U_i^j(c_i^1(s), c_i^2(s), c_i^3(s)) \right]
\]

subject to

\[ c_i^1 = \omega + p_1(a_i^0 - a_i^1) + q_{ij}b_i^j - q_{ji}l_i^j \]
\[ c_i^2(s) = \omega + p_2(s)(a_i^1 - a_i^2(s)) - f_{ij}(s)b_i^j + f_{ji}l_i^j \]
\[ c_i^3(s) = a_i^2(s)s \]
\[ a_i^1 + \nu_j l_i^j \geq b_i^j \]
\[ b_i^j \geq 0 \]
\[ l_i^j \geq 0 \]

At date 1, agent \( i \) has resources \( \omega + p_1a_i^0 \) and chooses holding \( a_i^1 \), lending \( \ell_i^j \) and borrowing \( b_i^j \). Given these decisions, his resources at date 2 is the endowment \( \omega \) and the value of his asset holdings \( p_2(s)a_i^1 \) as well the net value of the repo positions \( f_{ij}(s)\ell_i^j - f(s)b_i^j \). Equation (11) is the collateral constraint of agent \( i \). A borrower (an agent for which \( b > 0 \)) must hold one asset per unit of repo contract sold. He can buy these assets either in the spot market (\( a_1 > 0 \)) or in the repo market (\( l > 0 \)). In the latter case, however, only a fraction \( \nu_j \) of the asset purchased can be re-used. The collateral constraint also shows that a lender can take a short position on the spot market. Let indeed \( b = 0 \) and \( l > 0 \). Then, it can be that \( a_1 < 0 \) if \( \nu > 0 \). With re-use, a lender acquires ownership of the asset pledged by the lender and can then sell it. The only difference with a regular sale is the commitment to return the asset to the agent who initially sold it.

**Definition 2. Repo equilibrium**

An equilibrium is a system of spot prices \( p_1 \) and \( p_2 = \{p_2(s)\}_{s \in S} \), a pair of repo contracts \( (f_{12}, f_{21}) \in \mathcal{F}_{12}(p_2) \times \mathcal{F}_{21}(p_2) \) and their prices \( q_{12} \) and \( q_{21} \), and allocations \( \{c_i^1(s), a_i^1, \ell_i^j, b_i^j\}_{i=1,2,j \neq i}^{1,3,s \in S} \) such that

1. \( \{c_i^1(s), a_i^1, \ell_i^j, b_i^j\}_{i=1,3,s \in S}^{1,2,j \neq i} \) solves agent \( i = 1, 2 \) problem (7)-(13).
2. Markets clear, that is \( a_2^1 + a_1^2 = a \) and \( b_i^j = l_i^j \) for \( i = 1, 2 \) and \( j \neq i \)
3. For any contract \( \tilde{f} \notin \{f_{12}, f_{21}\} \), there exists a price \( q(\tilde{f}) \) such that agents do not trade this contract.
Points 1 and 2 are self-explanatory. Point 3 is a natural requirement to characterize the repo contracts traded in equilibrium. A repo contract can be part of an equilibrium if and only if agents do not wish to trade an alternative contract \( \tilde{f} \). For example, if \( \tilde{f} \in \mathcal{F}_{12}(p_2) \), the implicit equilibrium price \( q(\tilde{f}) \) must be too low (resp. too high) for agent 1 (resp. agent 2) to wish to sell (resp. to buy) this contract. Hence, with our equilibrium definition, all contracts are available to trade and agents select their preferred contracts taking prices as given.

2.3.3 State-Contingent Repos

We allow the repurchase price to depend on market observable, namely the payoff \( s \) of the asset used as collateral. This expands the space of feasible contracts but might be viewed as unrealistic. However, we argue that state-contingency of \( f \) ultimately reproduces the effect of margin calls or repricing on the terms of trade during the lifetime of a repo. In practice, counterparties quote an interest rate \( r \) for the transaction. The non-state contingent repurchase price then obtains as \( f = (1 + r)q(f) \) where \( q(f) \) is the repo sale price. Suppose now that the borrower pledges one unit of asset. The expected value of the asset is 100 and the borrower is to repay 80. If the asset price falls to 90, the lender calls a margin and requires the borrower to post more collateral\(^{15} \). After a margin call, the borrower must then pledge more asset to sustain the same level of borrowing. This is similar to leaving the quantity of asset unchanged and reducing the amount borrowed.

One may still wonder about the implications of imposing a constant repurchase price to our contract space. Essentially, default can become valuable even when condition (5) holds, because it introduces state-contingency in the payoff function. Using constraint (3) in the lowest state \( s \), borrower \( i \) could not pledge more than \( p_2(s)/(1 - \theta_1) \) without defaulting. Raising the fixed promised payment entails default costs in low states as before. However, it now has the additional benefit of increasing the amount pledged in high states. We discuss this trade-off in more details after Proposition (3).

\(^{15}\)See the ? guide for technical details
3 Equilibrium contract without re-use

In this section, we characterize the equilibrium when agents cannot re-use collateral, that is $\nu_1 = \nu_2 = 0$. Then, a repo contracts is a standard collateralized loan. To gain intuition, remember that agent 1 wants to borrow in period 1 by pledging to repay $c^2_{2,*} - \omega$ in period 2. Consider the following trade pattern. Agent 1 sells all his asset in a repo, that is $b^{12} = a$ and does not trade spot. The maximum per-unit payoff of the repo is $p_2(s) / (1 - \theta_1)$. Hence, in period 2 and state $s$, using his budget constraint, agent 2 consumption must satisfy

$$c^2_2(s) \leq \omega + \frac{ap_2(s)}{1 - \theta_1}.$$  

In low states $s$, this amount may fall short of $c^2_{2,*}$. The repurchase price should then be set at its maximum value $f(s) = p_2(s) / (1 - \theta_1)$ since agents are constrained. In high states however, this could raise agent 2 consumption too much. There $f(s)$ should be constant. We thus define $s^*$ as the solution to

$$c^2_2(s^*) = \omega + \frac{as^*}{v'(c^1_2,*)}.$$  

This is the minimal state where the first-best allocation can be financed. The second equality follows from the observation that $p_2(s) = s / v'(c^1_2(s))$ since agent 1 is the natural holder of the asset into period 3. Observe that $s^*$ is decreasing with $a$ and $\theta$. Therefore, it is easier to achieve the first best level of consumption the larger the stock of asset and the more agent 1 is able to commit. We have the following result.

**Proposition 3.** Define $p_2(s)$ as the unique solution - increasing in $s$ - to

$$\begin{cases} 
    p_2(s)v'(\omega - a \frac{p_2(s)}{1 - \theta_1}) - s = 0 & \text{if } s < s^* \\
    p_2(s) = s / v'(c^2_2,*) & \text{if } s \geq s^*
\end{cases}$$  

There is a unique equilibrium allocation with repo contract $f$ where:

1. If $s^* \geq \bar{s}$ ($a$ is low), $f(s) = p_2(s) / (1 - \theta_1)$ for all $s \in S$
2. If $s^* \in [\underline{s}, \bar{s}]$ (a is intermediate),
\[
f(s) = \begin{cases} 
\frac{p_2(s)}{1 - \theta_1} & \text{for } s \leq s^* \\
\frac{p_2(s^*)}{(1 - \theta_1)} & \text{for } s \geq s^* 
\end{cases}
\] 
(16)

3. If $s^* \leq \underline{s}$ (a is high), $f(s) = f^*$ for all $s \in S$ where $f^* \in \left[\frac{p_2(s^*)}{(1 - \theta_1)}, \frac{p_2(\bar{s})}{(1 - \theta_1)}\right]$.

In equilibrium, agents strictly prefer to trade repo over any combination of repo and spot trades in cases 1 and 2. They are indifferent to using a combination of both in case 3.

The equilibrium contract reflects the optimal use of the collateral value. As we explained, agent 1 can indeed pledge at most $p_2(s)/(1 - \theta_1)$ per unit of asset in state $s$. This amount increases in $s$ together with the collateral value $p_2(s)$. When the collateral value is low, for $s \leq s^*$, the borrowing constraint of agent 1 is binding and the repurchase price $f(s)$ is equal to this maximal amount. However, when the collateral value is high, agent 1 does not want to borrow above the first best amount. Hence, the repurchase price becomes flat for $s \geq s^*$. We call this the hedging motive. The proof of Proposition 1 in the Appendix formalizes this argument ensuring that agents do not want to trade another contract $\hat{f}$. Figure 1 plots the equilibrium repo contract, in the case $v(x) = x$.

It is interesting to emphasize why agents prefer trading repo rather than spot. Suppose indeed that agent 1 sells the asset spot in period 1 and buys it back at the spot market price $p_2(s)$ in period 2. This is formally equivalent to a repo contract $\hat{f}$ with $\hat{f}(s) = p_2(s)$. This alternative trade is dominated for two reasons. When the collateral value is low, agent 1 can increase the amount he pledges from $p_2(s)$ to $p_2(s)/(1 - \theta_1)$ with a repo. More importantly, when the collateral value is high, the equilibrium repo limits the repayment to agent 2 to the first best level.

**Non-state-contingent repos.**

When repurchase prices may be state-contingent, agents choose to trade a default-free repo, for which inequality (3) holds. Indeed, under assumption (5), deadweight costs from defaulting always exceed the implicit transfers through the partial recovery of the shortfall.
This is not true if we constrain repurchase prices to be constant across states, that is $f(s) = f$. As we explained before, the highest repurchase price without default is $f_{nd} = p_2(\bar{s})/(1 - \theta)$, the dashed red line on Figure 1. Suppose that $f_{nd}$ is traded in equilibrium and agent 1 considers allocating one unit of collateral to repo $f_d$ instead where $f_d > f_{nd}$. There exists a threshold $s(f_d) \in [\underline{s}, \bar{s}]$ so that the borrower defaults on repo $f_d$ in state $s < s(f_d)$ and repays otherwise. The effective payoff to the lender can be written

$$\hat{f}_d(s) = \begin{cases} p_2(s) + \alpha (f_d - p_2(s)) & \text{if } s < s(f_d) \\ f_d & \text{if } s \geq s(f_d) \end{cases}$$

There are now three effects. In low states $s < s(f_d)$, defaulting entails a cost $\pi f_{nd}$. In addition, through default, the net additional amount pledged in those states is $\hat{f}_d(s) - f_{nd}$. Assumption (5) states that the net benefit from these two first effects is negative. However, agents also benefit in high states $s \geq s(f_d)$ where the net additional amount pledged is $f_d - f_{nd}$. These gains are not present if the contract can be made state-contingent. Without state-contingency in the contract space, the trade-off between default and economy efficiency is no longer trivial. It is easy to realize that the equilibrium contract $f^*$ should then belong to $[f_{nd}, p_2(s^*)/(1 - \theta_1)]$. The lower bound is the default-free contract while the upper bound ensures the first-best level of consumption in states $s \geq s^*$. Intuitively, $f^*$ is closer to the upper bound if the non-pecuniary cost $\pi$ is low, the recovery rate $\alpha$ is high and the distribution $G$ is skewed towards high states.

### 3.1 Haircuts, liquidity premium, and repo rates

In this section, we derive the equilibrium properties of the liquidity premium and repo haircut. We compare the haircuts and liquidity premia of two assets with different risk profile. We also investigate the role of counterparty risk, as measured by $\theta$. We define the liquidity premium $\mathcal{L}$ as the difference between the spot price of the asset in period 1 and its holding value. We thus obtain

$$\mathcal{L} = p_1 - E[s]$$
The holding value $E[s]$ follows naturally from the preferences of agent 1. The liquidity premium is also the shadow price of the collateral constraint. Hence, whenever the asset is scarce and agents are constrained, the asset bears a positive liquidity premium. Using the equilibrium characterization, we can relate the liquidity premium to the repo contract and the allocation:

$$L = E[f(s)(u'(c^2_s(s)) - v'(c^1_s(s)))]$$

The liquidity premium is positive if there exist (low) states where agents are constrained because they cannot increase borrowing. In those states, $u'(c^2_s(s)) > v'(c^1_s(s))$ which implies $L > 0$.

The repo haircut is the difference between the spot market price and the repo price. Indeed, it costs $p_1$ to obtain 1 unit of the asset, which can be pledged as collateral to borrow $q$. So to purchase 1 unit of the asset, an agent needs $p_1 - q$
which is the downpayment or haircut\textsuperscript{16}.

\[ \mathcal{H} \equiv p_1 - q = E[(p_2(s) - f(s))u'(c_2'(s))] \]  

(17)

where the second equality follows from the first order condition of agent 1 with respect to spot and repo trades. Finally, the repo rate is

\[ 1 + r = \frac{E[f(s)]}{q} = \frac{E[f(s)]}{E[f(s)u'(c_2'(s))]} \]  

(18)

When agents are constrained (case i) and ii) of Proposition 3), we have \( u'(c_2'(s)) > u'(c_2'(s)) \) for \( s \in [\underline{s}, s^*] \) so that \( 1 + r < 1 + r^* \). Agent 2 would like to lend at the frictionless interest rate \( 1 + r^* \). However, agent 1 cannot increase borrowing since he runs out of collateral. The interest rate must then fall for agent 2 to be indifferent. Interestingly, \( r < r^* \) when the liquidity premium \( \mathcal{L} \) is strictly positive. Remember that a positive liquidity premium precisely indicates collateral scarcity. Net repo rates \( r \) can thus be negative for assets with large liquidity premium. This is consistent with market data as reported in ICMA (2013).\textsuperscript{17} We now derive the haircut and liquidity premium for the optimal price schedule \( f \).

**Corollary 4.** The haircut and liquidity premium are:

\[
\mathcal{L} = \int_{\underline{s}}^{s^*} \frac{s}{1 - \theta_1} \left[ \frac{u'(\omega + \frac{p_2(s)}{1-\theta})}{v'(\omega - \frac{p_2(s)}{1-\theta})} - 1 \right] dF(s)
\]

\[
\mathcal{H} = -\frac{\theta_1}{1 - \theta_1} \int_{\underline{s}}^{s^*} s dF(s) + \int_{s^*}^{\bar{s}} \left( s - \frac{s^*}{1 - \theta_1} \right) dF(s)
\]

when \( s^* \geq \underline{s} \), where \( p_2(s) \) is the period 2 spot market price defined in (15). When agents can reach the FB allocation in all states, that is \( s^* \leq \underline{s} \), the liquidity

\textsuperscript{16} An alternative but equivalent definition is \((p_1 - q)/q\).

\textsuperscript{17} The ICMA (2013) reports that “The demand for some assets can become so strong that the repo rate on that particular asset falls to zero or even goes negative. The repo market is the only financial market in which a negative rate of return is not an anomaly.” (p.12) and in footnote 6 “negative repo rates have been a frequent occurrence and can be deeply negative.” Also, see Duffie (1996), or Vayanos and Weill (2008).
premium is $L = 0$ and the haircut lies in the following range:

$$\mathcal{H} \in \left[ E[s] - \frac{s}{1 - \theta_1}, E[s] - \frac{s^*}{1 - \theta_1} \right]$$

As Figure 1 shows, the borrowing and hedging motives have opposite effects on the size of the haircut. In the states $s < s^*$ where agents are constrained, the borrower uses the maximum pledgeable capacity $p_2(s)/(1 - \theta_1)$ per unit while the asset price trades at $p_2(s)$. From expression (17), this contributes negatively to the haircut. However, in states $s \geq s^*$, agent 1 does not wish to borrow more than $c_{2,s}^2 - \omega$. Hence, he does not use the full collateral value of the asset. In particular, the repayment $f(s)$ is flat while the asset value $p_2(s)$ increases with $s$. This contributes positively to the haircut. The overall sign of the haircut depends on the weights on both regions in the distribution of $s$. Finally, observe that the haircut is not pinned down when $s^* \leq s$ since several (constant) repurchase prices $f$ are possible in equilibrium.

The liquidity premium captures the value of the asset as an instrument to borrow over and above its holding value. This premium is zero when agents are not constrained in any state, that is $s^* \leq s$, as shown by the expression above. When $s^* > s$, the liquidity premium is an average of the pledging capacity of the asset $s/(1 - \theta_1)$ multiplied by the wedge in marginal utilities

### 3.1.1 Counterparty risk

We now perform a comparative static exercise varying $\theta$ a proxy for counterparty quality. Indeed, a higher $\theta$ implies a higher punishment from defaulting and thus a superior ability to honor debt. Although there is no default in equilibrium, the equilibrium contract reflects default risk. Using the expression derived in Corollary 4, we obtain that haircuts increase with counterparty risk, or:

$$\frac{\partial \mathcal{H}}{\partial \theta_1} = -\frac{1}{(1 - \theta_1)^2} \int_{s}^{s^*} sdF(s) \leq 0$$

Indeed, as Figure 2 shows, a higher $\theta_1$ increases the amount a borrower can raise per unit of the asset pledged. This naturally leads to a decrease in the haircut, by
increasing the size of the region where $f(s) > p_2(s)$ while leaving the other region unchanged.

When it comes to the liquidity premium $L$, counterparty quality $\theta_1$ has an ambiguous effect. First, remember that agent 1 can pledge at most $ap_2(s)/(1 - \theta_1)$ in state $s$. Hence, an increase in $\theta$ raises the pledgeable amount. Agent 1 can thus borrow more in states $s < s^*$, which reduces the wedge $(c_2^s(s))/(c_1^s(s)) - 1$. This effect, similar to an increase in the asset available $a$, tends to reduce the liquidity premium. However, $\theta_1$ also increases the slope of the repurchase price $1/(1 - \theta_1)$ on those states where the agents are constrained. As more income can be pledged when this is most valuable, the asset becomes a better borrowing instrument, which raises its price. Observe that this second effect does not arise when we vary the asset supply $a$. Thus, counterparty quality $\theta_1$ can have a non-monotonic impact on the liquidity premium $L$.

\[\frac{s^*_L}{1 - \theta_L} = \frac{s^*_H}{1 - \theta_H}\]

**Figure 2:** Influence of $\theta$, with $\theta_H > \theta_L$

---

\[\text{This argument abstracts from the negative equilibrium impact of } \theta \text{ on the spot market price } p_2(s) \text{ which is pinned down by the relationship } p_2(s)v'(\omega - ap_2(s)/(1 - \theta)) - s = 0 \text{ for } s \leq s^* \text{. However, one can easily show that the net effect is positive, that is } \partial[p_2(s)/(1 - \theta)]/\partial \theta > 0.\]
3.1.2 Asset risk

We now want to compare haircuts and liquidity premium as a function of asset riskiness. For this purpose, we introduce two assets with different risk profiles but perfectly correlated payoffs\(^\text{19}\). We compute the liquidity premium of the safer asset relative to the riskier, the haircuts that both assets carry, and the repo rates. As before, \(s \sim F[\underline{s}, \bar{s}]\) but there are now two assets \(i = A, B\) with payoffs \(\rho_i(s)\):

\[
\rho_i(s) = s + \alpha_i(s - \mathbb{E}[s]),
\]

where \(\alpha_B > \alpha_A = 0\). With \(\alpha_A = 0\), asset \(A\) is our benchmark asset. Since \(\alpha_B > 0\), asset \(B\) has the same mean but a higher variance than asset \(A\). Indeed \(\text{Var}[\rho_i] = (1+\alpha)\text{Var}[s]\). We choose to consider two assets with perfectly correlated payoff to ignore the effect of risk sharing on the structure of the repo contract.

Agent 1 is endowed with \(a\) units of asset \(A\) and \(b\) units of asset \(B\), while agent 2 does not hold any of the assets. It is relatively straightforward to extend the equilibrium analysis of the previous section to this new economy with two assets. The set of available contracts consists of feasible repos using assets \(A\) and \(B\). For each asset \(i = A, B\), the repo contract \(f_i\) uses the maximum pledgeable capacity up to the state where the first best level of consumption can be reached. We then prove the following result.

**Proposition 5.** The safer asset \(A\) always has a higher liquidity premium and a lower haircut than the riskier asset \(B\).

The key intuition behind the result is the misallocation of collateral value induced by a mean preserving spread. Asset \(A\) and \(B\) have the same expected payoff. However, since \(\rho_B(s) - \rho_A(s) = \alpha_B(s - \mathbb{E}[s])\), the risky asset pays relatively more in high states (upside risk) and less in low states (downside risk). Since agents are constrained for low values of \(s\), this is precisely when collateral is valuable. Since the safe asset \(A\) pays more in these states, it carries a larger liquidity premium. We now turn to the haircut. In high states, the riskier asset \(B\) has a higher payoff

\(^{19}\)We can prove similar results, in the one asset case, by considering a mean preserving spread. However, we would then compare quantities across equilibrium rather than within an equilibrium as we do here.
which means that more income can be pledged compared to asset $A$. However, agent 1 does not wish to borrow over the first best level. Hence agents do not exploit the the higher collateral value of the risky asset in high states, implying a larger haircut. Observe that without this hedging motive, asset risk would have no impact on the haircut.

So far, repos are indistinguishable from collateralized loans. Indeed, with $\nu = 0$, the asset is immobile once pledged in a repo. The next two sections show that allowing for re-use delivers new predictions. First, re-use increases the borrowing capacity of agent 1. Second, the possibility to re-use collateral may lead to endogenous intermediation in equilibrium.

4 The multiplier effect of re-use

In this section, we analyze the impact of collateral re-use on equilibrium contracts and allocations. This is a natural feature of a repo trade where the collateral is sold to the lender. Re-use has been very much under scrutiny following the crisis (see Singh and Aitken, 2010) since a default on re-used collateral may affect several agents along a credit chain. While we do not model the consequence of such default cascades, we provide the foundations for this analysis by highlighting the benefits of re-use. The lender, agent 2 is now able to re-use collateral, that is $\nu_2 > 0$.

To understand the potential benefits, consider the equilibrium without re-use. Agent 2 (the lender) holds collateral pledged by agent 1. Re-use frees up a fraction $\nu_2$ of this collateral. Suppose agent 2 then sells $\epsilon$ units where $\epsilon$ is small to agent 1 at the equilibrium price $p_1$. The marginal gain for agent 1 is null since buying the asset is feasible without re-use. The marginal gain to agent 2 is

$$\frac{\partial U^2}{\partial \epsilon} = p_1 - E[p_2(s)u'(c_2^2(s))] = \eta_1^2$$

where $\eta_1^2$ is the shadow price of the asset for agent 2. Using the equilibrium
characterization, we obtain:

\[ \eta_1^2 = \frac{\theta_1}{1 - \theta_1} \int_{s}^{s^*} (u'(c_{2}^2(s)) - v'(c_{1}^2(s))) \, dF(s) \]

Hence, this marginal gain is strictly positive when \( s^* > s \) (agents are constrained) and \( \theta_1 > 0 \). To understand this last condition, observe that agent 1 may now sell in a repo the re-used asset he bought spot from agent 2. He can thus pledge \( p_2(s)/(1 - \theta_1) \) per unit. The net transfer however is

\[-p_2(s) + \frac{p_2(s)}{1 - \theta_1} = \frac{\theta_1}{1 - \theta_1} p_2(s)\]

since he first bought the asset spot from agent 2. This transfer is positive and increases agent 1’s borrowing only if \( \theta_1 > 0 \).

These steps can be repeated over multiple rounds. Agent 1 initially owns a unit of the asset. In the first round, he can pledge \( a p_2(s)/(1 - \theta_1) \), just as in the no re-use case. After this trade, agent 2 has \( \nu_2a \) units of re-usable asset. Given our argument above, agent 1 can then pledge an additional \( \frac{\theta_1}{1 - \theta_1} \nu_2 a p_2(s) \) in state \( s \). After this operation, agent 2 has \( (\nu_2)^2a \) units of re-usable asset. Iterating over these rounds infinitely, the total pledgeable amount per unit of asset in state \( s \) obtains:

\[ M_{12}p_2(s) := \frac{p_2(s)}{1 - \theta_1} + \sum_{i=1}^{\infty} (\nu_2)^i \frac{\theta_1}{1 - \theta_1} p_2(s) \]

\[ = \frac{1}{1 - \nu_2} \left[ \frac{1}{1 - \theta_1} - \nu_2 \right] p_2(s) \]

(19)

where we call \( M_{12} \) the borrowing multiplier between agents 1 and 2. This expression is strictly increasing in \( \nu_2 \) as long as \( \theta_1 > 0 \). Again, the role of trustworthiness \( \theta \) for re-use appears clearly.

In this analysis, we guessed that agent 2, the lender, returns the collateral. Indeed, our construction implicitly relied on a repo contract \( f_{12} \) with \( f_{12}(s) = p_2(s)/(1 - \theta) \). This satisfies the no-default constraint (4) of the lender, agent 2.

Let us now define \( s^*(\nu_2) \) as the minimal state above which agent 1 can pledge enough income to finance the first best allocation, that is:
\[ \omega + a M_{12} p_2(s^*(\nu_2)) = c_{2,s}^2. \]

We can then introduce the candidate equilibrium repo contract \( f(\nu_2) \) where:

\[
f(s, \nu_2) = \begin{cases} 
  \frac{p_2(s)}{1 - \theta_1} & \text{if } s < s^*(\nu_2) \\
  \frac{\nu_2(s - s^*(\nu_2))}{v'(c_{2,s}^1)} + \frac{s^*(\nu_2)}{v'(c_{2,s}^1)} & \text{if } s \geq s^*(\nu_2)
\end{cases}
\]

The following Proposition establishes that agents trade this contract in an equilibrium with re-use:

**Proposition 6. Collateral Re-use.**

Let \( \nu_2 \in (0, 1) \) be the fraction of collateral agent 2 can re-use. Collateral re-use is strictly preferred whenever \( \theta_1 > 0 \) and \( s^*(0) > \frac{s}{2} \) (the first-best allocation cannot be achieved without re-use). The (essentially) unique equilibrium repo contract is \( f(\nu) \) defined in (20). When \( \nu_2 \geq \nu^* \), agents reach the first-best allocation, where

\[
\nu^* = \frac{s^*(0) - \frac{s}{2}}{s^*(0) - (1 - \theta_1)\frac{s}{2}}.
\]

As we discussed before, when \( \theta_1 > 0 \), re-use strictly increases the amount agent 1 can pledge to agent 2. This is valuable when agents are constrained and want to expand borrowing in low states. From the expression of \( M_{12} \) in (19), it is clear that for \( \nu_2 \) high enough, the first-best allocation can even be financed in the lowest state \( \frac{s}{2} \). One can obtain the expression for \( \nu^* \) by setting \( s^*(\nu_2) = \frac{s}{2} \). The equilibrium repo contract has \( f(s, \nu_2) = p_2(s)/(1 - \theta_1) \) in those states \( s \leq s^*(\nu_2) \), where agents are still constrained. There, agents use the maximum pledgeable amount. On states \( s \geq s^*(\nu_2) \), the repo schedule is not flat anymore when \( \nu_2 > 0 \). The second component \( \nu_2(s - s^*(\nu_2)) \) corrects for the short position taken by agent 2 in the spot market. Indeed, in every round of re-use, agent 2 sells a fraction \( \nu_2 \) of the collateral he receives as a pledge from agent 1. He must then return this collateral in period 2 or equivalently cover his short position in the asset. Since he re-sells a fraction \( \nu_2 \) of every unit, it is not surprising that the second term is proportional.
to $\nu_2$.

It may seem that agent 1 would only be willing to engage in re-use if the haircut is negative. Indeed, buying 1 unit of asset from agent 2 to pledge it back yields a net gain of $-p_1 + p_F = -\mathcal{H}$ in period 1. This intuition proves correct when there is no uncertainty, that is $s = E[s] = 1$ for all $s$. Then, since $f = p_2/(1 - \theta_1)$, expression (17) shows that $\mathcal{H} \leq 0$ indeed. Intuitively, the borrower would benefit only if he increases consumption in period 1. When there is uncertainty, the above logic is incomplete since agent 1 may also gain by transferring consumption across states. If the haircut is positive, agent 1 benefits by decreasing his consumption in period 1 and in the low states of period 2 to increase it in the high states of period 2.

The liquidity premium $\mathcal{L}$ can exhibit non-monotonicity in the re-use factor $\nu_2$. Indeed, while re-use relaxes the collateral constraint, it also increases the amount pledgeable in states where agents are constrained. This last effect makes the asset more valuable and can increase the liquidity premium. These two effects are reminiscent of the comparative statics with respect to the commitment power $\theta$. Finally, our model predicts that re-use is helpful when collateral is most scarce (that is $s^*(0) > \bar{s}$) and there is evidence that this is indeed the case (see Fuhrer et al., 2015).

Remark 7. The multiplier effect and the role of $\nu_1$.

Since agent 2 is the natural lender, it seems that the re-use capacity of the borrower $\nu_1$ should play no role. The proof of Proposition 6 in the appendix shows that the pattern of trades we described is an equilibrium if $\nu_1$ is not too large. We provide an informal explanation here. Observe first that agent 2 is free to re-use the asset as he wishes. He may either re-sell it spot or re-pledge it in a repo to agent 1. In the discussion leading to Proposition 6, we implicitly guessed that he prefers the first option. We now argue that this is indeed the case if:

$$-p_2(s) + M_{12}p_2(s) \geq -\nu_1(1 - \theta_1)p_2(s) + \nu_1 M_{12}p_2(s)$$

The left hand side measures the gains from re-selling the asset spot. Agent 2 then gives up $p_2(s)$ units of consumption in period 2. However, agent 1 can now repledge the asset. Accounting for the multiplier effect, agent 2 consumption
increases by $M_1 p_2(s)$. The right hand side captures the gains from selling the asset in a repo with $f(s) = \nu_1(1 - \theta_1)p_2(s)$ to agent 1. Agent 2 only gives up $\nu_1(1 - \theta_1)p_2(s)$ units of consumption when he sells but agent 1 only has $\nu_1$ units available to repledge. Hence, for agent 2, the benefit from selling in repo is the smaller transfer from agent 2 to agent 1 in period 2 while the cost is the segregation of $(1 - \nu_1)$ units by agent 1. Intuitively, when $\nu_1$ is small, the cost dominates and agent 2 sells spot. Elementary transformations of this inequality yields the following condition.

$$\frac{\nu_1(1 - \nu_2)(2 - \theta_1)}{1 - \nu_2 \nu_1} < 1 \quad (21)$$

The left hand side is monotonic in $\nu_1$, the fraction of pledged collateral agent 1 can re-use. The equilibrium characterization in Proposition 6 is thus valid not only when $\nu_1 = 0$ but whenever (21) holds. When it does not hold, agent 2 re-sells in a repo to agent 1. In equilibrium, this will also affect the contract sold by agent 1 to agent 2. Although, the equilibrium contracts may change, the core intuition remains. Collateral re-use allows agent 2 to sell the asset back to agent 1, whether spot or repo, for him to increase the amount he borrows.

5 Collateral Re-use and Intermediation

In their guide to the repo market, Baklanova et al. (2015) state that “dealers operate as intermediaries between those who lend cash collateralized by securities, and those who seek funding”. To fix ideas, let us consider the following chain of trades. First, a hedge fund who needs cash borrows from a dealer bank through a repo. The dealer bank then taps in a money market fund (MMF) cash pool through another repo to finance the transaction. Figure 3 illustrates this pattern of repo intermediation. Since direct trading platforms such as Direct RepoTM in the US are available, it may seem puzzling that a significant share of the repo market is intermeditated. In this section, we explain these chain of trades based on heterogeneity in trustworthiness between the hedge fund and the dealer bank. A remarkable feature of this equilibrium is that intermediation arises endogenously.
although in our example, the hedge fund would be free to trade with the MMF\textsuperscript{20}.

We change the economy slightly for this purpose. For simplicity, we assume that agent 1 has linear preferences, that is:

\[ U^1(c_1, c_2, c_3) = c_1 + \delta c_2 + c_3 \]

This is a particular case of our general framework with \( v(x) = \delta x \). We also introduce a third type of agent named \( B \), for Banker. Agent \( B \) has no asset initially. He is endowed with \( \omega \) in period 1 and 2 and has the following preferences:

\[ U^B(c_1, c_2, c_3) = c_1 + \delta_B c_2 + c_3 \]

where \( \delta \leq \delta_B < u'(\omega) \). Under this assumption, agent \( B \) also wants to borrow from agent 2 but he has lower gains from trade than agent 1. We set \( \theta_B > \theta \) so that the Bank has a higher trustworthiness than agent 1. The corresponding greater borrowing capacity will explain why agent \( B \) can play a role as an intermediary.

We will say that there is intermediation when agent 1 chooses not to trade directly with 2 in a repo contract. The last section already discussed the role of the re-use factor of the borrower \( \nu_1 \). For simplicity, we thus set \( \nu_1 = 0 \) here.

\textsuperscript{20}Our analysis thus extends Infante (2015) and Muley (2015) which assume intermediation exogenously. There can also be institutional differences between the two trades involved. Indeed, repos between hedge funds and dealer banks are typically bilateral whereas dealer banks and hedge funds often trade via a tri-party agent (in the US, these are JP Morgan and BNY Mellon) which acts as a collateral custodian. See Federal Reserve Bank of New York (2010) for a discussion of Tri-Party repo. Although we do not model the distinction between bilateral and Tri-Party repo, our analysis still provides a fundamental explanation for this market segmentation.
5.1 Intermediation via spot trades

We assume first that agents 1 and B have the preferences, $\delta = \delta_B$ and only differ in their trustworthiness with $\theta_1 < \theta_B$. Although agents 1 and B have no gains from trade, we show that the latter plays an active role as an intermediary.

**Proposition 8.** Let $\delta = \delta_B$ and $\theta_1 < \theta_B$ and suppose the asset is too scarce to reach the first-best allocation. Then, in equilibrium, agent 1 sells his asset spot. Agent B buys the asset and iterates on repo trades with agent 2.

The striking feature in Proposition 8 is that agent 1, does not trade a repo with agent 2, the natural lender while he holds the asset. It means that there exists no repo contract $\tilde{f}_{12}$ that these agents wish to trade. Instead, in equilibrium, 1 sells the asset to agent B for the latter to use as collateral with agent 2. Observe that once he buys the asset from agent 1, agent B substitutes for agent 1 as a borrower with agent 2. In particular, the equilibrium repo contract $f_{B2}$ is the same as (20), replacing $\theta_1$ with $\theta_B$.

Without agent B, agent 1 would borrow in a repo from agent 2 as before. However, agent B may pledge more income to agent 2 due to its higher trustworthiness $\theta_B > \theta$. In a competitive equilibrium, agent B makes no profit as an intermediary. Hence, his higher borrowing capacity with agent 2 is fully reflected in the spot price he pays for the asset to agent 1. As a result, agent 1 now prefers to sell his asset and delegates borrowing to a more trustworthy agent. When $\delta = \delta_B$, intermediation takes place via a spot trade between agents 1 and B and not via a repo. Observe indeed that there are no direct gains from trade between 1 and B. As a result, agents 1 and B do not value the extra borrowing capacity from a repo when $\theta_1 > 0$. To the contrary, trading repo is costly because a fraction $1 - \nu_B$ of the asset could not be used by agent B to borrow from 2. This trade-off is no longer trivial when $\delta < \delta_B$ and a chain of repos may emerge as we show in the next subsection.

---

21As before, agents B and 2 do multiple rounds of repo thanks to re-use by agent 2. Whether agent 2 re-sells spot rather than repo the collateral pledged by agent B depends on the following condition:

$$\frac{\nu_B(1 - \nu_2)}{1 - \nu_2 \nu_B} (2 - \theta_B) < 1$$

This expression is similar to (21) replacing $\nu_1$ by $\nu_B$ and $\theta_1$ by $\theta_B$. 

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When the asset is not scarce, the first best allocation whereby \( u'(c_{2,t}) = \delta \) is attainable. In this case, other equilibrium trades are possible. In particular, agent \( B \) could be inactive if agent 1 has enough asset to compensate for his low trustworthiness \( \theta_1 \). An interesting implication of our result is thus that intermediation should be observed precisely when collateral is scarce. When \( \delta_B = \delta \), the Bank essentially acts as a proxy by “selling” his higher trustworthiness to the risky agent.

5.2 Chain of repos

We now let \( \delta < \delta_B \) and show that an equilibrium with a chain of repos exists. When \( \delta < \delta_B \), agents 1 and \( B \) have direct gains from trade. A repo sale can now be valuable because it increases the transfer with respect to a spot sale when \( \theta_1 > 0 \). Since a fraction \( (1 - \nu_B) \) of the asset pledged is segregated, this may dominate a spot sale only if \( \delta_B - \delta \) is large enough.

If agents trade in a chain of repo, agent \( B \) acts both as a lender with agent 1 and as a borrower vis a vis agent 2. This creates a competing use for the asset. Indeed, when he holds one unit, agent \( B \) may either sell the asset back to agent 1 for him to increase borrowing or use it to borrow from agent 2. A key observation is that in equilibrium, he will be marginally indifferent between these two options. Suppose for instance that he strictly prefers to re-use to borrow from agent 2. This means that some asset collateralizing trade between agent 1 and \( B \) is misallocated and should rather support trade between \( B \) and 2. As a result, agents 1 and \( B \) would rather trade spot as before. Intuitively, indifference is possible if the gains from trade between agents 1 and \( B \) (proportional to \( \delta_B - \delta \)) are not too different from those between agents \( B \) and 2 (proportional to \( u'(\omega) - \delta_B \)).

Finally, agent 1 must prefer trading in a repo with agent \( B \) rather than with agent 2 while gains from trade are larger with 2. However, with heterogeneity in re-use factors \( \nu \), one unit of pledged collateral can be redeployed at different rates by each counterparty. Indeed, we have shown that the multiplier between borrower \( i \) and lender \( j \) is:

\[
M_{ij} = \frac{1}{1 - \nu_j} \left[ \frac{1}{1 - \theta_i - \nu_j} \right] \quad i = B, 2
\]
In equilibrium, agent 1 will prefer to trade with $B$ if the larger borrowing multiplier compensates for the lower gains from trade. We may now state the exact conditions under which a chain of repo can arise in equilibrium.

**Proposition 9. Intermediation equilibrium.**

An intermediation equilibrium with a chain of repos $f_{1B}$ and $f_{B2}$ exists iff

$$\frac{1}{1 - \theta_B} - \frac{1}{1 - \theta_1} \geq (1 - \nu_B)M_{B2}$$

$$\delta \geq \delta_B \geq \bar{\delta}$$

where the thresholds verify $\delta < \bar{\delta} < \delta < u'(\omega)$ and depend on all the parameter values. Agents 1 and $B$ trade using repo $f_{1B}$ given by

$$f_{1B}(s) = \frac{s}{1 - \theta_1} \quad \forall s \in [s, \bar{s}]$$

Observe first that the repo contract $f_{1B}$ between agents 1 and $B$ does not reflect any hedging motive since both agents are risk neutral. Uncertainty does not play any crucial role for the rest of the equilibrium characterization. To simplify the interpretation of Proposition 9, we thus assume that $s = 1$ for the discussion. When involved in a repo chain, agent $B$ acquires re-usable collateral from agent 1. He may either re-resell this collateral to agent 1 to increase borrowing or re-pledge the collateral to agent 2. As we discussed, in equilibrium, he must be marginally indifferent between both usages. The indifference condition can be written as follows:

$$M_{B2}[u'(c_2^2) - \delta_B] = (M_{1B} - 1)(\delta_B - \delta)$$

The left hand side is the gain from pledging collateral to agent 2. It is equal to the gains from trade times the borrowing multiplier $M_{B2}$ between the two agents. The left hand side is the gain from re-selling the asset to agent 1. The term in factor of the gains from trade is $M_{1B} - 1$ since the asset must first be sold for agent 1 to borrow, if it is initially held by agent $B$. Condition (24) should now be clearer. If $\delta_B$ is too close to $\delta$, the right hand side is necessarily smaller than the left hand side. In this case, agent 1 and $B$ would rather trade spot as in Proposition (8).

We must still understand why agent 1 should trade with agent $B$ rather than
with agent 2. He can indeed sell one unit of asset in repo $f_{1B}$ to either agent. We argue that the first option dominates the second if:

$$\frac{\delta_B - \delta}{1 - \theta_1} + \nu_H M_{B2}(u'(c^2_2) - \delta_B) \geq \frac{u'(c^2_2) - \delta}{1 - \theta_1} + \nu_2(M_{B2} - 1)(u'(c^2_2) - \delta_B)$$

The left hand side (resp. right hand side) measures the gains from selling the asset in a repo to agent $B$ (resp. 2). The first component is the direct gain from trade. This is obviously larger with agent 2 since $u'(\omega) > \delta_B$. However, with re-use, there are also indirect gains (the second term) since the collateral can be redeployed. If agent 1 sells to agent $B$, $\nu_H$ units can support borrowing with agent 2. If sold to agent 2, $\nu_2$ units can be re-used. These re-use gains may be larger with $B$ if $\nu_B > \nu_2$. The possibility to re-use collateral thus explains why seemingly dominated trades (here between 1 and $B$) can take place.\(^{22}\) Equilibrium condition (23) obtains from straightforward manipulation of the inequality above. When we set $\nu_2 = 0$, condition (23) nicely reads as a cost-benefit analysis of intermediation. Indeed, it collapses to:

$$1 - \nu_B \leq \frac{\theta_B - \theta_1}{1 - \theta_1}$$

The left hand side is the fraction of collateral immobilized by intermediation. The right hand side is the (normalized) extra borrowing capacity $\theta_B - \theta_1$ of agent $B$.

To summarize, an agent may become a dealer if he is more trustworthy than the natural borrower and more efficient at re-deploying collateral than the natural lender. Our analysis thus shows that repo intermediation arises endogenously out of fundamental heterogeneity between traders. Existing models of repo intermediation typically take the chain of possible trades as exogenous. Our approach is helpful to rationalize several features of the repo market. First, we can explain why intermediating repo is still popular despite the emergence of direct trading.

\(^{22}\)Condition (23) is equivalent to

$$M_{1B}(\delta_B - \delta) \geq M_{12}(u'(c^2_2) - \delta)$$

From the point of view of agent 1, borrowing from agent $B$ dominates if the multiplier $M_{1B}$ is larger than $M_{12}$ although gains from trade are smaller $(\delta_B - \delta \leq u'(c^2_2) - \delta)$. Again this is possible only if $\nu_B > \nu_2$.\(^{34}\)
platforms. Second, in exogenous intermediation models, dealers typically gain and collect fees by charging higher haircuts to borrowers. In our model, the haircut paid by the borrower to the bank may very well be smaller than the one paid by the bank to the lender. Using data from the Australian repo market, Issa and Jarnecic (2016) show that this is indeed the case in most transactions.

6 Conclusion

We analyzed a simple model of repurchase agreement with limited commitment and price risk. Unlike a combination of sale and repurchase in the spot market, a repo contract provides insurance against the asset price risk. We introduce counterparty risk as heterogeneous cost from defaulting on the promised repurchase price. We showed that the repo haircut is an increasing function of counterparty risk and a decreasing function of the asset inherent risk. Safe assets naturally command a higher liquidity premium than risky ones. Our model targets repos since we allow agents to re-use collateral. We showed that re-use increases borrowing through a multiplier effect. In addition, it can explain intermediation whereby trustworthy agents borrow on behalf of riskier counterparties.

Our simple model delivers rich implications about the repo market but leaves many venues for future research. We argued that counterparty risk is a fundamental determinant for the terms of trade in repo contracts. It would be interesting to analyze the impact of clearing on repo market activity since clearing often implies novation by a central counterparty. Novation bears some similarities with intermediation although terms of trades cannot be adjusted and risk may be concentrated on a single agent. When it comes to re-use, besides the limit on the amount of collateral that can be re-deployed, we assumed a frictionless process. Traders establish and settle positions smoothly although many rounds of re-use may be involved. This may not be the case anymore in the presence of frictions in the spot market for instance. Recent theoretical papers have shown that secured lending markets can be fragile. Although we did not investigate this aspect in the present work, we believe collateral re-use may add to this fragility.
Appendices

A  Equilibrium analysis of spot trade only

We prove the following Proposition that characterize spot trade equilibria.

**Proposition 10.** When agents can only trade spot, there exists a threshold \( \bar{a}_{\text{spot}} \) such that

1. **Low asset quantity:** if \( a < \bar{a}_{\text{spot}} \), then agent 1 sells his entire asset holdings at date 1. The liquidity premium \( L \) is strictly positive.

2. **High asset quantity:** if \( a \geq \bar{a}_{\text{spot}} \), then agent 1 sells less than \( a \) at date 1. The liquidity premium is \( L = 0 \).

Deriving the first order conditions, the following system of equations characterize the equilibrium.

\[
\begin{align*}
    c_1^1(s) &= \omega + as, \\
    c_2^1(s) &= \omega - p_2(s)(a_1^2 - a_2^2(s)), \\
    -p_2(s)v'(c_1^2(s)) &= s + \xi_1^1(s), \\
    -p_2(s)u'(c_2^2(s)) &= \delta s + \xi_2^2(s), \\
    -p_1 + E \left[ p_2(s)v'(c_2^1(s)) \right] + \xi_1^1 &= 0, \\
    -p_1 + E \left[ p_2(s)u'(c_2^2(s)) \right] + \xi_2^2 &= 0, \\
    \xi_1^1\xi_2^2 &= 0
\end{align*}
\]

where \( \xi_i^t \) is the Lagrange multiplier on the no-short sale constraint of agent \( i \) in period \( t \). Given that \( u'(\omega) > v'(\omega) \), one can easily check that \( \xi_1^t(s) = 0 \) for all \( s \). This is natural since agent 1 who does not discount period 3 payoffs is the natural holder of the asset. By the same logic, we have that \( \xi_2^t = 0 \). Agent 2 must buy a positive quantity of the asset since otherwise gains from trade are left on the table. From equation 27, it is easy to realize that \( p_2(s) \) is increasing in \( s \). Moreover there exists \( \hat{s}(a_1^2) \) such that \( a_2^2(s) \) is equal to 0 for \( s \leq \hat{s}(a_1^2) \) and solves \( p_2(s)u'(\omega + p_2(s)(a_1^2 - a_2^2(s))) = \delta s \) otherwise. Agent 2 carries positive holdings of the asset into period 3 in those high states \( s > \hat{s}(a_1^2) \) where re-selling everything would increase too much his period 2 consumption. Focusing now on period 1, we are left to pin down \( a_1^2 \), the quantity agent
2 initially buys from agent 1.

\[-p_1 + E \left[ p_2(s)v'(c_2^1(s)) \right] + \xi_1^1 = 0,\]
\[-p_1 + E \left[ p_2(s)u'(c_2^2(s)) \right] = 0,\]
\[c_2^1(s) = \omega - p_2(s)a_1^2\]
\[c_3^1(s) = 2\omega + as\]

so

\[\xi_1^1 = E \left\{ p_2(s) \left[ u'(c_2^2(s)) - v'(c_2^1(s)) \right] \right\} \tag{29}\]

To solve for the equilibrium price \(p_2(s)\) and the quantity sold \(a_1^2\), let us introduce the following system:

\[p_2(s)v'(\omega - p_2(s)a_1^2) = s\]
\[G(a_1^2) = E \left\{ p_2(s) \left[ u'(\omega + p_2(s)a_1^2) - v'(\omega - p_2(s)a_1^2) \right] \right\}\]

The first equation implicitly defines \(p_2(s)\) as a function of \(s\) and \(a_1^2\), using equation 27. The Implicit Function Theorem shows that \(p_2(s)\) depends negatively on \(a_1^2\). The total derivative of \(G\) with respect to \(a_1^2\) is equal to

\[G'(a_1^2) = \int_{\bar{s}}^{s(a_1^2)} \left[ \frac{\partial p_2(s)}{\partial a_1^2} \left\{ u'(\omega + p_2(s)a_1^2) + a_1^2 p_2(s) u''(\omega + p_2(s)a_1^2) \right\} + p_2(s)^2 u''(\omega + p_2(s)a_1^2) \right] dF(s)\]

This expression is strictly negative if the coefficient of relative risk aversion of \(u\) is less than 1.

Define then \(\bar{a}_{\text{spot}}\) as the unique solution to \(G(a_1^2) = 0\). Two cases are then possible: i) \(a \geq \bar{a}_{\text{spot}}\) and \(\xi_1^1 = 0\) and \(a_1^2 = \bar{a}_{\text{spot}}\) or ii) \(a < \bar{a}_{\text{spot}}\) and \(\xi_1^1 > 0\) that is \(a_1^2 = a\).

## B Proofs

### B.1 Proof of Proposition 3

In the absence of re-use (\(\nu = 0\)), the set of no-default repo contracts for agent \(i \in \{1, 2\}\) at a given spot market price schedule \(p_2 = \{p_2(s)\}_{s \in S}\) is:

\[\mathcal{F}_i(p_2) = \left\{ f \in C^0[\bar{s}, \tilde{s}] \mid 0 \leq f(s) \leq \frac{p_2(s)}{1 - \theta_i} \right\}\]

We proceed in three steps. First we characterize the equilibrium repurchase contract \(f \in \mathcal{F}_1(p_2)\) for a given spot price schedule \(p_2\), using the fact that agents must not be willing to trade
any other feasible contract. Then we characterize the spot market price \( p_2 \) compatible with the equilibrium. Finally, we back our claim that agents do not trade repo inducing default when assumption (5) holds.

Observe first that agent 2 needs not borrow, that is \( b_{21} = 0 \) and that spot trading is redundant, that is \( a_1^1 = a_2^1(s) = a \) for all \( s \) wlog. Indeed, agent 1 is the natural borrower since \( u'(\omega) > v'(\omega) \) and the payment schedule from a spot transaction \( p_2 \) is included in the set of feasible repos \( F(p_2) \). Hence, we will only consider a repo contract where agent 1 is the borrower that we call \( f \) for simplicity.

The equilibrium conditions when agents trade repo \( f \) are:

\[
\begin{align*}
-p_1 + E[p_2(s)v'(c_2^1(s))] + \gamma_1^1 &= 0, \\
-p_F + E[f(s)v'(c_2^1(s))] + \gamma_1^1 &= 0, \\
-p_1 + E[p_2(s)u'(c_2^2(s))] + \gamma_2^1 &= 0, \\
-p_F + E[f(s)u'(c_2^2(s))] &= 0, \\
-p_2(s)v'(c_2^1(s)) - s &= 0, \\
\xi_1^1\xi_2^1 &= 0, \\
c_1^1(s) &= \omega - f(s)b_{12}^1, \\
c_2^2(s) &= \omega + f(s)b_{12}^2
\end{align*}
\]

We used the fact that agent 1 will be the marginal holder of the asset into period 3. We can also derive the marginal willingness to pay for any contract \( \tilde{f} \in F_{12}(p_2) \) for both agents. In other words, we derive the minimum (resp. maximum) price \( \tilde{q}^{1}_{12}(\tilde{f}) \) and \( \tilde{q}^{2}_{12}(\tilde{f}) \) at which agent 1 (resp. agent 2) is ready to sell (resp. to buy) an infinitesimal amount of contract \( \tilde{f} \).

\[
\begin{align*}
\tilde{q}^{1}_{12}(\tilde{f}) &= E[\tilde{f}(s)v'(c_2^1(s)) + \gamma_1^1] \\
\tilde{q}^{2}_{12}(\tilde{f}) &= E[\tilde{f}(s)u'(c_2^2(s))]
\end{align*}
\]

For agents not to trade contract \( \tilde{f} \) in equilibrium, the following inequality must hold:

\[
\tilde{q}^{2}_{12}(\tilde{f}) \leq \tilde{q}^{1}_{12}(\tilde{f}) \tag{30}
\]

Indeed, if this inequality holds, there is an equilibrium price \( \tilde{q}_{12}(\tilde{f}) \in [\tilde{q}^{1}_{12}(\tilde{f}), \tilde{q}^{2}_{12}(\tilde{f})] \) such that agents’ optimal trade in \( \tilde{f} \) is 0. We will use this inequality to show that the equilibrium \( f \) is the contract characterized in Proposition 3.
B.1.1 Characterization of the equilibrium repo contract

There are two cases.

i) \( \gamma_1^1 = 0 \): agent 1 is unconstrained.

Then agents 1 and 2’s (marginal) valuation for any contract \( \tilde{f} \in \mathcal{F}_1(p_2) \) must coincide, that is:

\[
E \left[ \tilde{f}(s) u'(c_2^2(s)) \right] = E \left[ \tilde{f}(s) u'(c_2^1(s)) \right]
\]

(31)

where \( c_2^2(s) = \omega + f(s)b^{12} \). Suppose there is an open interval \( (s_1, s_2) \in S \) such that for all \( s \in (s_1, s_2) \), \( u'(c_2^2(s)) - u'(c_2^1(s)) = 0 \) and has a constant sign. Let us then consider the piece-wise linear schedule \( \tilde{f} \) such that \( \tilde{f}(s_1) = \tilde{f}(s_2) = \tilde{f}(\bar{s}) = 0 \) and \( \tilde{f}(s_1/2 + s_2/2) = s_1 \). The schedule \( \tilde{f} \in \mathcal{F}_1 \) would violate equality (31). It means that there cannot be an open interval on which \( u'(c_2^2(s)) - u'(c_2^1(s)) \neq 0 \). Hence, by continuity, we must have for all \( s \in S \), \( u'(c_2^2(s)) = v'(c_2^1(s)) \), that is \( c_2^2(s) = c_2^1(s) \). This means that \( f \) is constant and in particular that agent 2 can finance \( c_{2,s}^1 \) in the lowest state \( s \):

\[
c_{2,s}^1 = \omega + f(s)b^{12} \leq \omega + a \frac{p_2(s)}{1 - \theta_1} = \omega + a \frac{s}{v'(c_{2,s}^1)(1 - \theta_1)}
\]

where we can replace \( p_2(s) = s/v'(c_{2,s}^1) \), using the spot market equilibrium condition in period 2 and the fact that \( c_2^1(s) = c_{2,s}^1 \). This may hold, only if \( s^* \leq \bar{s} \). In that case, although the equilibrium allocation is unique, the contracts traded are not. The expression of \( c_2^2(s) \) only pins down\(^{23}\) the product \( b^{12}f \), and the repurchase price \( f \) may lie anywhere in the interval \( [\frac{s}{v'(c_{2,s}^1)}, \frac{s}{v'(c_{2,s}^1)}'] \).

ii) \( \gamma_1^1 > 0 \): agent 1 is constrained.

This means that \( b^{12} = a \). Rewriting (30) using equilibrium conditions, we obtain:

\[
E \left[ \left( f(s) - \tilde{f}(s) \right) \left( u'(c_2^2(s)) - u'(c_2^1(s)) \right) \right] \geq 0
\]

(32)

Let us now define a partition of \( S \) as follows

\[
S^+(p_2) = \left\{ s \in S \mid \omega + a \frac{p_2(s)}{1 - \theta_1} \geq c_{2,s}^1 \right\}, \quad S^-(p_2) = \left\{ s \in S \mid \omega + a \frac{p_2(s)}{1 - \theta_1} < c_{2,s}^2 \right\}
\]

Intuitively, \( S^+(p_2) \) is the union of intervals (by continuity) where the FB allocation is attainable given \( p_2 \). We have \( S^+(p_2) \cup S^+(p_2) = S \) also by continuity. We argue first that if \( f(s) = s^*/(1 - \theta_1) \) for \( s \in S^+(p_2) \). If \( f \) lies below this constant, by definition of \( s^* \), we have \( u'(c_2^1(s)) - v'(c_2^2(s)) > 0 \). Any \( f \) lying slightly above \( \tilde{p} \) would violate (32). A similar argument can be applied to show that \( f \) cannot lie above \( p_2(s^*)/(1 - \theta_1) \) for \( s \in S^+(p_2) \). Now, we

\(^{23}\)In addition, agent 2 could also buy the asset spot to sell it in a repo \( F_2 \). In any case, having agent 1 sell \( a \) units of contract \( \tilde{p} = s^*/(1 - \theta) \) is an equilibrium since agents do not (strictly) want to trade another contract.
argue that \( f(s) = p_2(s)/(1 - \theta_1) \) for \( s \in S^-(p_2) \). Indeed, by definition, for all \( s \in S^-(p_2) \),
\[ u'(c^{2*}_2(s)) - v'(c^{2*}_2(s)) > 0 \]
so that any schedule \( \bar{p} \) above \( \bar{p} \) is feasible and would again violate (32). Hence, we have fully defined the equilibrium \( f \) as a function of \( p_2 \).

**B.1.2 Characterization of the spot market price**

We now characterize the fixed point defining equilibrium \( p_2 \). Given equilibrium trades and the equilibrium contract traded, we have:

\[
\begin{align*}
    p_2(s) v'(\omega - \frac{p_2(s)}{1-\theta_1}) &= s & s \in S^-(p_2) \\
    p_2(s) v'(c^{2*}_2(s)) &= s & s \in S^+(p_2)
\end{align*}
\]

We have that \( c^2_1(s) < c^{2*}_1(s) \) for \( s \in S^-(p_2) \). Suppose there exists \( s^+ \in S^+(p_2) \). Since \( p_2(s) > p_2(s^+) \) for \( s > s^+ \), we have that \([s^+, \hat{s}] \in S^+(p_2)\). In this case, \( S^+(p_2) \) is an interval containing the larger elements of \( S \). We are left to show that its minimal element is \( s^* \) defined in (14). Clearly, \( s^* \in S^+(p_2) \). Consider now \( \hat{s} < s^* \). By definition of \( s^* \), we have that

\[
\omega + a \frac{\hat{s}}{v'(c^{2*}_1(s))} < c^2_1(s)
\]

In words, the first best allocation cannot be reached if the spot market price is equal to its “fundamental value” that is \( p_2(\hat{s}) = \hat{s}/v'(c^{2*}_2(s)) \). This means that \( \hat{s} \in S^-(p_2) \) as otherwise, we would have \( f(\hat{s}) = s^*/(1 - \theta_1) \) and \( p_2(\hat{s}) = \hat{s}/v'(c^{2*}_2(s)) \).

To conclude, the equilibrium contract \( f \) and spot market price \( p_2 \) verify the following equations

\[
\begin{align*}
    \text{If } s < s^*, & \quad \begin{cases} p_2(s) v'(\omega - \frac{p_2(s)}{1-\theta_1}) - s = 0 \\
                          f(s) = \frac{p_2(s)}{1-\theta_1} \end{cases} \\
    \text{If } s \geq s^*, & \quad \begin{cases} p_2(s) = s/v'(c^{2*}_2(s)) \\
                          f(s) = \frac{p_2(s^*)}{1-\theta_1} \end{cases}
\end{align*}
\]

**B.1.3 No default-prone contracts**

Consider now a repo contract \( \tilde{f} \) such that agent 1 defaults in some states of the world, that is \( \tilde{f}(s) \) violates (3) for some \( s \). If \( f \) is the equilibrium contract, we can focus on contracts such that \( \tilde{f}(s) = f(s) \) for \( s \geq s^* \) since it is not possible to improve over \( f \) on this region. Let us now define \( S_d = \{ s \in [s, s^*] | \tilde{f}(s) \text{ violates (3)} \} \) and \( S_{nd} = [s, s^*]\setminus S_d \). Agents do not trade contract \( \tilde{f} \)
in equilibrium if and only if
\[
\int_{S_d} \left( f(s) - \tilde{f}(s) \right) \left( u'(c^2_2(s)) - v'(c^2_2(s)) \right) dG(s) + \\
\int_{S_d} \left( f(s) - \alpha \tilde{f}(s) - (1 - \alpha)p_2(s) \right) \left( u'(c^2_2(s)) - v'(c^2_2(s)) \right) dG(s) - \pi \int_{S_d} \tilde{f}(s)v'(c^2_2(s))dG(s) \geq 0
\]

The first line is condition (32) for a no-default contract. The second line corresponds to the states \( S_d \) where the borrower defaults. Observe that the realized payoff to the lender is only \( p_2(s) + \alpha(\tilde{f}(s) - p_2(s)) \). In addition, the borrower incurs the non-pecuniary cost (the second term of the second line). This inequality holds if
\[
\int_{S_d} \left( f(s) - (1 - \alpha)p_2(s) \right) \left( u'(c^2_2(s)) - v'(c^2_2(s)) \right) dG(s) \geq \int_{S_d} \alpha \tilde{f}(s) \left( u'(c^2_2(s)) - v'(c^2_2(s)) \right) dG(s) \\
- \pi \int_{S_d} \tilde{f}(s)v'(c^2_2(s))dG(s)
\]

Using that \( u'(c^2_2(s)) \leq u'(\omega) \) and \( v'(c^2_2(s)) \geq v'(\omega) \), we derive the following upper bound for the right hand side:
\[
[\alpha(u'(\omega) - v'(\omega)) - \pi v'(\omega)] \int_{S_d} \tilde{f}(s) dG(s)
\]

under assumption (5), this term is negative. Hence we proved that the inequality above holds and that agents do not want to trade default-prone repo contracts.

### B.2 Proof of Proposition 5

**Proof.** Building on the case with one asset, we can characterize the equilibrium as follows. Define \( s^{**} \) as the minimal state where the first best allocation can be reached.
\[
\omega + \frac{a\rho_A(s^{**}) + b\rho_B(s^{**})}{(1 - \theta_1)v'(c^2_{2,s})} = c^2_{2,s}.
\]

Then the repayment schedule for asset \( i \)
\[
f_i(s) = \begin{cases} 
    \frac{p_{2,i}(s)}{(1 - \theta_1)\rho_i(s^{**})} & \text{for } s \leq s^{**}, \\
    \frac{p_{2,i}(s)}{(1 - \theta_1)v'(c^2_{2,s})} & \text{for } s > s^{**}.
\end{cases}
\]

where \((p_{2,A}(s), p_{2,B}(s))\) are the spot market prices of asset \( A \) and \( B \) respectively in period 2, state \( s \). They are defined as follows for \( i = A, B \):
\[
\begin{align*}
    p_{2,i}(s) &= \left( \omega + \frac{a\rho_A(s) + b\rho_B(s)}{(1 - \theta_1)} \right) - \rho_i(s) = 0 & s \leq s^{**} \\
    p_{2,i}(s) &= \rho_i(s) & s > s^{**}
\end{align*}
\]
for $s < s^{**}$.

The liquidity premium for asset $i = A, B$ is

$$L_i = \int_{s}^{s^{**}} \frac{\rho_i(s)}{1 - \theta} \left[ \frac{u'(c^2_i(s))}{v'(c^2_i(s))} - 1 \right] dF(s)$$

Hence,

$$\mathcal{L}_{a,b} = L_A - L_B$$

$$= \int_{s}^{s^{**}} \frac{s - \rho_\alpha(s)}{1 - \theta} \left[ \frac{u'(c^2_\alpha(s))}{v'(c^2_\alpha(s))} - 1 \right] dF(s)$$

$$= -\frac{\alpha}{1 - \theta} \int_{s}^{s^{**}} (s - E[s]) \left[ \frac{u'(c^2_\alpha(s))}{v'(c^2_\alpha(s))} - 1 \right] dF(s)$$

$$> 0$$

where the inequality follows from the fact that the integral is negative over the integration range.

The haircut as a function of $\alpha$ is:

$$H_i(\alpha) = p_{1,i} - q_i$$

$$= E\left[ (p_{2,i}(s) - f_i(s))v'(c^2_\alpha(s)) \right]$$

$$= E[p_{2,i}(s)v'(c^2_\alpha(s))] - \int_{s}^{s^{**}} \frac{p_{2,i}(s)}{1 - \theta} v'(c^2_\alpha(s))dF(s) - \int_{s^{**}}^{s^{**}} \frac{p_{2,i}(s^{**})}{1 - \theta} v'(c^2_\alpha(s^{**}))dF(s)$$

$$= E[s] - \int_{s}^{s^{**}} \frac{\rho_i(s)}{1 - \theta} dF(s) - \int_{s^{**}}^{s^{**}} \frac{\rho_i(s^{**})}{1 - \theta} dF(s)$$

$$= E[s] - \int_{s}^{s^{**}} \frac{(1 + \alpha_i)s - \alpha_i\mu}{1 - \theta} dF(s) - \int_{s^{**}}^{s^{**}} \frac{(1 + \alpha_i)s^{**} - \alpha_i\mu}{1 - \theta} dF(s)$$

$$= E[s] + \frac{\alpha_i\mu}{1 - \theta} - \frac{(1 + \alpha_i)}{1 - \theta} \left[ \int_{s}^{s^{**}} sdF(s) + \int_{s^{**}}^{s^{**}} s^{**} dF(s) \right]$$

The term in brackets is less than $E[s]$ therefore, for all assets $A$ and $B$ such that $\alpha_A < \alpha_B$ we obtain

$$H_A < H_B$$

i.e. the safe asset always commands a lower haircut than the risky asset.
Proof of Proposition 6

Given that $\nu_1 = 0$, the same arguments apply to establish that agent 2 does not borrow in a repo so that we need to consider only one repo contract $f(\nu_2) \in F_{12}(p_2)$. However, spot trades are non-trivial because agent 2 can now re-sell collateral pledged by agent 1. The equilibrium conditions write:

\[-p_1 + E[p_2(s)v'(c_2^1(s)))] + \gamma_1^1 = 0,
-q_{12} + E[f(s, \nu_2)v'(c_2^1(s))] + \gamma_1^1 = 0.
-p_1 + E[p_2(s)u'(c_2^2(s))] + \gamma_2^2 = 0,
-q_{12} + E[f(s, \nu_2)u'(c_2^2(s))] + \nu_2 \gamma_1^2 = 0.
\]

We can write agent 2 consumption as

\[c_2^1(s) = \omega - f(s, \nu_2)b^{12} + p_2(s)(a_1^1 - a)\]

\[c_2^2(s) = \omega + f(s, \nu_2)b^{12} + p_2(s)a_1^2\]

We only look at the case where the collateral constraint binds as otherwise agents may reach the first-best allocation and the analysis is straightforward. In this case:

\[a_1^1 = b^{12}\]  \hspace{1cm} (33)
\[a_1^2 = -\nu_2 \ell^{21}\]  \hspace{1cm} (34)

Using clearing in the spot market, we have $a_1^1 + a_1^2 = a$. Market clearing for repo requires $b^{12} = \ell^{21}$. Summing (33) and (34) we obtain

\[a_1^1 = b^{12} = \frac{a}{1 - \nu_2}\]
\[a_1^2 = -\nu_2 b^{12} = \frac{\nu_2}{1 - \nu_2} a\]

We can write agent 2 consumption as

\[c_2^2(s) = \omega + \frac{a}{1 - \nu_2} (f(s, \nu_2) - \nu_2 p_2(s))\]

We can then adapt the proof of the no re-use case, observing that $f(s, \nu_2) \leq p_2(s)/(1 - \theta_1)$ as before and replacing $f(s)$ by $(f(s, \nu_2) - \nu_2 p_2(s))/(1 - \nu_2)$. As a consequence, the repurchase schedule finances the first-best consumption profile $(c_2^{1,*}, c_2^{2,*})$ whenever possible and hits the borrowing limit otherwise. We thus have:

\[\bar{p}(s, \nu) = \begin{cases} \frac{p_2(s)}{1 - \theta} & \text{if } s < s^*(\nu) \\ s^*(\nu) & \text{if } s \geq s^*(\nu) \end{cases} \]

\[v(s - s^*(\nu)) / v(c_2^{2,*}) \]
where \( s^*(\nu_2) \) is implicitly defined as follows

\[
\omega + \frac{as^*(\nu_2)}{(1 - \nu_2)v'(c^2_{21})} \left[ \frac{1}{1 - \theta} - \nu_2 \right] = c^2_{21}.
\]

Since \( v \rightarrow \frac{1 - (1 - \theta)\nu}{1 - \nu} \) is increasing in \( v \), \( s^*(\nu_2) \) is decreasing in \( \nu_2 \) and \( \lim_{\nu_2 \rightarrow 1} s^*(\nu_2) < 0 \).

Assuming now that \( \nu_1 > 0 \), we provide a formal argument for the claim in Remark ???. Agent 2 does not want to sell in a repo if for all \( \tilde{f}_{21} \in F_{21}(\nu_2) \), we have:

\[
E[\tilde{f}_{21}(s)u'(c^2_{2}(s))] + \gamma^2 \geq E[\tilde{f}_{21}(s)v'(c^2_{2}(s))] + \nu_1 \gamma^1
\]

Using the equilibrium characterization, we obtain the following inequality:

\[
E \left[ \tilde{f}_{21}(s) \left( u'(c^2_{2}(s)) - v'(c^2_{2}(s)) \right) \right] \geq \frac{1}{1 - \nu_2} E \left[ (\nu_1(f(s, \nu_2) - \nu_2p_2(s)) - f(s, \nu_2) + p_2(s)) \left( u'(c^2_{2}(s)) - v'(c^2_{2}(s)) \right) \right]
\]

Using the expression for \( f(s, \nu_2) \) we derived and \( \tilde{f}_{21} = \nu_1(1 - \theta_1)p_2(s) \) (the contract for which the inequality above is the most difficult to satisfy), we obtain:

\[
(1 - \theta_1)\nu_1 \geq \frac{(1 - \nu_1 \nu_2)(1 - \theta_1) - (1 - \nu_1)}{(1 - \theta_1)(1 - \nu_2)}
\]

\[
\Leftrightarrow \nu_1(1 - \nu_2)(1 - \theta_1) \geq \nu_1(1 - \nu_2) - \theta_1(1 - \nu_1 \nu_2)
\]

\[
\Leftrightarrow \theta_1(1 - \nu_2 \nu_1) \geq \nu_1(1 - \nu_2) \theta_1(2 - \theta_1)
\]

This is equivalent to condition (21).

\( \square \)

### B.4 Proof of Proposition 8 and 9

In this section, we prove the following result, that nests Proposition 7 and 8. Define first \( s^*(b, \theta_i, \nu) \) for \( i = 1, B \), as the solution in \( s^* \) to:

\[
u' \left( \omega + \frac{bs^*}{1 - \nu} \left[ \frac{1}{1 - \theta_i} - \nu \right] \right) = \delta_i,
\]

and for \( i = 1, B \), define the repo contract \( f_{i2}(b) \) implicitly as a function of the amount borrowed \( b \):

\[
f_{i2}(b, \nu_2, s) = \begin{cases} 
\frac{\nu_2(s)}{1 - \theta_i} & \text{if } s < s^*(b, \theta_i, \nu_2) \\
\frac{s^*(b, \theta_i, \nu)}{s} + \frac{s^*(b, \theta_i, \nu_2)}{s} & \text{if } s \geq s^*(b, \theta_i, \nu_2)
\end{cases}
\]

As before, \( s^*(b, \theta_i, \nu_2) \) is the threshold in \( s \) above which marginal rates of substitution between
a type \( i \) and 2 agents can be equalized, given, in particular, the amount of asset \( b \) available. Since agent 1 is the marginal holder of the asset into period 3, we know that \( p_2(s) = s/\delta \).

**Proposition.** Let \( f_{1B} \) be the repo contract given by:

\[
f_{1B}(s) = \frac{p_2(s)}{1 - \theta_1} \quad \forall s \in [\underline{s}, \bar{s}]
\]

and \( \hat{b} \) the solution to:

\[
\int_{\underline{s}}^{\bar{s}} \left[ u'(\omega + \frac{\hat{b}}{1 - \nu_2} \left( \frac{1}{1 - \theta_B} - \nu_2 \right)) - \delta_B \right] p_2(s)dF(s) = \frac{(\delta_B - \delta)\theta_1}{(1 - \nu_B)(1 - \theta_1)} \frac{(1 - \theta_B)(1 - \nu_2)}{1 - (1 - \theta_B)\nu_2} E[p_2(s)]
\]

(37)

The equilibrium features intermediation iff \( \hat{b} > 0 \). Three cases are then possible:

1. \( \hat{b} > a \). Agent 1 sells the asset spot to \( B \) who borrows \( b^{B2}_* = a/(1 - \nu_2) \) from agent 2 using repo \( f_{B2}(a, \nu_2) \). [Proposition 8]

2. \( \hat{b} \in [\nu_B a, a] \). Agent 1 uses a combination of a spot and repo sale with \( f_{1B} \) with \( B \). Agent \( B \) borrows \( b^{B2}_* = \hat{b}/(1 - \nu_2) \) from agent 2 using repo \( f_{B2}(b^{B2}_*, \nu_2) \)

3. \( \hat{b} \in [0, \nu_B a] \). Agent 1 borrows from \( B \) using repo \( F_{1B} \) and \( B \) borrows \( b^{B2}_* = \hat{b}/(1 - \nu_2) \) from agent 2 using repo \( f_{B2}(b^{B2}_*, \nu_2) \). [Proposition 9]

In case 2 and 3, the following condition is necessary:

\[
\frac{1}{1 - \theta_B} - \frac{1}{1 - \theta_1} \geq (1 - \nu_B) \frac{1 - \nu_2(1 - \theta_B)}{(1 - \nu_2)(1 - \theta_B)}
\]

In all three cases the amount \( b^{1B}_* \) borrowed by agent 1 from \( B \) is given by:

\[
b^{1B}_* = \frac{a - (1 - \nu_2)b^{B2}_*}{1 - \nu_B}
\]

(38)

**Proof.** Under our conjecture, agents 1 and \( B \) may trade in a repo \( f_{1B} \) and agents \( B \) and 2 can trade in a repo \( f_{L2} \). As usual, agents may also trade in the spot market. We then derive the conditions and characterize the repo contracts \( f_{LH} \) and \( f_{H2} \) for this conjecture to be an equilibrium.

**Step 1: Agents problem and first order conditions**

Observing that agent 1 will be the final holder of the asset, as before, we can write his
optimization problem as follows:

\[
\max_{a_1^B, b_1^B} \omega + p_1 (a - a_1^B) + q_1 b_1^B + E \left[ \delta (\omega + p_2(s)a_1^B - f_1(s)b_1^B) + a.s \right] \\
\text{s.t.} \quad a_1^B \geq b_1^B \quad (\gamma_1^B) \\
\quad b_1^B \geq 0 \quad (\xi_1^B)
\]

While the problem of agent \( H \) is:

\[
\max_{a_1^H, b_1^H} \omega - p_1 a_1^B - q_1 b_1^H + \delta_B \left[ \omega + p_2(s)a_1^H + f_1(s)b_1^H \right] \\
\text{s.t.} \quad a_1^B + \nu_B \ell_1^B \geq 0 \quad (\gamma_1^B) \\
\quad \ell_1^B \geq 0 \quad (\xi_1^B) \\
\quad b_1^H \geq 0 \quad (\xi_1^B)
\]

Recall that \( a_1^B \) is the spot market trade of agent \( B \). The variable \( \ell_1^B \) is the amount agent \( B \) lends to 1. Every unit of loans yields agent \( B \) a fraction \( \nu_B \) of re-usable asset. These units can be re-sold spot, which decreases \( a_1^B \), or re-pledged to agent 2, which increases \( b_2^B \). Finally, agent 2 solves

\[
\max_{a_1^2, \ell_2^B} \omega - p_1 a_1^2 - q_2 \ell_2^B + E \left[ u \left( \omega + sa_1^2 + f_2(s)\ell_2^B \right) \right] \\
\text{s.t.} \quad a_1^2 + \nu_2 \ell_2^B \geq 0 \quad (\gamma_1^2) \\
\quad \ell_2^B \geq 0 \quad (\xi_2^B)
\]

Let us now write down the first order conditions for our 3 agents:

\[
-p_1 + \delta E[p_2(s)] + \gamma_1^1 = 0 \quad (39) \\
q_1 - \delta E[f_1(s)] - \gamma_1^1 + \xi_1^B = 0 \quad (40) \\
-p_1 + \delta_B E[p_2(s)] + \gamma_1^B = 0 \quad (41) \\
-q_1 + \delta_B E[f_1(s)] + \nu_B \gamma_1^B + \xi_1^B = 0 \quad (42) \\
+q_2 - \delta_B E[f_2(s)] - \gamma_1^B = 0 \quad (43) \\
-p_1 + E \left[ f_2(s)u'(c_2^2(s)) \right] + \gamma_1^2 = 0 \quad (44) \\
-q_2 + E \left[ f_2(s)u'(c_2^2(s)) \right] + \nu_2 \gamma_1^2 = 0 \quad (45)
\]

Market clearing implies that \( b_{ij}^s = \ell_{ji}^s \) for each pair of agents \( (i,j) \). Hence, we only use the notation \( b \) in the following. Observe that we introduced the positivity constraint on the amount borrowed by 1 to \( B \) as these agents may not use a repo transaction but a spot trade exclusively.
A quick examination shows that all three collateral constraints bind, that is \( \gamma_1 > 0 \), \( \gamma_B^1 > 0 \) and \( \gamma_1^2 > 0 \). This implies that:

\[
\begin{align*}
\gamma_1^1 &= b^{1B} \\
\gamma_1^B + \nu_B b^{1B} &= b^{B2} \\
\gamma_1^2 + \nu_2 b^{B2} &= 0
\end{align*}
\]

while market clearing for the asset yields:

\[
\gamma_1^1 + \gamma_1^B + \gamma_1^2 = a
\]

Using this last equations together with the collateral constraints above, we obtain equation (38), that is

\[
a = (1 - \nu_B)b^{1B} + (1 - \nu_2)b^{B2}
\]

Quick manipulations of equations (39) to (45) give the following expressions for the Lagrange multipliers associated to the collateral constraints:

\[
\begin{align*}
\gamma_1^2 &= \frac{1}{(1 - \nu_2)} E \left[ (f_{B2}(s) - p_2(s)) \left( u'(c_2^B(s)) - \delta_B \right) \right] \\
\gamma_1^B &= \frac{1}{(1 - \nu_2)} E \left[ (f_{B2}(s) - \nu_2 p_2(s)) \left( u'(c_2^B(s)) - \delta_B \right) \right] \\
\gamma_1^1 &= \gamma_1^B + (\delta_B - \delta) E[p_2(s)]
\end{align*}
\]

Let \( b^* \) be the amount of asset available for the repo between \( 1H \) and \( 2 \) so that \( b^{B2}_* = b^*/(1 - \nu_2) \)

\[
c_2^B(s) = \omega + b^{B2}_*(f_{B2}(s, b^*, \nu_2) - \nu_2 p_2(s))
\]

**Step 2 : Equilibrium Repo contracts**

i) Equilibrium repo contract \( f_{1B} \) between 1 and \( B \)

With usual equilibrium selection argument, agents 1 and \( B \) are not willing to trade contract \( \tilde{f}_{1B} \) if and only if

\[
\delta E[\tilde{f}_{1B}(s)] + \gamma_1^1 \geq \delta_B E[\tilde{f}_{1B}(s)] + \nu_B \gamma_1^B
\]

If \( b_{1B} > 0 \), from equations (40) and (42), we obtain

\[
(\delta_B - \delta) E \left[ f_{1B}(s) - \tilde{f}_{1B}(s) \right] \geq 0
\]
which, if \( \delta_B > \delta_1 \), may only hold if

\[
f_{1B}(s) = \frac{p_2(s)}{1 - \theta_L}, \quad \forall s.
\]

If \( b_{1B} = 0 \) (i.e. agents 1 and B trade only spot) then we must have:

\[
\frac{(\delta_B - \delta_1)\theta_1}{1 - \theta_1} E[p_2(s)] \leq (1 - \nu_B)\gamma_1^B
\]

ii) Equilibrium repo contract \( f_{B2} \) between B and 2.

For a given amount \( b^* \) of asset available, agents B and 2 trade as in the previous section replacing \( a \) by \( b^* \). Using our previous results, the equilibrium repo contract is \( f_{B2}(b^*, \nu_2) \) defined in (35).

Step 3: Determination of \( b^* \)

We now determine the endogenous amount \( b^* \) available for the repo trade between 1H and 2 in order to describe the equilibrium completely. This determines in particular whether agents 1 and B trade spot.

i) \( b_{1B} = 0 \).

Since agent 1 collateral constrain binds, it must be that \( a_1^1 = 0 \) and \( a_{1B}^B = a_1^{1} - \nu_2 \). This implies that \( b^* = a \) and \( b_{B2} = \frac{a}{1 - \nu_2} \). Agent 2 consumption is

\[
c_2^B(s) = \begin{cases} 
\omega + \frac{ap_2(s)}{1 - \nu_2} \left[ \frac{1}{1 - \nu_2} - \nu_2 \right] & \text{if } s < s^*(a, \nu_2, \theta_B) \\
c_2^B, & \text{if } s \geq s^*(a, \nu_2, \theta_B)
\end{cases}
\]

This is an equilibrium if \( \xi_{1B} + \xi_{B1} \geq 0 \). Using equations (40) and (42), the condition writes

\[
\gamma_1^1 \geq \nu_B\gamma_1^B + \frac{\delta_B - \delta}{1 - \theta_1} E[p_2(s)]
\]

which, using our derivations above, can be rewritten:

\[
\frac{1 - (1 - \theta_B)\nu_2}{(1 - \theta_B)(1 - \nu_2)} \int_{s^*(a, \theta_2, \nu_2)}^{s^*(a, \theta_2, \nu_2)} [u' (c_2^B(s)) - \delta_B] p_2(s) dF(s) \geq \frac{(\delta_B - \delta)\theta_1}{(1 - \nu_B)(1 - \theta_1)} E[p_2(s)]
\]

Since the mapping

\[
a \rightarrow \int_{s^*(a, \theta_2, \nu_2)}^{s^*(a, \theta_2, \nu_2)} [u' (c_2^B(a, s)) - \delta_B] p_2(s) dF(s)
\]
is decreasing in its argument, the condition above is equivalent to condition \( \hat{b} > a \) of case 1. 

ii) \( b_1B > 0 \)

In this case, we obtain two expressions for \( \gamma_1^B \) which impose the following equality:

\[
1 - \frac{(1 - \theta_B)\nu_2}{(1 - \theta_B)(1 - \nu_2)} \int_{\bar{s}}^{s^*(a,\theta_B,\nu_2)} \left[ u' \left( c_2^B(s) \right) - \delta_B \right] p_2(s) \, dF(s) = \frac{(\delta_B - \delta)\theta_1}{(1 - \nu_B)(1 - \theta_1)} E[p_2(s)]
\]

and pins down the amount \( b^* \in [0,a] \) available for \( B \) to use in repo \( f_{B2} \). Suppose first that \( b^* \in [\nu_Ba, a] \). Then it must be that agent \( 1H \) buys a fraction of the asset from \( 1L \). If they only trade in a repo, the maximum amount of free collateral available to \( 1H \) is \( \nu_Ba \). Suppose now that \( b^* \in [\nu_Ba, a] \), then using equation (38) and agent 1 collateral constraint, we obtain \( a_1^1 = b^{1B} > a \). Since agent \( 1L \) initially owns \( a_0^1 = a \), it means that he is a net buyer of the asset (through\( B \) re-selling in the repo). Hence, agent 1 and \( B \) only use repo \( f_{1B} \).

**Step 4**: No profitable contract between 1 and 2

We need to check that intermediation is optimal, that is agents 1 and 2 do not want to trade directly a contract \( \tilde{f}_{12} \). This requires

\[
\delta E[\tilde{f}_{L2}(s)] + \gamma_1^1 \geq E[\tilde{f}_{12}(s)u'(c_2^2(s))] + \nu_2 \gamma_1^2
\]

We can rewrite the condition as

\[
(1 - \nu_2)\gamma_1^2 \geq E \left[ \left( \tilde{f}_{12}(s) - p_2(s) \right) \left( u'(c_2^2(s)) - \delta \right) \right]
\]

Using the expression of \( \gamma_1^2 \), we obtain:

\[
E \left[ \left( f_{B2}(s) - p_2(s) \right) \left( u'(c_2^2(s)) - \delta_B \right) \right] \geq E \left[ \left( \tilde{f}_{12}(s) - p_2(s) \right) \left( u'(c_2^2(s)) - \delta \right) \right]
\]

or,

\[
E \left[ \left( f_{B2}(s) - \tilde{f}_{12}(s) \right) \left( u'(c_2^2(s)) - \delta_B \right) \right] \geq \frac{\theta_1(\delta_B - \delta)}{1 - \theta_1} E[p_2(s)]
\]

In the LHS, we use \( \tilde{f}_{12} = p_2(s)/(1 - \theta_1) \) to find the tightest bound:

\[
\left( \frac{1}{1 - \theta_H} - \frac{1}{1 - \theta_L} \right) \int_{\bar{s}}^{s^*(b^*,\theta_B,\nu_2)} p_2(s) \left( u'(c_2^2(s)) - \delta_B \right) \geq \frac{\theta_{L}(\delta_B - \delta_{L})}{1 - \theta_L} E[p_2(s)]
\]

where the last line follows from the expression for \( \gamma_1^B \) derived above. We know from (39) - (42) that the RHS lies below \( (1 - \nu_B)\gamma_1^B \) and that it is equal when \( b_{1B} > 0 \). In this latter case,
we can rewrite the necessary condition above as

\[
\left(1 - \frac{1 - \theta_B}{1 - \theta_1}\right) \frac{1}{1 - \nu_2(1 - \theta_B)} \geq \frac{1 - \nu_B}{1 - \nu_2}
\]

which is sufficient condition (9). The condition will also be necessary whenever \(b^{UP} > 0\). \(\Box\)
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