Dynamic equilibrium with heterogeneous agents and risk constraints

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Abstract

We examine the impact of risk-based portfolio constraints on asset prices in a standard exchange economy model where agents have different risk aversion. Constrained agents scale down their benchmark portfolio and behave locally like power utility investors with risk aversion that depends on current market conditions. We characterize the equilibrium using the consumption share of the constrained agents and provide explicit existence results. The imposition of constraints on active market participants dampens fundamental shocks when they bind, a result that challenges recent studies that suggest that risk management rules serve to amplify aggregate fluctuations. The results also dispute the belief that capital regulations make financial crises larger and more costly, as constraints are more likely to bind in bad times. Constraints may give rise to equilibrium asset pricing bubbles, a result that is associated to the risk aversion distribution across agents and the severity of the constraint.

Keywords: Asset pricing bubbles; Endogenous regimes; General equilibrium; Risk constraints; Stochastic volatility.

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1 Introduction

Despite the widespread presence of (balance sheet) constraints based on risk measures, such as those based on value-at-risk, in the banking and portfolio management industries (see Jorion (1997) and Duffie and Pan (1997)\textsuperscript{1} there is, surprisingly, very few academic studies analyzing their effect in equilibrium settings.

Recent contributions consist of (i) models which point to an amplifying effect on the volatility of stock prices, specially in ‘bad times’, but that usually assume exogenous credit markets or risk neutrality of market participants (see Danielsson, Shin and Zigrand (2009) and references therein), or (ii) models that are based on (asymptotic) approximations of equilibrium quantities, an approach that is not suitable to derive existence results (see Leippold, Trojani and Vanini (2006)).

In this paper we want to understand how the presence of risk constraints impact equilibrium quantities by modeling a full fledged yet tractable general equilibrium model. Results will help in interpreting empirical regularities in the time series of stock returns and volatility and in understanding the effectiveness of regulatory standards.

It is well known that taking into account portfolio constraints in a pure exchange equilibrium setting is a challenging task. Most contributions, such as Detemple and Murthy (1997), Basak and Cuoco (1998) and Basak and Croitoru (2000), obtain existence results in models where all agents have logarithm utility.

We depart from the logarithm paradigm because it is well known that in such a setting, stock prices and return volatilities do not change with the imposition of constraints. To keep matters simple, we assume that there are only two classes of agents: unconstrained agents with power utility who are free to choose the composition of their portfolio, and agents who have logarithmic utility and are subject to constraints.

We propose a parsimonious way to specify the constraints such that they induce portfolios that represent general risk constraints. The constraint forces the agent to scale down his benchmark unconstrained portfolio, and this reduction in risk taking induces a position that is ‘locally’ akin to the policy of a power utility agent whose risk aversion depends on current market conditions.

Methodologically, we determine the equilibrium by identifying a suitable consumption sharing rule which, due to the nature of the constraint, will follow an autonomous process whose coefficients can be determined in closed form, in contrast to economies with heterogeneous agents and frictions such as borrowing or proportional constraints (see Chabakauri (2009) and Gårleman and Pedersen (2009)).

Utilizing this insight, we are able to fully analyze the properties of the equilibrium and provide explicit existence results. In particular, the interest rate and the market price of risk are determined in closed form as functions of the consumption sharing rule only, and thus, solving the model

\textsuperscript{1}We note that their wide presence in the financial world is mainly due to agency problems, default risk or budgeting practices that are encouraged by regulators through capital requirement rules.

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amounts to compute a single linear ordinary differential equation which describes the price dividend ratio, avoiding the need for approximate solutions whose accuracy may be hard to evaluate.

We illustrate the results by studying, in a framework with one risky asset, two different constraint sets which proxy for different evaluation and updating frequencies in risk limits and exposures. In the first example, the constraint imposes a lower bound on the portfolio’s mean-variance ratio, that is, it sets a specific risk/return tradeoff which will induce a constant reduction from the mean variance efficient portfolio. In the second example, the constraint imposes an upper bound on the portfolio’s volatility. It is a pure risk constraint (equivalent to the limited expected loss (LEL) family of constraints), which will induce a time varying reduction from the benchmark unconstrained portfolio that resembles the policy induced by a relative VaR constraint.

Our main results are summarized as follows.

First, we show that risk constraints can increase the effect of negative shocks on return premia and that negative shocks to fundamentals make risk constraints more likely to bind, lowering the interest rate and raising the market price of risk. The effects in the interest rate and the market price of risk depend on the tightness of the constraint and are more pronounced in bad times, in line with the empirical literature (see Ferson and Harvey (1991)).

Second, when the constraint is imposed on the less risk averse agent, who holds a levered position in the stock, the volatility of the stock price decreases because constraints narrow the ‘effective’ risk aversion distribution across agents when they bind, thus restraining an efficient risk sharing whose dynamic evolution is partly responsible for the volatility of the stock price. This insight is in contrast to recent studies (see Danielsson, Shin and Zigrand 2009 and references therein) that suggest that risk management rules used by active market participants, who are presumably more risk tolerant, serve to amplify aggregate fluctuations. Since constraints bind in bad times, it challenges the belief that current capital regulations (such as those based on Basel II accords) make systemic financial crises larger and more costly.

Third, when the constraint is imposed on the more risk averse agent, who holds a reduced position in the stock, the volatility of the stock price may increase depending on the constraint type. Under a mean-variance constraint, the ‘effective’ risk aversion distribution across agents is widened and fundamental shocks are amplified. The result mirrors Bhamra and Uppal (2009), who show that in an economy with no constraints and heterogeneous agents, the volatility is increasing in the dispersion of risk aversion. Under a volatility constraint, fundamental shocks may be amplified but not in all states where the constraint binds. In both cases, the increase in volatility of the stock price is coupled with an increase in the volatility of the interest rate. Quantities in the economy with mean-variance constraints are consistent with empirical findings (see Mele (2007) and references therein), as lower price dividend ratios and higher stock return volatilities happen in bad times (the so-called leverage effect).

Fourth, we complement the results of Hugonnier (2009), who shows that portfolio constraints
can generate equilibrium pricing bubbles even if the economy includes unconstrained arbitrageurs. In particular, we show how the emergence of bubbles in equilibrium hinges on the risk aversion distribution across agents and the severity of the constraint, two dimensions that determine the cost of the constraint. We remark that, even though an equilibrium with a price system that contains bubbles may seem a contradiction in itself, because of the violation of the law of one price, bubbles are not incompatible with the existence of an equilibrium because unconstrained agents cannot exploit the arbitrage opportunities due to standard wealth admissibility constraints (see Loewenstein and Willard (2000 a,b)).

We also extend our baseline model to models where beliefs across agents may be heterogeneous and there are multiple securities. When agents hold heterogeneous beliefs, the model shares some resemblance with Gallmeyer and Hollifield (2008), who examine the effect of short sale constraints on asset returns. In contrast to our portfolio constraint set, which allows for a complete characterization of the regions in which the constraint is active using exogenous quantities, Gallmeyer and Hollifield (2008) need to restrict the sign of the stock price volatility process in order solve the model via Monte Carlo simulations.

Contrary to models with general position constraints, it is relatively straightforward to introduce multiple risky assets. Individual stock prices satisfy linear partial differential equations with known coefficients. There is one caveat. When the equilibrium contains bubbles, a pricing kernel fails to exists, and thus, there are infinitely many ways to represent individual stock prices.

Literature review. Our work is related to various strands of literature. We briefly review the most relevant.

The partial equilibrium implications of risk constraints in dynamic settings have been studied in Basak and Shapiro (2001), Cuoco, He, and Isaenko (2007), and more recently, Leippold, Trojani and Vanini (2006) and Pirvu and Zitković (2008), among others. Basak and Shapiro (2001) argue that constrained investors who face VaR limits are induced to take on a larger risk exposure and losses in states which are more costly. A drawback of their model is given by the fact that the portfolio’s VaR is never reevaluated after the initial date, as Cuoco, He, and Isaenko (2007) remark, noting that if the trader satisfies the specified risk limit at all times, rather than only at the initial date, no unappealing incentives arise, and the policy generated is a dynamically consistent risk reduction process that scales down the unconstrained benchmark and that behaves qualitatively similar to the one studied in this paper. Leippold, Trojani and Vanini (2006) use perturbation methods to study the policy of a power utility agent under VaR constraints and stochastic market coefficients, and find that, in contrast to Cuoco, He, and Isaenko (2007) constraints may increase the risk exposure in high-volatility states due to the anticipatory effect that arises from the hedging demand term. They also approach the general equilibrium problem using an asymptotic approximation in the neighborhood of logarithm utility, a method proposed in Kogan and Uppal (2001), and find that
the presence of constraints may generate a lower interest rate and a higher market price of risk.

Pirvu and Zitkovic (2008) apply a solution technique for the investor’s problem similar to ours and find that for agents with logarithm preferences facing a class of risk constraints that includes VaR, tail-value-at-risk, and limited expected loss, the optimal policy consists of the mean variance efficient portfolio scaled down by a risk reduction process that depends on current market conditions, as in the optimal portfolio policies in this paper.

This paper is related to models of exchange economies with heterogeneous agents, where the source of the heterogeneity is the difference in risk aversion across agents. In frictionless economies, Wang (1996) examines the term structure of interest rates and risk premium in an exchange economy similar to our unconstrained benchmark, restricting one agent to have logarithm preferences and the other to square-root power utility. Longstaff and Wang (2008) are able to find closed forms for all equilibrium quantities, extending Wang (1996), and focus on how variations in the size of the credit market relate to variations in expected stock returns.

Within the category of equilibrium models with portfolio constraints, we highlight the models of Basak and Cuoco (1998), Garleanu and Pedersen (2009) and Chabakauri (2009).

We solve the general case of the restricted stock market participation model in Basak and Cuoco (1998) and find two results of interest: (i) if stockholders have lower risk aversion than non-stockholders, as the empirical literature seem to imply, the calibration of equilibrium quantities using the (non stationary) consumption share of the non-stockholders may be very transitory as non-stockholders are expected to disappear fast, and (ii) the model fails to produce significant volatility in excess of the volatility of dividends.

Garleanu and Pedersen (2009) incorporate margin constraints in an exchange economy with heterogeneous agents to show how the presence of margins may lead to deviations of the law of one price, a feature that also arises in our model and that we associate to the cost of the constraint and standard admissibility constraints on wealth. They exemplify their model using an economy with one risky asset which requires the solution of a single nonlinear ordinary differential equation for the price dividend ratio. The region in which the constraint is active depends on endogenous functions, and is determined jointly with the stock price, hence, existence results are not readily available. In contrast, our model requires the solution of a linear ordinary differential equation for the price dividend ratio, and the regions with binding constraints depend on exogenous quantities.

Chabakauri (2009) studies a dynamic exchange economy where investors share identical CRRA preferences (and thus, under no frictions, a no trade equilibrium obtains with constant equilibrium quantities which facilitates the comparison across constrained and unconstrained economies) where the constrained agent faces a borrowing constraint that is always binding. In contrast to our model, all equilibrium quantities are computed numerically, and consequently, explicit existence results are not available. The equilibrium when agents have risk aversion lower than unity may generate higher stock return volatilities and procyclical price dividend ratios, a result that resemble our model when
the unconstrained agent is the less risk averse agent.

There are models that explicitly deal with risk constraints which may fall within the so-called behavioral strand of the finance literature, as they study their implications in environments that allow for noise traders to have a significant impact on market prices and credit markets are not endogenized. Rytchkov (2008) considers a model with a constrained risk averse agent and noise traders. The enforcement of the constraint is given by an exogenous process, a feature that can make the market substantially less liquid and more volatile. Danielsson, Shin and Zigrand (2009) features a risk neutral agent that is subject to a volatility constraint which is always binding, and noise traders. Constraints increase the volatility of stock prices and generate a countercyclical Sharpe ratio, due to an amplification of shocks generated by the constrained trader’s demand response to price shocks, which shifts according to changes in his ‘risk appetite’, a behavior that will also arise in our model incarnated in the risk reduction process that scales down the unconstrained benchmark portfolio. Contrary to these papers, we emphasize that the presence of the constraint dampens rather than amplifies fundamental shocks in an environment where the constrained agent is a net borrower. If the presence of the constraint increases the volatility of returns, it will also increase the volatility of the interest rate.

The paper is also related to papers that deal with balance sheet constraints in the macroeconomic literature, such as Aiyagari and Gertler (1999), who show that binding margin requirements might induce stock prices to overreact to shocks, and general equilibrium models with incomplete markets in discrete time, such as Dumas and Lyasoff (2008), who solve for equilibrium in various incomplete market settings using binomial trees.

Finally, we note that the pricing kernel in the economy has a functional form that resembles the pricing kernel in an economy with a representative agent who features the external habit preferences in Campbell and Cochrane (1999). Indeed, the equilibrium quantities suggest that a (near) stationary consumption share may play the role of the surplus consumption ratio process, which is the key driver in that literature. In this line of research, Găreleanu and Panageas (2008), who study an overlapping generation model, and Guvenen (2009), who studies an augmented restricted stock market participation model, have also suggested that their models have a reduced form which is similar to habit formation models.

The remainder of the paper is structured as follows. Section 2 presents the assumptions about the economy, the traded assets and the agents. In Section 3 we solve the investors’ problem and characterize the equilibrium. Section 4 investigates the equilibrium implications using the two proposed examples. Section 5 extends the model to economies with heterogeneous beliefs and multiple risky assets. Section 6 concludes. Proofs and technical results are gathered in the Appendix.
2 The economy

2.1 Information structure

We consider a continuous time economy with infinite time horizon. The uncertainty in the economy is represented by a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) supporting a standard Brownian motion denoted by \(B\). The filtration \(\mathbb{F} = (\mathcal{F}_t)\) is the augmentation of the filtration generated by the Brownian motion. We let \(\mathcal{F} = \bigcup_{t \geq 0} \mathcal{F}_t\) determine the true state of nature.

2.2 Securities

There is a single consumption good which serves as the numéraire. The financial market consists of two assets: a locally riskless bond in zero net supply and one risky asset\(^2\) in positive supply of one unit.

The price of the riskless asset evolves according to

\[
S^0_t = 1 + \int_0^t S^0_s r_s ds,
\]

for some instantaneous interest rate process \(r \in \mathbb{R}\) which is to be determined in equilibrium. The risky asset is a claim to a strictly positive dividend process of the form

\[
\delta_t = \delta_0 + \int_0^t \delta_s (\mu_s ds + \sigma_s dB_s),
\]

for some exogenously given constants \((\mu_\delta, \sigma_\delta) \in \mathbb{R} \times \mathbb{R}_+\). The ex-dividend price process of the risky asset is denoted by \(S\) and evolves according to

\[
S_t = S_0 + \int_0^t S_s (\mu_s ds + \sigma_s dB_s) - \int_0^t \delta_s ds, \tag{1}
\]

for some initial value \(S_0 \in \mathbb{R}_+\) and some drift and volatility processes \((\mu, \sigma) \in \mathbb{R} \times \mathbb{R}\) which are to be determined in equilibrium. The process

\[
\theta_t = \sigma_t^{-1} (\mu_t - r_t), \tag{2}
\]

corresponds to the market price of risk associated with the source of risk in the model.

2.3 Agents

The economy is populated by two price-taking agents indexed by \(k \in \{1, 2\}\) with homogeneous beliefs about the state of the economy. Agent \(k\) maximizes his expected utility over strictly positive

\(^2\)We extend the model to multiple risky assets in Section 5.
consumption plans,

\[ U_k(c) = E \left[ \int_0^\infty e^{-\rho t} u_k(c_{kt}) \, dt \right], \tag{3} \]

where \( \rho \) is the rate of subjective time preference and preferences are given by \( u_1(c) = \frac{c^{1-\gamma} - 1}{1-\gamma} \) and \( u_2(c) = \log c \). We assume that agent 2 represents a financial institution/money manager who faces portfolio constraints which limits the amount of risk which can be held while trading. The assumption of logarithm preferences is necessary to obtain a simple characterization of optimality under portfolio constraints\(^5\) and may be natural choice to describe the objective function of a money manager because an agent with logarithm preferences maximizes the growth rate of a fund (see Hakansson (1970) and Karatzas and Shreve (1998), p. 150.).

In order to ensure that investors’ expected utilities are uniformly bounded given the dividend process, we impose a growth condition given by

\[ \rho > \max \left( 0, (1 - \gamma)(\mu_\delta - \frac{1}{2}\gamma\sigma^2_\delta) \right). \]

Agent 2 is initially endowed with \( \beta \in \mathbb{R} \) units of the riskless asset and \( \alpha \leq 1 \) units of the risky asset,

\[ w_2 = \beta + \alpha S_0, \tag{4} \]

and consequently, agent 1’s initial endowment is given by \( w_1 = S_0 - w_2 \). Initial short positions in the riskless asset are allowed as long as initial wealth \( w_k, k \in \{1, 2\} \), is strictly positive when computed at equilibrium prices.

An admissible trading strategy is a process \((\phi^0, \phi) \in \mathbb{R}^2\) satisfying for all \( T \in [0, \infty) \),

\[ \int_0^T (\phi_t \sigma_t)^2 \, dt + \int_0^T |\phi^0_t r_t + \phi_t \mu_t| \, dt < \infty, \]

and

\[ W_t = \phi^0_t + \phi_t \geq 0, \]

where \( W_t \) denotes the agent’s wealth, the process \( \phi^0 \) represents the amount invested in the riskless asset and the process \( \phi \) represents the amount invested in the risky asset. The requirement of nonnegative wealth is standard in the literature and rules out the possibility of doubling strategies\(^4\).

A consumption plan \( c \) is said to be feasible if there exists an admissible trading strategy \((\phi^0, \phi)\)

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\(^4\)See Dybvig and Huang (1988) and Loewenstein and Willard (2000a).
such that the associated wealth process satisfies the dynamic budget constraint

\[ dW_t = (\phi_0^t r_t + \phi_t \mu_t - c_t) \, dt + \phi_t \sigma_t dB_t \]
\[ = r_t W_t dt + W_t \pi_t \sigma_t (dB_t + \theta_t dt) - c_t dt \]

where the process \( \pi = \phi/W \) represents the proportion of wealth invested in the risky asset for ease of notation. Feasible plans for agent 2 have the additional requirement that the trading strategy must belong to a portfolio constraint set, which is described in the next subsection.

We complete this part by introducing the concept of equilibrium in the economy.

**Definition 1.** An equilibrium is a price system \((S^0, S)\) and a set of consumption plans and trading strategies \(\{c_k, (\phi^0_k, \phi_k)\}\) such that: (i) The consumption plan \(c_k\) maximizes the agent’s utility in (3) and is financed by the trading strategy \((\phi^0_k, \phi_k)\) subject to admissibility and portfolio constraints and (ii) the securities and goods markets clear at all times,

\[ \phi^0_1 + \phi^0_2 = 0, \quad \phi_1 + \phi_2 = S, \quad c_1 + c_2 = \delta. \]

### 2.4 The constraint set

We model constraints such that they generate portfolio policies similar to those induced by constraints based on commonly used risk measures (e.g. VaR). While constraints are not endogenized in the model, we note that their wide use in the financial world is due to agency problems, default risk, the need to allocate scarce capital and budgeting practices that are encouraged by regulators through capital requirement rules.

The constraint set is a deterministic convex function of two statistics, the portfolio’s net return, \(\pi \sigma_t \theta_t\), and the portfolio’s volatility, \(\pi \sigma_t\), given by

\[ C = \{ \pi \in \mathbb{R} : a_1 \pi \sigma_t \theta_t + a_2 (\pi \sigma_t)^2 \leq a_3, \quad \forall t \in [0, \infty) \}, \quad (5) \]

where \((a_2, a_3)\) are nonnegative constants.

We use two examples in this paper which represent, parsimoniously, more complex forms. The parameters \((a_1, a_2, a_3)\) are specified to serve as proxy for structural restrictions which induce optimal policies that differ on the way market conditions enter on the risk taking ability of the agent.

The first example consists of a lower bound on the portfolio’s mean-variance ratio specified by

\[ (a_1, a_2, a_3) = (-1, L_1, 0), \quad C = \{ \pi \in \mathbb{R} : -\pi \sigma_t \theta_t + L_1 (\pi \sigma_t)^2 \leq 0 \}, \]

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5The set in \(\mathbb{E}\) resembles the general form of risk constraints in Pirvu and Zitković (2008, see definition 2.9.), who use a deterministic convex function of the portfolio’s return and the portfolio’s volatility to represent constraints on VaR, tail-value-at-risk (TVaR), and limited expected loss (LEL), among others.
with $L_1 > 0$. This constraint imposes a specific risk/return tradeoff, which, as seen in the next section, will induce a constant reduction from the mean variance efficient portfolio.

On the other hand, the second example considers a pure risk constraint which consists of an upper bound on volatility, specified by

$$(a_1, a_2, a_3) = (0, 1, L_2^2), \quad \mathcal{C} = \{ \pi \in \mathbb{R} : (\pi \sigma_t)^2 \leq L_2^2 \},$$

with $L_2 > 0$. This constraint will induce a state dependent reduction from the mean variance efficient portfolio. Volatility constraints are equivalent to the family of limited expected loss constraints (LEL) (see Pirvu and Zitković (2008), equation 2.20), and are also studied in Gârleanu and Pedersen (2007) and Danielsson, Shin and Zigrand (2009) as VaR constraints and in Rytchkov (2008), as approximations to dynamic margin constraints.

3 Constructing an equilibrium

In this section we gather some results about individual optimality and show that an equilibrium can be easily determined by building a suitable consumption sharing rule. We also determine conditions under which the equilibrium stock price includes a bubble component.

3.1 Individual optimality

Agent 1 faces a complete market and thus his state price density is uniquely given by

$$\xi_{1t} = e^{-\int_0^t (r_s + \frac{1}{2} \theta_s^2) ds - \int_0^t \theta_s dB_s},$$

where the market price of risk $\theta$ is the process defined in (2). It is well known that the solution of the unconstrained agent’s problem is given by

$$c_{1t} = (e^{\rho t y_1 \xi_{1t}})^{-\frac{1}{\gamma}}, \quad (7)$$

$$\pi_{1t} = \sigma_t^{-1} \left[ \theta_t + h_{1t} (\xi_{1t} W_{1t})^{-1} \right], \quad (8)$$

where the process

$$W_{1t} = \frac{1}{\xi_{1t}} E \left[ \int_t^\infty \xi_{1s} c_{1s} ds \left| \mathcal{F}_t \right. \right] = \frac{1}{\xi_{1t}} \left[ H_{1t} - \int_0^t \xi_{1s} c_{1s} ds \right], \quad (9)$$

represents the agent’s wealth along the optimal path, $h_1$ is the integrand in the stochastic integral representation of the martingale $H_1$ and the strictly positive constant $y_1$ is chosen in such a way that $W_{10} = w_1$.

On the other hand, since agent 2 faces portfolio constraints, the process $\xi_1$ no longer identifies
the unique arbitrage free state price density. However, due to the fact that the agent has logarithm preferences, his optimal plans have a very familiar structure.\footnote{See the proof of Proposition 1 in the Appendix for details.}

The consumption policy is given by

\[ c_{2t} = \rho W_{2t}, \tag{10} \]

and is financed by a portfolio that solves the mean-variance program given by

\[ \sup_{\pi \in \mathcal{C}} \left\{ \pi \sigma_t \theta_t - \frac{1}{2} (\sigma_t \pi)^2 \right\}. \tag{11} \]

In Proposition 1 we provide closed form solutions of this program for the mean-variance and volatility constraints. In general, the optimal policy is given by \( \pi_{2t} = \kappa_t \pi_{t}^{mv} \) where \( \pi_{t}^{mv} \) corresponds to the policy chosen by an unconstrained agent, i.e., the mean variance efficient portfolio, defined by \( \sigma_t \pi_{t}^{mv} = \theta_t \), and the process \( \kappa_t \in [0,1] \) is the reduction in risk taking induced by the constraint.

**Proposition 1.** The optimal portfolio under the mean-variance constraint is given by a constant fraction of the mean variance efficient portfolio,

\[ \pi_{2t} = \kappa \pi_{t}^{mv}, \quad \kappa = \frac{1}{1 + (L_1 - 1)^+}, \tag{12} \]

whereas the optimal portfolio policy under a volatility constraint features a time varying risk reduction,

\[ \pi_{2t} = \kappa_t \pi_{t}^{mv}, \quad \kappa_t = \frac{1}{1 + (|\theta_t|/L_2 - 1)^+}. \tag{13} \]

Interestingly, the optimal policy can be interpreted as the portfolio of a power utility agent whose (higher) relative risk aversion might depend on the current market coefficients and has no hedging demand.

The mean-variance constraint is active when \( L_1 \) is higher than one (if \( L_1 < 1 \), the constraint is slack when the agent chooses the mean-variance efficient portfolio). This is a very stylized example which we use to represent a ‘sluggish’ agent that keeps a fixed risk reduction policy regardless of market conditions.

On the other hand, the constraint on volatility induces a time varying risk reduction policy that binds when the (absolute value of the) market price of risk is high, a feature that is present in the policy generated by a constraint on (relative) VaR, as we see next.

The VaR of a position \( \pi \) (for a fix horizon \( \tau \)) is the positive part of the upper \( \alpha \)--percentile of the projected relative wealth loss distribution, \( \text{VaR}^\alpha(\pi) = \zeta_\alpha^+ \), where \( \zeta_\alpha : \mathbb{P} \left[ \frac{W_{t+\tau} - W_{t}}{W_{t}} \geq \zeta_\alpha \right| \mathcal{F}_t \right] = \alpha \), and the portfolio constraint imposes an upper bound, \( \text{VaR}^\alpha(\pi) \leq a_v \), where \( a_v \in (0,1) \) is an exogenous risk limit.
As shown in Cuoco, He and Isaenko (2007), when the risk position is reevaluated dynamically and the projected relative wealth loss distribution is lognormal, the risk reduction process induced by the constraint is given by
\[
\kappa_t = \min \left( 1, \frac{\theta_t r + N^{-1}(\alpha) + \sqrt{(\theta_t r + N^{-1}(\alpha))^2 + \phi(r_t)}}{|\theta_t r|} \right),
\]
where \( \phi(r) = 2(r \tau - \log(1 - a_v)) \) and \( N^{-1} \) denotes the inverse function of the standard Normal cumulative distribution function.

Note that the VaR constraint, for commonly used parameter values, also binds when the market price of risk is high, however, it induces a risk reduction process that depends on the market price of risk and the interest rate in a nonlinear fashion, a feature that might generate discontinuities in equilibrium quantities.

The fact that the risk reduction process in (13) depends only on the (absolute value of the) market price of risk is the key to a tractable characterization of equilibrium, as we will see next.

### 3.2 Determination of equilibrium

Because one of the agents faces portfolio constraints, the usual construction of a representative agent as a linear combination of the individual utility functions with constant weights is impossible. Despite this, an equilibrium can be easily determined by building a suitable consumption sharing rule,

\[ s_{2t} = c_{2t}/\delta_t, \]

from the optimality conditions of both agents. The consumption share process follows an Itô process defined by

\[ ds_{2t} = s_{2t} \mu_{s2}(\cdot) dt + s_{2t} \sigma_{s2}(\cdot) dB_t, \]

where the coefficients \((\mu_{s2}, \sigma_{s2})\) are determined jointly with the interest rate and the market price of risk.

We briefly discuss the steps which lead to the equilibrium. The unconstrained agent’s state price density is obtained from the first order condition of the unconstrained agent in equation (7) as

\[ \xi_1(t, s_{2t}, \delta_t) = e^{-\rho t} y_1^{-1}(1 - s_{2t})^{-\gamma} \delta_t^{-\gamma}, \]

Note that Pirvu and Zitković (2008) show that the portfolio induced by tail-value-at-risk, limited expected loss and other risk constraints under time varying market coefficients, is given by \( \pi_t = \kappa_t \pi_t^{mv} \), where \( \kappa \) depends on the market through \((W_t, r_t, \theta_t)\) and lives in the interval \([0, 1]\).
and thus, an application of Ito’s lemma to this function identifies the market price of risk and
the interest rate as functions of the drift and diffusion terms in the consumption share and the
dividend dynamics. We then use the optimal consumption policy of the constrained agent in (10)
and the definition of the consumption share to obtain \( s_{2t} = \rho W_{2t}/\delta_t \), whose dynamics provides two
additional equations.

Since the risk reduction process \( \kappa \) depends on \( \theta \) or is a constant, the market price of risk is
obtained from a nonlinear equation given by

\[
\theta_t = \gamma \sigma_\delta - \gamma \frac{s_{2t}}{1 - s_{2t}} (\kappa(\theta_t) \theta_t - \sigma_\delta).
\]  

(14)

The solution in (14), \( \theta(s_2) \), is then used to compute the interest rate and the pair \( (\mu_{s_2}, \sigma_{s_2}) \). All
quantities are thus available in closed form as functions of the consumption share rule only.

Proposition 2 formalizes the above discussion and provides expressions for the interest rate, the
market price of risk and the coefficients in the consumption share dynamics.

**Proposition 2.** In equilibrium, the state price density is given by

\[
\xi_1(t, s_{2t}, \delta_t) = e^{-\rho t} \vartheta_1^{-1} (1 - s_{2t})^{-\gamma} \delta_t^{-\gamma},
\]  

(15)

and the consumption plans of the two agents are determined by

\[
c_{1t} = (1 - s_{2t}) \delta_t, \quad c_{2t} = s_{2t} \delta_t.
\]  

(16)

The market price of risk and the interest rate are given by

\[
\theta(s_{2t}) = \frac{1}{1 - (1 - \kappa(s_{2t})) R(s_{2t}) s_{2t} R(s_{2t}) \sigma_\delta},
\]  

(17)

\[
\tau(s_{2t}) = \rho + \mu \delta R(s_{2t}) + (P(s_{2t}) - R(s_{2t})) s_{2t} \Phi(s_{2t}) \theta(s_{2t}) + \frac{P(s_{2t}) R(s_{2t})}{2} \left[ (s_{2t} \Phi(s_{2t}))^2 - \sigma_\delta^2 \right],
\]  

(18)

where the functions \( \Phi(\cdot) \), \( R(\cdot) \) and \( P(\cdot) \) are defined by

\[
\Phi(s_{2t}) = - (1 - \kappa(s_{2t})) \theta(s_{2t}),
\]  

(19)

\[
R(s_{2t}) = \frac{\gamma}{1 + (\gamma - 1) s_{2t}},
\]  

(20)

\[
P(s_{2t}) = R(s_{2t})^2 \left[ 2 s_{2t} + \frac{1 + \gamma}{\gamma^2} (1 - s_{2t}) \right].
\]  

(21)

The consumption share of the constrained agent obeys

\[
ds_{2t} = s_{2t} \mu_{s_2} (s_{2t}) dt + s_{2t} \sigma_{s_2} (s_{2t}) dB_t,
\]  

(22)
where

\[ \mu_{s_2}(s_{2t}) = f(s_{2t}) - \rho + \sigma_{s_2}(s_{2t})(\theta(s_{2t}) - \sigma_{\delta}), \tag{23} \]

\[ f(s_{2t}) = r(s_{2t}) + \sigma_{\delta}\theta(s_{2t}) - \mu_{\delta}, \tag{24} \]

\[ \sigma_{s_2}(s_{2t}) = \kappa(s_{2t})\theta(s_{2t}) - \sigma_{\delta}, \tag{25} \]

and its starting point, \(s_{20} \in (0, 1)\) is the solution to the equation

\[ (1 - \alpha)\rho^{-1}\delta_0 s_{20} = \beta + \alpha E\left[ \int_0^\infty \xi_1(t, s_{2t}, \delta_t)(1 - s_{2t})\delta_t dt \right]. \tag{26} \]

The fact that equilibrium quantities depend only on the consumption share has two important consequences. First, the consumption share is an autonomous process, a fact that will allow us to present explicit existence results. Second, as we will see later in the section, it facilitates the computation of the stock price, as the price dividend ratio can be obtained from a single linear ordinary differential equation.

This result is in contrast to equilibrium models with position constraints, such as Garleanu and Pedersen (2009)\(^\text{10}\), where the diffusion term \(\sigma_{s_2}()\) depends explicitly on the stock price volatility. This implies that the consumption share process and the stock price form a forward-backward system of equations which must be solved for simultaneously in order to characterize every equilibrium quantity.

**Existence of equilibrium.** The equilibrium satisfies Definition 1 when two conditions are met: (i) there is \(s_{20} \in (0, 1)\) which solves the equation (26), and (ii) the consumption share process \(s_{2t}\) never reaches either zero or one for \(t \in [0, \infty)\).

Condition (i) implicitly restricts the size of the initial position, \((\alpha, \beta)\), in such a way that the initial endowments of the agents are strictly positive. For instance, in the restricted participation model of Basak and Cuoco (1998)\(^\text{11}\) the existence of equilibrium requires \(\alpha = 0\), which in turn implies a restriction on the size of the initial debt position, \(\beta < \rho^{-1}\delta_0\), to ensure that the stockholder is not so deeply in debt at the initial time that he can never pay back from the dividend supply, i.e., \(w_1 > 0\). Furthermore, the equilibrium will be uniquely determined if equation (26) has only one positive root in \((0, 1)\).

Condition (ii) indicates that boundaries cannot be reached when the process starts from \(s_{20} \in (0, 1)\), otherwise, the consumption policies would not be optimal and equilibrium would fail to exist. As an example, take the case in which the consumption share of agent 2 reaches 0 with positive probability, the utility of agent 2 would be minus infinity, which implies that the conjectured policy

\(^{10}\)See also Chabakauri (2009) and Dumas and Lyasoff (2008) for examples in continuous and discrete time, respectively.

\(^{11}\)The restricted market participation model of Basak and Cuoco (1998) is a limiting case in the set in \(\{a_1, a_2, a_3\} = (0, 1, 0)\).
would never constitute an equilibrium. We note that this condition can be verified independently of the stock price due to fact that the consumption share is an autonomous process.

**Stock price and bubbles.** Since financial markets clear in equilibrium, the value of the stock price is determined by the sum of the individual wealth processes in (9) and (10),

\[
S_t = W_{1t} + W_{2t} = E \left[ \int_t^\infty \frac{\xi_1 s}{\xi_{1t}} \delta_s ds \middle| \mathcal{F}_t \right] + \rho^{-1} s_{2t} \delta_t - E \left[ \int_t^\infty \frac{\xi_1 s}{\xi_{1t}} s_{2t} \delta_s ds \middle| \mathcal{F}_t \right].
\]

On the other hand, the state price density of the unconstrained agent, \( \xi_1 \), is the unique nonnegative process such that the deflated stock price is a nonnegative local martingale and because of this, also a supermartingale (see Karatzas and Shreve (1988), p. 36), i.e.,

\[
\xi_{1t} S_t + \int^t_0 \xi_{1s} \delta_s ds \geq E \left[ \int^\infty_0 \xi_{1s} \delta_s ds \middle| \mathcal{F}_t \right].
\]

This result implies that the stock price must be at least as large as the expected value of its future dividends. If this inequality is strict, we can think of the stock price as being composed of two parts: a fundamental value given by

\[
E \left[ \int_t^\infty \frac{\xi_1 s}{\xi_{1t}} \delta_s ds \middle| \mathcal{F}_t \right],
\]

and a bubble component given by

\[
b_t = S_t - E \left[ \int_t^\infty \frac{\xi_1 s}{\xi_{1t}} \delta_s ds \middle| \mathcal{F}_t \right] = \rho^{-1} s_{2t} \delta_t - E \left[ \int_t^\infty \frac{\xi_1 s}{\xi_{1t}} s_{2t} \delta_s ds \middle| \mathcal{F}_t \right].
\]

which represents the difference between the asset’s price and the present value of its dividends. This is consistent with the traditional definition of bubbles used by Santos and Woodford (1997) and Loewenstein and Willard (2000 a,b) among others. We remark that bubbles are not incompatible with the existence of an equilibrium because, as noted by Loewenstein and Willard (2000a) and Jarrow, Protter and Shimbo (2007), unconstrained agents may not be able to fully exploit the arbitrage opportunities due to standard wealth admissibility constraints (see also Hugonnier (2009) for explicit examples).

The expression in (28) says that a bubble component arises when the cost of the optimal consumption plan to the constrained agent is larger than the cost of the same plan for the unconstrained agent. The latter corresponds to the plan’s lowest cost replicating strategy.

The notion that the emergence of bubbles is associated to how costly the constraint is for agent 2 is reinforced using the following representation:
Proposition 3. The bubble term in (28) is given by

\[ b_t = \delta_t^\gamma (1 - s_{2t})^\gamma \int_t^\infty e^{-\rho(s-t)} \left[ \lambda(s_{2t}, \delta_t) - E[\lambda(s_{2s}, \delta_s) | \mathcal{F}_t] \right] ds, \]  

(29)

where

\[ \lambda(s_{2t}, \delta_t) = s_{2t}(1 - s_{2t})^{-\gamma} \delta_t^{1-\gamma} \]  

(30)
is a nonnegative local martingale whose dynamics obey

\[ \lambda_t = \lambda_0 + \int_0^t \lambda_s \Phi(s_{2s}) dB_s. \]  

(31)

The function in (30) corresponds to the ratio of the agents’ marginal utilities, a variable that identifies the stochastic weight of the constrained agent in the representative agent construction of Cuoco and He (1994). To facilitate the mapping with our equilibrium, we note that the functions \( R(\cdot) \) and \( P(\cdot) \) in (20) and (21) correspond to the relative risk aversion and relative prudence of the representative agent with stochastic weights.

Equation (29) says that \( b_t \) is necessarily a nonnegative process, as \( \lambda_t \) is a nonnegative local martingale and hence a supermartingale. Furthermore, it shows that the equilibrium will be free of bubbles when \( \lambda_t \) is a true martingale, a condition that is necessary and sufficient (see Hugonnier (2009), Theorem 1).

The intuition of the result is clear and will be confirmed when we examine the equilibrium under volatility constraints: the stock price contains a bubble if and only if the weight of the constrained agent is strictly decreasing in expectation, i.e., bubbles arise in equilibrium if the portfolio constraint is so costly to the constrained agent that it strictly benefits the unconstrained agent.

Price dividend ratio and volatility of returns. The price dividend ratio is given by a function of the consumption share, \( p : (0, 1) \rightarrow \mathbb{R}_+ \), that solves the boundary value problem that arises from the pricing equation in (2),

\[ (f(x) - \rho)xp'(x) + \frac{1}{2}(x\sigma_{s_2}(x))^2p''(x) - f(x)p(x) + 1 = 0, \]  

(32)

subject to boundary conditions at \( \{0, 1\} \) which depend on the type of constraint under consideration and are derived in the next section. We note that the equation in (32) follows from the explicit dependance of the stock price in (27) on \((\delta, s_2)\) and the fact that the consumption share is an

\footnote{We note the convenience of the system \((\delta, s_2)\) to characterize equilibrium quantities, in contrast to \((\delta, \lambda)\), since the process in (31) is autonomous only when both agents have logarithm preferences, in which case its diffusion is given by \( \lambda \Phi \left( \frac{1}{1+\lambda} \right) \).}

\footnote{See Basak and Cuoco (1998), Basak and Croitoru (2000), Wu (2006), Gallmeyer and Hollifield (2008), Hugonnier (2009) and Chabakauri (2009) for examples of equilibrium models with portfolio constraints that have used this construction.}
autonomous process.

An application of Ito’s lemma to the price process \( S_t = \delta_t p(s_{2t}) \) reveals that the volatility of the stock returns has two components,

\[
\sigma_t = \sigma_\delta + \sigma_p(s_{2t}). \tag{33}
\]

The first term is the volatility of dividends and is referred to as the fundamental component, whereas the second term is the volatility of the price dividend ratio,

\[
\sigma_p(s_{2t}) = s_{2t} \sigma_{s_2}(s_{2t}) \frac{p'(s_{2t})}{p(s_{2t})} \tag{34}
\]

and is referred to as the excess volatility component.

We finish this section with a couple of observations. Since equilibrium quantities are functions of the consumption share only, we use \( s_{2t} \) as the state variable that summarizes the general state of the economy, and thus, as in other models with a single driving state variable (e.g., the surplus consumption ratio in the external habit model of Campbell and Cochrane (1999) or the consumption share in the overlapping generations model of Gärleanu and Panageas (2008)), the dynamics of the consumption share determines time series patterns in equilibrium quantities, such as persistence and ‘mean reversion’.

Furthermore, as in Mele (2007) and Bhamra and Uppal (2009), any variable positively (negatively) correlated with \( s_{2t} \) is termed ‘countercyclical’ when \( \sigma_{s_2}(x) < (>)0 \). In particular, we note that the excess volatility term in (34) is positive when \( \sigma_{s_2}(x) < (>)0 \) and \( p'(x) < (>)0 \), that is, only when the price dividend ratio is ‘procyclical’.

4 Analysis of equilibrium

In this section, we study the implications of the two types of constraints in equilibrium. We show that the imposition of constraints on market participants which are more risk tolerant dampens fundamental shocks. We also determine the conditions under which the stock price includes a bubble component under volatility constraints.

4.1 Equilibrium under mean-variance constraints

Using the fact that risk reduction is constant when the agent is subject to a mean-variance constraint, the market price of risk in (17) is a continuous and positive function of the consumption share given by

\[
\theta(s_{2t}) = \frac{\gamma \sigma_\delta}{1 + (\gamma \kappa - 1)s_{2t}}. \tag{35}
\]
The expression in (35) is higher than its unconstrained counterpart for the same distribution of consumption, and, interestingly, would correspond to the market price of risk of an economy with no constraints but where the risk aversion distribution across agents is given by \( \{ \gamma, \frac{1}{\kappa} \} \).

However, unlike an unconstrained economy with heterogeneous agents, a high market price of risk does not imply a relatively high interest rate (see Kogan, Makarov and Uppal (2007), Proposition 5.). On the contrary, Figure 1 shows that in an economy with constraints, the market price of risk rises and the equilibrium interest rate decreases, to induce the unconstrained agent to scale up his position in the risky asset, as we verify later. We note also that the impact of the constraint is asymmetric as it is more pronounced in bad times and increases as the constraint becomes tighter.

The existence result and risk sharing. When \( \kappa \in (0, 1] \), the consumption share obeys

\[
\frac{d s_{2t}}{s_{2t}} = \mu_{s_2}(s_{2t})dt + (\kappa \theta(s_{2t}) - \sigma_\delta) d B_t \tag{36}
\]

with

\[
\mu_{s_2}(x) = (\gamma - 1) \frac{1 - x}{1 + (\gamma - 1)x} \mu_\delta + \frac{\gamma}{1 + (\gamma - 1)x} \sigma_\delta^2 - \frac{\gamma}{2[1 + (\gamma - 1)x]^3} \sigma_\delta^2
\]

\[
+ (\kappa - 1)(\gamma - 1) \frac{\gamma^2 (1 - x) x^2 [\gamma + \gamma \kappa - 2 + 2(\gamma - 1)(\gamma \kappa - 1)x]}{2[1 + (\gamma - 1)x]^3[1 + (\gamma \kappa - 1)x]^2} \sigma_\delta^2
\]

\[
- (\gamma \kappa - 1) \frac{(1 - x)[1 - \gamma + (\gamma \kappa - 1)x]}{[1 + x(\gamma \kappa - 1)]^2} \sigma_\delta^2.
\]

which shows that both the drift and volatility of the consumption share are zero at the two extreme ends, i.e., when one of the agents owns the whole economy, and thus, \( \{0, 1\} \) are absorbing states for \( s_2 \). We show, however, that is possible to establish the following existence result.

**Proposition 4.** Suppose that the process in (36) has a starting point in \( (0, 1) \), the equilibrium exists as the boundary points \( \{0, 1\} \) cannot be reached in finite time.

The existence result in Proposition 4 hinges on the fact that is possible to invoke standard pathwise comparison arguments such that the process in (36) lives in \( (0, 1) \).

In Figure 2 we plot both the drift and the volatility of \( s_2 \) for the baseline parameter values and different levels of the constraint. If the unconstrained agent is more risk averse \( (\gamma > 1) \), depending on parameter values, the drift can be uniformly positive or can be positive for values of \( s_{2t} \) below some threshold and negative for values greater than that threshold. The latter situation, which occurs when the constraint is tight enough, implies a certain type of mean-reverting behavior for \( s_2 \).

The volatility of \( s_2 \) is given by a humped shape function in an economy with no active constraints, whereas when the constraint is present, its volatility decreases with the severity of the constraint.
When the unconstrained agent is less or equally risk averse than the constrained agent ($\gamma \leq 1$), the drift of $s_2$ is always negative, indicating that agent 2 is steadily losing share of the economy. The volatility of $s_2$ is given by a inverted humped shape function in an economy with no active constraints, and importantly, the presence of the constraint increases its exposure to shocks.

The above discussion reveals that, as the consumption volatility of the constrained agent is given by

$$\sigma_\delta + \sigma_{s_2}(\cdot) = \kappa \theta(\cdot),$$

the portfolio that finances his optimal consumption plan may depart drastically from the unconstrained benchmark. Recall that when there are no constraints ($\kappa = 1$), agent 2 would hold a leveraged position in the stock and a short position in the riskless asset if the other agent is more risk averse ($\gamma > 1$), or would hold a reduced (but positive) position in the stock and become a net lender otherwise.

Intuitively, the constraint changes the portfolio allocations in equilibrium as if it were ‘modifying’ the risk aversion distribution across agents. To see this, note that the function $\kappa \theta(\cdot)$ corresponds to the volatility of consumption growth of an agent with risk aversion $\frac{1}{\kappa} > 1$ in an unconstrained economy where the risk aversion distribution across agents is given by $\{\gamma, \frac{1}{\kappa}\}$ (see Figure 3 for a comparison across economies).

This implies that, as shown in Figure 4, when the unconstrained agent is more risk averse, ($\gamma > 1$), the ‘effective’ risk aversion distribution across agents is narrowed, to the extent that if the constraint is tight enough, agent 2’s relative position may change from borrower to net lender, whereas when the unconstrained agent is less risk averse, ($\gamma < 1$), the constraint acts as if it were widening the risk aversion distribution, and therefore, the constrained agent is forced to take an even smaller position in the stock. This is the mechanism behind the dampening/amplifying effect of the constraint on the volatility of returns, as we will see next.

**Price dividend ratio and the volatility of returns.** One advantage of this example is the fact that the presence of bubbles can be easily ruled out.

**Proposition 5.** The stock price is given by its fundamental value for $\kappa \in (0, 1]$, consequently, the price dividend ratio is represented by

$$p(s_2) = E\left[\int_t^\infty e^{-\int_t^u f(s_2u)du} \frac{\bar{M}_s}{\bar{M}_t} ds\right] = \bar{E}\left[\int_t^\infty e^{-\int_t^u f(s_2u)du} ds\right]$$

where the discount rate $f(\cdot)$ is defined in (24) and the density of the measure $\bar{\mathbb{P}}$ is given by the exponential martingale

$$\bar{M}_t = e^{-\frac{1}{2} \int_0^t (\theta(s_2) - \sigma_\delta)^2 ds - \int_0^t (\theta(s_2) - \sigma_\delta) dB_s}.$$
solution of the ODE in (32) and can be viewed as an extension of the Gordon growth formula, i.e., the price dividend ratio reflects the investors’ risk adjusted expectation about the future state of the economy, and hence, the discount rate to prevail in the future. The higher the future discount rate \( f(\cdot) \), the lower the price dividend ratio.

The representation in (37) allow us to apply Lemma 1 in Mele (2007) which provides a sufficient condition for monotonicity of the price dividend ratio based on the probabilistic representation of the derivative term \( p'(s_{2t}) \),

\[
p'(s_{2t}) = -\mathbb{E} \left[ \int_t^\infty e^{-\int_t^s g(s_{2u})du} p(s_{2u}) f'(s_{2u})ds \right] \Gamma_t \tag{38}
\]

with \( g(x) = f(x) - f'(x) \). The expression in (38) says that if the discount rate \( f(\cdot) \) is monotonically increasing (decreasing) in \( s_2 \), then the price dividend ratio is monotonically decreasing (increasing) in the consumption share. This implies that the excess volatility component in equation (34) is positive if \( \sigma_{s2}(x) < (>)0 \) and \( f'(x) > (<)0 \), that is, when \( f(\cdot) \) is countercyclical.\(^{15}\)

In order to quantify the effects of the constraint, we solve for the price dividend ratio in (32). We obtain two boundary conditions by passing to the limit in (32) at \( s_2 \to \{0, 1\} \). The boundaries are given by

\[
p(0) = \frac{1}{\rho - (1 - \gamma)(\mu - \frac{1}{2}\gamma\sigma_2^2)},
\]

which corresponds to the price dividend ratio in an economy with a single unconstrained agent, and

\[
p(1) = \frac{1}{\rho},
\]

which corresponds to the price dividend ratio that would prevail in an economy where there is single constrained agent.

The left panel in Figure 5 shows a key result. When the unconstrained agent is more risk averse (\( \gamma > 1 \)), the constraint forces agent 2 to decrease its position in the stock, narrowing the ‘effective’ risk aversion distribution, which leads to a decrease in volatility as the constraint restrains an efficient risk sharing whose dynamic evolution is partly responsible for the volatility of the stock price.

Note also that if the constraint is sufficiently tight, (\( \kappa < \frac{1}{\gamma} \)), the volatility in a constrained economy would be lower than the volatility of an otherwise equivalent unconstrained economy in any state of nature. We note that the monotonicity of the discount rate is the same (the discount rate \( f(\cdot) \) is decreasing in the consumption share which makes \( p'(\cdot) > 0 \)), meaning that the change in sign of the excess volatility term in equation (34), from positive under no constraints to negative

\(^{14}\)Care must be taken because (37) is not a valid representation of the stock price dividend ratio in the presence of bubbles.

\(^{15}\)Bhamra and Uppal (2009) show that in an unconstrained economy, (\( \kappa = 1 \)), countercyclicality in the discount rate \( f(\cdot) \) obtains when \( P(s_{2t}) \leq 1 + \frac{\mu}{\sigma_2^2} \).
in an economy with constraints, comes through the different sign in the diffusion term of the consumption share, that is, the constraint renders the discount rate \( f(\cdot) \) ‘procyclical’.

On the other hand, as seen in the right panel in Figure 5 when the unconstrained agent is less risk averse \((\gamma < 1)\), fundamental shocks are amplified because a countercyclical discount rate \( f(\cdot) \) induces a positive excess volatility term. As the constraint tightens, the volatility increases because of the widening effect that constraint has on the ‘effective’ risk aversion distribution. The result mirrors Bhamra and Uppal (2009), who show that in an economy with no constraints and heterogeneous agents, the volatility is increasing in the dispersion of risk aversion.

The result is also consistent with the empirical literature\(^{16}\) by displaying lower price-dividend ratios and higher stock return volatilities in bad times, when the share of the constrained agent is high. Due to the dynamics of the consumption share process, the equilibrium is characterized by long periods of moderation with low volatility.

We note also that the market leverage ratio \(|\phi_{02}|/S_t\) increases with the dispersion in the ‘effective’ risk aversion as Figure 6 shows. This quantity is a good proxy for the volatility of the interest rate, as in economies with no constraints (see Longstaff and Wang (2008)), meaning that an increase in volatility of stock returns is coupled necessarily with a higher interest rate volatility.

Quantities at time zero. As agents in economies with different levels of the constraint face different equilibrium quantities, one has to be careful with the comparison results using the consumption share as state variable.

One exercise that partially addresses this drawback consists on comparing equilibrium quantities across economies at time 0. This exercise answers the question of what is the instantaneous effect of imposing the constraint keeping fixed two exogenous quantities, the distribution of wealth across agents \(0 < \alpha < 1\) (we set \(\beta = 0\)), and the level of aggregate endowment \((\delta_0)\). The results confirm our previous analysis\(^{17}\).

This static exercise also allows us to compare the instantaneous effect of the constraint on price levels. Intuitively, since the constraint binds at time zero, security prices must change to increase agent 1’s stock demand. Since markets have to clear, the unconstrained agent modifies his consumption-savings mix in a way that reflects income and substitution effects which stem from the fact that his utility is not logarithmic.

An income effect is driven by an increase in his current wealth due to the increase in expected returns, which drives him to consume more today and save less (and prevails when \(\gamma > 1\)), whereas a substitution effect goes on the opposite direction, meaning that an increase in the return today makes current consumption more costly compared to future consumption, which leads him to consume less today and save more for the future (and prevails when \(\gamma < 1\)). Since the supply of

\(^{16}\)Mele (2007) (and references therein) documents that there is strong evidence in the US that the price dividend ratio decreases more during recessions than they increase during expansions and the volatilities of the price dividend ratio and returns are countercyclical.

\(^{17}\)Specific results are available on request.
aggregate consumption is fixed, to clear the consumption good market the price must move in order to offset the change in the consumption-saving decision.

If $\gamma < 1$ the demand for current consumption must increase which occurs if the value of stock price increases relative to an unconstrained economy, which is the result in the right panel of Figure 7. Conversely, if $\gamma > 1$, the demand for savings decreases and the demand for consumption rises, which leads to a decrease in level of the stock price, which is precisely what we have in the left panel of Figure 7.

4.2 Equilibrium under volatility constraints

Using the optimal portfolio policy under volatility constraints in (13) and the equation for the market price of risk in (14), gives the following result.

Proposition 6. The market price of risk is given by

$$\theta(s_{2t}) = \gamma \sigma_{\delta} \frac{1 - \varepsilon s_{2t}}{1 - s_{2t}} I(R(s_{2t}) > \varepsilon) + \frac{\gamma \sigma_{\delta}}{1 + (\gamma - 1)s_{2t}} I(R(s_{2t}) \leq \varepsilon),$$

(39)

where we express the risk bearing capacity of the constrained agent in units of the volatility of dividends\(^{18}\), $L_2 = \varepsilon \sigma_{\delta}$, with $\varepsilon \geq 0$.

The presence of the constraint generates an equilibrium with two regions in the space defined by the consumption share, which can be completely determined using exogenous parameters. The first term in (39) is the market price of risk in the region where the constraint is active, whereas the second term corresponds to the market price risk in the region where the constraint does not bind. The active region, $R(s_{2t}) > \varepsilon$, can be characterized in terms of an upper (lower) bound on the consumption share process, $s_{2t} < (>) s_{2t}^*$ if $\gamma > (\gamma <) 1$, with $s_{2t}^* = \frac{\gamma - \varepsilon}{\varepsilon(\gamma - 1)}$.

To see how the interaction between the risk bearing capacity and the risk aversion of the unconstrained agent determine the region in which the constraint is active, we take the example in which the unconstrained agent is more risk averse, $\gamma > 1$. If $\varepsilon \in [0, 1]$, the constraint is active in all states of nature. As one increases the risk bearing capacity of agent 2, such that $\varepsilon \in (1, \gamma)$, the constraint binds in the region determined by $s_{2t} < s_{2t}^*$, since $0 < s_{2t}^* < 1$. Finally, when $\varepsilon \geq \gamma$, the constraint is never active. The case when $\gamma < 1$ can be described in similar terms. Note also that if the constrained agent has logarithm preferences, ($\gamma = 1$), the constraint is active only when $\varepsilon < 1$, that is, when the risk bearing capacity of the constrained agent is less than the volatility of the market, $\sigma_{\delta}$.

The two-regime structure reveals a reassuring result: the constraint limits the trader’s risk exposure in ‘bad times’. We recall that when agent 1 is more risk averse, ($\gamma > 1$), the constrained agent would hold most of the stock, and low total wealth states would correspond to low levels of

\(^{18}\)This is done for notational convenience, as the active region will depend only on the risk aversion parameter ($\gamma$) and the severity of the constraint ($\varepsilon$).
the consumption share, whereas when agent 1 is less risk averse, \((\gamma < 1)\), the constrained agent would hold most of his wealth in the riskless asset and low total wealth states would coincide with the upper region.

In broad terms, the constraint acts in the same direction of our first example, increasing the market price of risk and decreasing the interest rate, as Figure 8 shows. This change induces the unconstrained agent to scale up his position in the risky asset. Note however, there is a sharp change in behavior of both quantities in states where the consumption share is large if the constraint binds.

**The existence result and risk sharing.** The two-regime structure is reflected in the dynamics of the consumption share process. Overall, they share qualitative features with the dynamics in [36], in the sense that they critically depend on the tightness of the constraint and the risk aversion distribution across agents. The dynamics of the consumption share are given by

\[
\frac{ds_{2t}}{s_{2t}} = \mu_{s_2}(s_{2t})dt + \sigma_{s_2}(s_{2t})dB_t \tag{40}
\]

with

\[
\mu_{s_2}(x) = \begin{cases} 
(\gamma - 1) \frac{1-x^{1+(\gamma-1)x}}{1+(\gamma-1)x} \mu_{\delta} + (\gamma - 1)^\gamma \frac{[2-\varepsilon(2-\varepsilon)x]x-1}{2[1-x]1+(\gamma-1)x} \sigma_{\delta}^2 \\
+(1-\varepsilon)\sigma_{\delta}^2 \left(1-\gamma \frac{1-\varepsilon x}{1-x}\right), & R(x) > \varepsilon, \\
(\gamma - 1) \frac{1-x^{1+(\gamma-1)x}}{1+(\gamma-1)x} \mu_{\delta} + \gamma \frac{\gamma[1+\gamma+(\gamma-1)(1+2\gamma)x]}{2[1+(\gamma-1)x]^2} \sigma_{\delta}^2 \\
-(\gamma - 1) \frac{(1-x)[1-\gamma+(\gamma-1)x]}{[1+x(\gamma-1)]^2} \sigma_{\delta}^2, & R(x) \leq \varepsilon,
\end{cases}
\]

and

\[
\sigma_{s_2}(x) = -(1-\varepsilon)\sigma_{\delta} 1\{R(x) > \varepsilon\} + \sigma_{\delta} \left[\frac{\gamma}{1+(\gamma-1)x} - 1\right] 1\{R(x) \leq \varepsilon\}. \tag{42}
\]

We note that the drift diverges to minus infinity at one if the constraint binds and this extreme behavior counterbalances the effect of the linear diffusion in [42], such that the process never reaches one from the interior. This observation in conjunction with standard comparison arguments is behind the following existence result.

**Proposition 7.** Suppose that the process in [40] has a starting point in \((0,1)\), the equilibrium exists as the boundary points \(\{0,1\}\) cannot be reached in finite time.

For completeness, we plot in Figure 9 both the drift and the diffusion of \(s_2\) for the baseline parameter values and different levels of the constraint. When the unconstrained agent is more risk averse \((\gamma > 1)\), the drift may display a type of mean-reverting behavior if the constraint binds, whereas the volatility of \(s_2\) is given by a humped shape function in the region where the constraint is slack, which shifts to a linear function in the active region. Note that when the constraint binds, the diffusion term does not depend on the risk aversion distribution.
If the unconstrained agent is less or equally risk averse than the unconstrained agent ($\gamma \leq 1$), the drift of $s_2$ is always negative, and the volatility of $s_2$ displays an inverted humped shape in the region with binding constraints. Its exposure to shocks is increased in the region where the constraint is active, where the diffusion function is a linear function of the consumption share.

The constrained agent’s risk reduction process

$$\kappa(s_{2t}) = \frac{\varepsilon}{\gamma} \frac{1 - s_{2t}}{1 - \varepsilon s_{2t}} 1\{R(s_{2t}) > \varepsilon\} + 1\{R(s_{2t}) \leq \varepsilon\},$$

shows how constrained market participants may appear to become more risk averse in response to deteriorating market conditions, that is, negative shocks induce a decrease in the portfolio position with respect to an otherwise unconstrained agent. The risk reduction goes up over time as the ‘cycle’ improves.

Figure 10 shows the corresponding portfolio strategies. When the unconstrained agent is more risk averse, ($\gamma > 1$), the constrained agent curtails his position in the risky asset in states where his consumption share is low, and as the constraint tightens (a decreasing $\varepsilon$), his relative position may be changed, forcing the constrained agent to become a net lender. On the other hand, when the unconstrained agent is less risk averse, ($\gamma < 1$), the constraint is active in states where the consumption share is high, forcing the constraint agent to decrease his position in the stock further.

**Bubbles in equilibrium.** The characterization of the price dividend ratio requires more steps than the case with constant risk reduction, in particular, when the constraint is always binding or when $\gamma < 1$ and the constraint binds with positive probability, the presence of a singularity at $s_2 = 1$ in equilibrium quantities might suggest the existence of a bubble in the stock.

We verify the ‘martingality’ property of the weighting process $\lambda$ in (30) by exploiting the explicit dependence of the volatility of the weighting process in the consumption share. We summarize the result in the following proposition,

**Proposition 8.** The stock price contains a bubble if the constraint is always binding and $\gamma \geq 1$. The stock price is free of bubbles if (i) the constraint is not always binding and $\gamma > 1$, or (ii) the constraint binds with positive probability, with $\gamma < 1$ and $(1 - \gamma) (\mu_\delta - \frac{1}{2} \gamma \sigma_\delta^2) - \frac{1}{2} (1 - \varepsilon)^2 \sigma_\delta^2 > 0$.

Proposition 8 confirms the intuition put forward in Section 3; the stock contains a bubble component depending on how costly the constraint is for agent 2.

The cost of the constraint is determined primarily by (i) the severity of the constraint, and (ii) the risk aversion distribution across agents. The importance of both dimensions is evident from the fact that the region in which the constraint binds is completely determined by $(\gamma, \varepsilon)$.

The first dimension is easily seen by noting that when both agents share the same risk aversion ($\gamma = 1$), a bubble emerges as soon as the constraint binds, $\varepsilon < 1$.

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19. This behavior goes in line of the model of Danielsson, Shin and Zigrand (2009), were negative shocks induce feedbacks that might appear as if ‘asset sales beget asset sales’.
To see the relevance of the risk aversion distribution, we examine the limiting case \( \varepsilon = 0 \). We let the agents differ in risk aversion, which corresponds to the general case of the restricted participation model of Basak and Cuoco (1998). Proposition 8 says that a bubble arises as soon as the stockholder, agent 1, is more risk averse than the non-stockholder. In contrast, the stock is free of bubbles when agent 1 is less risk averse than the non-stockholder.

These results focus on stock price bubbles, but bubbles may also arise on the price of the riskless asset. Indeed, Heston, Loewenstein and Willard (2007) shows that the absence of bubble on the riskless asset is equivalent to the requirement that the local martingale \( S^0_t \xi_{1t} \) is a true martingale, and define the bubble in the riskless asset for a given horizon \( T \), the process \( b^0_t = S^0_t \left( 1 - E_t \left[ \frac{S^0_T}{S^0_t} \xi_{1T} \right] \right) \). It is straightforward to apply the conditions in Proposition 8 to the limiting case \( \varepsilon = 0 \), because the weighting process corresponds to the candidate risk neutral density, \( S^0_t \xi_{1t} = \lambda_t / \lambda_0 \).

Overall, conditions in Proposition 8 complement the findings in Hugonnier (2009, see Proposition 6), who shows that when both agents have logarithm preferences and constrained agents face borrowing constraints, the stock price and the riskless asset contain a bubble component.

When all agents have logarithm preferences, bubbles can be characterized in closed form, and their relative sizes with respect to the equilibrium prices increase with the tightness of the constraint. In the general case, simulations \(^20\) show that they are also increasing in the level of risk aversion.

**The volatility of returns.** As in our first example, in order to solve for the price dividend ratio in (32) and obtain the volatility term in (33), we obtain the boundary at 0 by passing to the limit in equation (32) which yields

\[
p(0) = \frac{1}{\rho - (1 - \gamma)(\mu_\delta - \frac{1}{2} \gamma \sigma_\delta^2)}.
\]

This quantity corresponds to the price dividend ratio in an economy with a single unconstrained agent. If the constraint is slack at \( s_2 = 1 \), the boundary corresponds to

\[
p(1) = \frac{1}{\rho},
\]

however, if the constraint binds as \( s_2 \to 1 \), which happens if \( \varepsilon < 1 \), the boundary at 1 cannot be \( p(1) = \frac{1}{\rho} \), since this limit would not correspond to the limiting price dividend ratio with a single constrained agent whose volatility of wealth is given by \( \varepsilon \sigma_\delta \leq \sigma_\delta \). In this case, we obtain a second condition by differentiating equation (32) and evaluating at \( s_2 = 0 \), to obtain

\[
p'(0) = \gamma \left( \frac{1}{\rho} - p(0) \right).
\]

\(^{20}\)Details of the simulations are available on request.
The left panel in Figure 11 displays the volatility when the unconstrained agent is more risk
averse than the constrained agent, ($\gamma > 1$). Note that the volatility decreases with respect to
the unconstrained economy, and if the constraint is tight enough, the volatility is lower than its
unconstrained economy counterpart in all states.

This is a key result that mirrors the outcome in the example with constant risk reduction. The
presence of the constraint dampens rather than amplifies fundamental shocks in an environment
where the constrained agent is a net borrower and the constraint binds.

The right panel in Figure 11 presents the volatility when $\gamma < 1$ and shows a pattern that differs
from the constant risk reduction case.

The volatility is increased in the region where the constraint is active, but as the wealth of the
constrained agent increases, there is an extreme change in the discount rate $f(\cdot)$, a large discounting
asymmetry, that changes the sign in the slope of the price dividend ratio. Since the diffusion term
in the consumption share dynamics in (40) is negative, an upward sloping price dividend ratio turns
negative the excess volatility term in (34).

We close the section noting that the effect of the constraint on the size of credit market resembles
that of the constant risk reduction example. Figure 12 shows there is an increase in the volatility
of the interest rate in the states in which the constraint is active when the unconstrained agent is
less risk averse.

5 Generalizations and applications

In this section, we show that is straightforward to characterize the equilibrium with risk constraints
in settings with heterogeneous beliefs and multiple risky assets. We also show that the presence of
bubbles gives rise to multiple equilibria in stock prices.

5.1 Introducing heterogeneous beliefs

We introduce heterogeneity in beliefs about the evolution of the aggregate dividend in an environ-
ment with risk constraints. We assume, without loss of generality, that the beliefs of agent 1 are
represented by the objective measure and let

$$
\eta_t = e^{-\int_0^t \frac{1}{2} \mu_s^2 ds - \int_0^t \mu_s dB_s}
$$

denote the density process of agent 2’s probability measure $P_2$ with respect to $P$. In the expression,
$\bar{\mu}$ represents the investors’ disagreement on the mean endowment growth rate, normalized by its
risk,

$$
\bar{\mu} = \frac{\mu_2 - \mu_1}{\sigma_2},
$$
where \( \mu_\delta \) and \( \mu_2 \delta \) represent the beliefs of agent 1 and 2, respectively. Note that \( \bar{\mu} \) is positive when agent 1 is more optimistic.\(^{21}\)

The divergence in beliefs is modeled in a reduced form through the processes \((\eta, \bar{\mu})\) and since all computations are done under the subjective measure of agent 1, the utility function of the second agent is state dependent and given by \( u_2(\eta, c) = \eta \log c \). The construction of equilibrium is similar to that of an economy with homogeneous beliefs. However, now agents trade also due to differences in beliefs. For simplicity, we set \( \bar{\mu} \) to be a constant.\(^{22}\)

The next proposition details the market of risk under volatility constraints.

**Proposition 9.** The market price of risk is given by

\[
\theta(s_{2t}) = R(s_{2t})(\sigma_\delta + s_{2t}\bar{\mu})I_{\{s_{2t} \in S_u\}} + \gamma \sigma_\delta \frac{1 - \varepsilon s_{2t}}{1 - s_{2t}} I_{\{s_{2t} \in S_1\}} + \gamma \sigma_\delta \frac{1 + \varepsilon s_{2t}}{1 - s_{2t}} I_{\{s_{2t} \in S_2\}},
\]

where \( \varepsilon \geq 0, \) and \( S_u = \{x \in (0, 1) : |R(x)(\sigma_\delta + x\bar{\mu}) - \bar{\mu}| \leq \varepsilon \sigma_\delta\}, \) \( S_1 = \{x \in (0, 1) : R(x)(\sigma_\delta + x\bar{\mu}) - \bar{\mu} > \varepsilon \sigma_\delta\}, \) and \( S_2 = \{x \in (0, 1) : -R(x)(\sigma_\delta + x\bar{\mu}) + \bar{\mu} > \varepsilon \sigma_\delta\}. \)

The constraint set induces an equilibrium with possibly three regions which depend on the tightness of the constraint, the divergence of beliefs, the volatility of dividends and the risk aversion distribution. When the divergence of beliefs is zero, the model collapses to the case analyzed in Section 4.2.

We note that the tractability of the baseline case is translated to the case with heterogeneous beliefs: the regions where the constraint is active are completely determined by known quantities. The set \( S_u \) describes the region in which the constraint does not bind, whereas the constraint is active when the consumption share enters \( S_1 \) or \( S_2. \)

The reason why there is a third region is intuitive. The constraint may curtail the risk taking behavior of agent 2 also when he holds a short position in the stock, which (most likely) happens when the unconstrained agent is more optimistic, \( \bar{\mu} > 0. \) Take the case of homogeneous preferences \( (\gamma = 1), \) the constraint is active in an upper region defined by \( \{S_1 : s_{2t} > 1 - (1 - \varepsilon) \frac{\sigma_\delta}{\bar{\mu}}\}, \) and in a lower region, given by \( \{S_2 : s_{2t} < 1 - (1 + \varepsilon) \frac{\sigma_\delta}{\bar{\mu}}\}, \) where the agent holds a short position in the stock.

Despite the fact that heterogeneity in beliefs may generate rich patterns in volatility in unconstrained economies (it is relatively easy to obtain non monotonic price dividend ratios when there are divergence of beliefs), we note that the central result in the paper stands, for most cases of

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\(^{21}\)See Basak (2005) for a general model in which investors observe the aggregate dividend and have incomplete but symmetric information on its dynamics, \( \bar{\mu} \) follows directly from the endowment process and agents’ priors.

\(^{22}\)This example corresponds to what has been termed dogmatic beliefs, because it can be rationalized by assuming that each investor \( k \) is so confident in his prior that he completely ignores any information from the output process, and keeps the same belief throughout.

\(^{23}\)The model shares some resemblance with the model of Gallmeyer and Hollifield (2008) who study the effect of short-sale constraints on asset returns. However, Gallmeyer and Hollifield (2008) need to restrict the sign of the stock price volatility process and solve the model via Monte Carlo simulations.
interest, in the sense that the presence of risk constrained agents decreases (the absolute value of) the excess volatility term in (34).

5.2 The limits of the restricted stock market participation model

As we mentioned in Section 4, the limiting case $\varepsilon = 0$ in Proposition 6 recovers the restricted stock market participation of Basak and Cuoco (1998). In this model, restricted agents choose a consumption process with no covariation with aggregate consumption and as a consequence, the unrestricted agents or stockholders must absorb all the aggregate consumption risk into their consumption stream.

This model has had some appeal, as simple calibrations show that it could help resolve some of the empirical asset pricing puzzles, because it generates a sizable and countercyclical market price of risk and low interest rates (see Basak and Cuoco (1998) p. 326). However, recent studies have pointed out some drawbacks: (i) as argued in Guvenen (2009) and Gomes and Michaelides (2008), since non-stockholders in Basak and Cuoco (1998) consume out of wealth, and receive interest payments from market participants, the mechanism works quantitatively only if non-stockholders own a large fraction of aggregate wealth, a fact that is not backed by the data (Guvenen (2009) adds labor as a second source of income for non-stockholders), and, (ii) Hugonnier (2009) revisits this model for the homogeneous case ($\gamma = 1$) and shows that a calibration that involves non-stockholders owning a large fraction of wealth is really short lived as the fraction of constrained agents is expected to decrease very fast.

The general case sheds light on two issues that have not addressed previously and are of interest since there is a large body of empirical work documenting heterogeneity in the EIS across the population;

(i) As seen in Figure 13, the model increases the risk premium, but fails to produce a significant positive excess volatility term. When the stockholder is more risk averse ($\gamma > 1$), the stock price contains a bubble, and the excess volatility term is negative for all states, in contrast to the analytical result derived in Mele (2007).

(ii) The relative extinction of non stockholders in the homogeneous case, ($\gamma = 1$), depends only on level of volatility of the market (or equivalently, the volatility of dividends, $\sigma_\delta$), the higher the volatility, the faster the speed of disappearance. When agents are heterogeneous, the speed at which the non-stockholders disappear not only depends on the volatility of the dividend but also on the

---

24Note that this model is also recovered by setting $\varepsilon = 0$ in the model with heterogenous beliefs.
25To the best of our knowledge, this is the first paper that provides existence results for the case in which the stockholder has arbitrary risk aversion.
26Using Basak and Cuoco (1998)'s ($\mu_\delta, \sigma_\delta) = (0.018, 0.036)$ and logarithmic utility, non-stockholders need to hold approximately 83% of wealth to match the risk free rate and market price of risk. As Hugonnier (2009) reports, the fraction of constrained agents is expected to decrease by almost 10% over the first ten years.
27Studies, e.g. Malloy, Moskowitz and Vissing-Jorgensen (2009) and references therein, show that non-stockholders have an elasticity of intertemporal substitution (EIS) that is lower than the EIS of stockholders.
risk aversion of the stockholders and the growth of aggregate dividends. In particular, simulations of
the expected consumption share for different horizons show that for levels of aggregate consumption
volatility which generate positive levels of the interest rate (among them, the Basak and Cuoco
(1998) parameters), the speed at which the fraction of constrained agents decreases is much faster
than the homogeneous case when \( \gamma < 1 \) (almost doubling it when \( \gamma = 1/2 \)), and slower when \( \gamma > 1 \).
Since stockholders have higher EIS, we conclude that the calibration of equilibrium quantities using
the (non stationary) consumption share of the non-stockholders may indeed be very transitory.

5.3 An economy with multiple risky assets

In this section, we make use of the following vectorial notation: \( a^\top \) denotes transposition, \( \| \cdot \| \)
denotes the Euclidean norm in \( \mathbb{R}^n \) and \( 1_n \) is a \( n \)-dimensional vector of ones. We denote by \( B \) an
\( n \)-dimensional standard Brownian motion.

Contrary to models with general position constraints, it is relatively straightforward to introduce
multiple risky assets in an environment where some agents face risk constraints. In particular, under
some additional assumptions, pertaining the dividend process and the initial endowment of agents,
the equilibrium will be structurally identical to the equilibrium with one risky asset.

In this economy, the \( i \)-th risky asset is a claim to a strictly positive dividend process \( \delta_{it} \), such
that the ex-dividend price process of the \( i \)-th risky asset, denoted by \( S_i \), evolves according to

\[
S_t^i = S_0^i + \int_0^t S_s^i (\mu_s^i ds + \sigma_s^i dB_s) - \int_0^t \delta_s^i ds, \tag{43}
\]

for some initial value \( S_0^i \in \mathbb{R}_+ \) and some drift and volatility processes \( (\mu^i, \sigma^i) \in \mathbb{R} \times \mathbb{R}^n \) which are
determined in equilibrium.

The process

\[
\theta_t = \sigma_t^{-1} (\mu_t - r_1 1_n),
\]
denotes the vector of relative risk premium associated with the sources of risk in the model, here
\( \mu \in \mathbb{R}^n \) and \( \sigma \in \mathbb{R}^{n \times n} \) denote the drift and the volatility of the price vector, respectively, obtained
by stacking up the individual drifts and volatilities of the stock prices.

We make two additional assumptions. First, we assume that the aggregate endowment follows a
geometric Brownian motion due to the presumed i.i.d. property of aggregate consumption growth,

\[
\delta_t = \sum_{i=1}^n \delta_{it} = \delta_0 + \int_0^t \delta_s \left( \mu_s ds + \sigma_s^\top dB_s \right)
\]
such that the dividend of security \( i \) is given by \( \delta_{it} = \delta_t x_{it} \), where \( x_{it} \) is the share in aggregate
endowment of dividend \( i \) (see Appendix C for an example).

Second, we assume that agent 2 is initially endowed with \( \beta \in \mathbb{R} \) units of the riskless asset and
a positive fraction $\alpha \leq 1$ of the market portfolio,

$$w_2 = \beta + \alpha \sum_{i=1}^{n} S_i^0 > 0.$$  

The following proposition reveals that the equilibrium with $n$ risky assets can be constructed in the same way of the equilibrium with one risky asset. For simplicity, we focus on the two examples of the baseline case, and thus, the vector $\theta$ solves a nonlinear system of equations. Quantities depend only on the consumption share of the constrained agent.

**Proposition 10.** The state price density and optimal consumption plans are given by the expressions in (15) and (16), respectively. The market prices of risk and the interest rate are given by

$$\theta(s_{2t}) = \frac{1}{1 - (1 - \kappa(s_{2t}))} R(s_{2t}) s_{2t}^j R(s_{2t}) \sigma_\delta,$$

$$r(s_{2t}) = \rho + \mu_\delta R(s_{2t}) + (P(s_{2t}) - R(s_{2t})) s_{2t} \Phi(s_{2t})^T \theta(s_{2t})$$

$$+ \frac{1}{2} P(s_{2t}) R(s_{2t}) \left[ \|s_{2t} \Phi(s_{2t})\|^2 - \|\sigma_\delta\|^2 \right],$$

where $\Phi(s_{2t}) = -(1 - \kappa(s_{2t})) \theta(s_{2t})$. The consumption share of the constrained agent obeys

$$ds_{2t} = s_{2t} \mu_{s_{2t}}^j (s_{2t}) dt + s_{2t} \sigma_{s_{2t}}^j (s_{2t})^T dB_t,$$  

(44)

where

$$\mu_{s_{2t}}^j (s_{2t}) = f(s_{2t}) - \rho + \sigma_{s_{2t}}^j (s_{2t})^T (\theta(s_{2t}) - \sigma_\delta),$$

$$f(s_{2t}) = r(s_{2t}) + \sigma_\delta^T \theta(s_{2t}) - \mu_\delta,$$

$$\sigma_{s_{2t}}^j (s_{2t}) = \kappa(s_{2t}) \theta(s_{2t}) - \sigma_\delta,$$

and its starting point, $s_{20} \in (0, 1)$ is the solution to the equation

$$(1 - \alpha) \rho^{-1} \delta_0 s_{20} = \beta + \alpha E \left[ \int_0^\infty \xi_1(t, s_{2t}, \delta_t)(1 - s_{2t}) \delta_t dt \right].$$

Importantly, the assumption that the agents’ initial endowments depend only on the market portfolio allows us to simplify the determination of equilibrium, as the starting point of the consumption share process, $s_{20} \in (0, 1)$, depends only on the primitives of the economy. Hence, the consumption allocation, the interest rate, and the relative risk premia are computed exactly as in the economy with one risky asset.

Note that individual stock returns satisfy a two factor capital asset pricing model where the
weighting process, defined in (30), plays the role of the second factor:

$$
\mu^i_t - r_t = R(s_{2t}) \left[ \text{cov} \left( \frac{dS^i_t}{S^i_t}, \frac{d\delta_t}{\delta_t} \right) - s_{2t} \text{cov} \left( \frac{dS^i_t}{S^i_t}, \frac{d\lambda_t}{\lambda_t} \right) \right].
$$

(45)

**Stock prices.** Since the consumption share in (44) is an autonomous process, we can test for the presence of bubbles in the market portfolio using the same procedure of the economy with one risky asset.

When the equilibrium is free of bubbles, the process $S^0_t \xi_{1t}$ defines an equivalent martingale measure, and stock prices are given by their fundamental values:

$$
S^i(s_{2t}, \delta_t, x_t) = \delta_t E \left[ \int_t^\infty e^{-\rho(s-t)} \left( \frac{1 - s_{2s}}{1 - s_{2t}} \right)^{-\gamma} \left( \frac{\delta_s}{\delta_t} \right)^{1-\gamma} x_{is} ds \left| \mathcal{F}_t \right. \right].
$$

(46)

On the other hand, in an economy with a bubble in the market portfolio, i.e.,

$$
\sum_{i=1}^n S^i_t = W_{1t} + W_{2t} = E \left[ \int_t^\infty \frac{\xi_{1s}}{\xi_{1t}} \delta_s ds \left| \mathcal{F}_t \right. \right] + b_t,
$$

stock prices are not uniquely determined.

The intuition follows from the fact that the state price density $\xi_1$ is the unique nonnegative process such that the deflated stock prices are nonnegative local martingales and because of this, also supermartingales,

$$
\xi_{1t} S^i_t + \int_0^t \xi_{1s} \delta_s^i ds \geq E \left[ \int_0^\infty \xi_{1s} \delta_s^i ds \left| \mathcal{F}_t \right. \right].
$$

If the inequality is strict, which happens when the bubble term of the market portfolio is nonzero, individual stock prices can be represented as

$$
S^i(s_{2t}, \delta_t, x_t) = \delta_t E \left[ \int_t^\infty e^{-\rho(s-t)} \left( \frac{1 - s_{2s}}{1 - s_{2t}} \right)^{-\gamma} \left( \frac{\delta_s}{\delta_t} \right)^{1-\gamma} x_{is} ds \left| \mathcal{F}_t \right. \right] + b^i_t,
$$

where the first part corresponds to the fundamental component and $b^i_t$ denotes the bubble component of the $i$ - th risky asset. Note however that the only equilibrium restriction on the price system is given by $\sum_{i=1}^n b^i_t = b_t$, there are no restrictions as to the way the value of $b_t$ is split among the risky assets.

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28 An empirical study of the effect of risk constraints in asset prices would require the identification of empirical proxies for our state variables, a topic that we leave for future research. We note that Adrian, Moench, and Shin (2009) study, in a reduced form framework, the ability of variables based on balance sheet constraints of financial intermediaries in pricing the cross-section and time-series of asset prices.

29 See an example for two risky assets and mean variance constraints in Appendix C.

30 The multiplicity of equilibria due to bubbles was first shown in Hugonnier (2009) in a revealing application with two risky assets, homogeneous agents ($\gamma = 1$), and volatility constraints. The example is a cautionary tale for models.
6 Concluding remarks

In this article, we study a continuous-time, pure exchange economy populated by two groups of agents. Agents in the first group have logarithmic preferences and face risk-based portfolio constraints which forces them to behave locally as power utility investors with a relative risk aversion coefficient that depends on current market conditions, while agents in the second group have arbitrary CRRA preferences and are unconstrained.

The equilibrium is very tractable, as the consumption sharing rule follows an autonomous process whose coefficients can be determined in closed form. This allows us to provide explicit existence results and solve the model by computing a single linear ordinary differential equation which describes the price dividend ratio.

We show that the imposition of constraints on market participants which are more risk tolerant dampens fundamental shocks. This insight is in contrast to recent studies that suggest that risk management rules serve to amplify aggregate fluctuations, and also, sheds doubts on the belief that current capital regulations (such as those based on Basel II accords) makes systemic financial crises larger and more costly.

The presence of agents who are subject to portfolio constraints may give rise to bubbles in equilibrium even though there are unconstrained agents in the economy who can exploit the induced arbitrage opportunity. The emergence of bubbles depends critically on the risk aversion distribution across agents and the severity of the constraint, two dimensions that determine the cost of the constraint in equilibrium.

We show that is straightforward to introduce risk constraints in settings with heterogeneous beliefs and multiple risky assets.

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with portfolio constraints which contain bubbles as, in particular, it shows that depending on the way the bubble is partitioned across the assets, the variations of key equilibrium quantities, such as stock volatilities, correlation and equity premia, can be substantial.
A Proofs

Proof of Proposition 4. The constrained agent solves the program

$$\sup_{c,\pi \in A(w_2)} E \left[ \int_0^\infty e^{-\rho t} \log (c_{2t}) dt \right],$$

subject to

$$\log(W_{2t}) = \log(W_{20}) + \int_0^t \left( r_s + \pi_s^\top \sigma_s \theta_s - \frac{1}{2} \left\| \sigma_s^\top \pi_s \right\|^2 - c_{2s} \right) ds + \int_0^t \pi_s^\top \sigma_s dB_s$$

where $A(w_2) = \{(\pi, c) : \pi \in \mathcal{C}_t$ and $W_{2t}^c \pi, c \geq 0$ for $t \in [0, \infty)\}$. Using the objective function and the budget constraint, the problem can be expressed as the maximization of

$$E \left[ \int_0^\infty e^{-\rho t} \left( \log (\alpha_t) + \log (W_{20}) + \int_0^t \left( r_s + \pi_s^\top \sigma_s \theta_s - \frac{1}{2} \left\| \sigma_s^\top \pi_s \right\|^2 - \alpha_s \right) ds \right) dt \right]$$

$$= E \left[ \int_0^\infty e^{-\rho t} \left( \int_0^\infty e^{-\rho s} ds \right) dt \right]$$

where we have used a consumption policy of the form $c_{2t} = \alpha_t W_{2t}$.

The problem is solved by a pointwise optimization of

$$\sup_{\alpha > 0} \{ \log (\alpha) - \rho^{-1} \alpha \},$$

which admits a unique solution given by $\alpha = \rho$, and the mean variance program

$$\sup_{\pi \in \mathcal{C}_t} \left\{ \pi^\top \sigma_t \theta_t - \frac{1}{2} \left\| \sigma_t^\top \pi_t \right\|^2 \right\}. \quad (47)$$

Since $\mathcal{C}_t$ is a closed convex subset of $\mathbb{R}^n$, the mean variance problem in (47) admits a unique solution given by

$$\sigma_t^\top \pi_t = \Pi \left[ \theta_t \mid \sigma_t^\top \mathcal{C}_t \right]. \quad (48)$$

where $\Pi$ denotes the projection operator, defined by $\Pi \left[ x \mid y \in \mathcal{J} \right] = \inf_{y \in \mathcal{J}} \frac{1}{2} \| y - x \|^2$.

We solve the mean variance program in (47) for our two examples:

(i) Mean variance constraint. Let $(a_1, a_2, a_3) = (-1, L, 0)$ and $x = \pi^\top \pi$. The Karush-Kuhn-Tucker (KKT) conditions of the projection problem are $- (\eta + 1) \theta + (1 + 2\eta L) x = 0, \eta \left[ x^\top \theta - L \| x \|^2 \right] = 0,$
with complementary slackness and $\eta \geq 0$. The problem is solved by

$$\eta = L^{-1} (L - 1)^+,$$

$$\sigma_t^\top \pi_t = \frac{1}{1 + (L - 1)^+} \theta_t,$$

$$\kappa = \frac{1}{1 + (L - 1)^+}.$$

(ii) **Volatility constraint.** Let $(a_1, a_2, a_3) = (0, 1, L^2)$. The KKT conditions of the projection problem are $-\theta + (1 + 2\eta) x = 0$, $\eta \left( \|x\|^2 - L^2 \right) = 0$, with complementary slackness and $\eta \geq 0$. The problem is solved by

$$\eta_t = \frac{1}{2} \left( \frac{\|\theta_t\|}{L} - 1 \right)^+, \quad \sigma_t^\top \pi_t = \kappa_t \theta_t, \quad \kappa_t = \frac{1}{1 + (\|\theta_t\|/L - 1)^+}.$$

**Mapping the result into the dual approach of Cvitanić and Karatzas (1992).** The agent’s problem is transformed into an unconstrained consumption and portfolio choice problem in a fictitious economy with a modified market price of risk and interest rate. Let $\beta_t : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be the support function of the set $-\mathcal{C}_t$, that is, the convex function defined by $\beta_t (\nu) = \sup \left\{ -\pi^\top \nu : \pi \in \mathcal{C}_t \right\}$, where $\mathcal{B}$ is its effective domain. The implicit state price density faced by agent 2 is given by

$$\xi_{2t} = e^{-\int_0^t (r_s + \beta_t(\nu_s) + \frac{1}{2} \|\theta_{2s}\|^2) ds - \int_0^t \theta_{2s}^\top dB_s}, \quad (49)$$

where $\theta_{2t} = \theta_t + \sigma_t^{-1} \nu_t$. Optimality conditions imply that $\nu$ is defined by the relation $\nu_t = \arg \min_{\nu \in \mathcal{B}_t} \left\{ \frac{1}{2} \|\theta_t + \sigma_t^{-1} \nu_t\|^2 + \beta_t (\nu) \right\}$, and the optimal consumption plan and trading strategy of the constrained agent are given by

$$c_{2t} = \frac{e^{-pt}}{y_2 \xi_{2t}}, \quad \pi_{2t} = (\sigma_t^{-1})^\top \theta_{2t},$$

thus, we have that the wealth process along the optimal path is given by $W_{2t} = \rho^{-1} c_{2t}$.

We recover the mean-variance program of the primal problem in (47) using the definition of the support function and Fenchel’s duality theorem (see Rockafellar 1970, Theorem 31.1) which imply that

$$\inf_{\nu \in \mathcal{B}_t} \left\{ \frac{1}{2} \|\theta_t + \sigma_t^{-1} \nu\|^2 + \beta_t (\nu) \right\} = \sup_{\pi \in \mathcal{C}_t} \left\{ \pi^\top \sigma_t \theta_t - \frac{1}{2} \|\sigma_t^\top \pi\|^2 \right\}, \quad (50)$$

as the conjugate functions of $f(\nu) = \frac{1}{2} \|\theta_t + \sigma_t^{-1} \nu\|^2$ and $g(\nu) = -\beta_t (\nu)$ are $f^*(\pi) = -\pi^\top \sigma_t \theta_t + \frac{1}{2} \|\sigma_t^\top \pi\|^2$ and $g^*(\pi) = 0$, for $\pi \in \mathcal{C}_t$, respectively.

Hiriart-Urruty and Lemaréchal (2001, Theorem 3.1.1), show that the projection operator satisfies

$$\left( \sigma_t^\top \varpi - \sigma_t^\top \pi_t \right)^\top \left( \sigma_t^\top \pi_t - \theta_t \right) \leq 0.$$

Taking the maximum on the left hand side gives $\max_{\varpi \in \mathcal{C}_t} \left\{ \left( \varpi - \pi_t \right)^\top y_t \right\} = 0$, where $y_t = \sigma_t \left( -\theta_t + \sigma_t^\top \pi_t \right)$. In conjunction with the definition of the support function, this implies that the vector $y_t \in \mathcal{B}_t$. Note
that the vector \( y_t \) attains the infimum on the left hand side of equation \( [50] \) and it follows that
\[
\nu_t = y_t = -(1 - \kappa_t) \sigma_t \theta_t.
\]

**Proof of Proposition 2.** We construct a consumption sharing rule, \( s_{2t} \), such that
\[
c_{1t} = (1 - s_{2t}) \delta_t
\]
and \( c_{2t} = s_{2t} \delta_t \) and whose dynamics follow
\[
ds_{2t} = s_{2t} \mu_{s_{2t}, t} dt + s_{2t} \sigma_{s_{2t}, t} dB_t.
\]

An application of Ito’s lemma to the process
\[
s_{2t} = c_{2t} / \delta_t = \rho W_{2t} / \delta_t,
\]
where \( W_{2t} \) is the wealth process of the constrained agent along the optimal path, with dynamics
\[
dW_{2t} / W_{2t} = (r_t + \kappa_t \theta_t^2 - \rho) dt + \kappa_t \theta_t dB_t,
\]
yields,
\[
\mu_{s_{2t}, t} = r_t + \sigma \delta \theta_t - \mu - (\kappa_t \theta_t - \sigma \delta) (\theta_t - \sigma \delta), \tag{51}
\]
\[
\sigma_{s_{2t}, t} = \kappa_t \theta_t - \sigma \delta. \tag{52}
\]

The first order condition of the unconstrained agent in equation \( [7] \) identifies the unconstrained agent’s state price density as
\[
\xi_1(t, s_{2t}, \delta_t) = e^{-\rho t} y_1^{-1} (1 - s_{2t})^{-\gamma \delta_t^{-\gamma}},
\]
and thus, an application of Ito’s lemma to this function identifies the market price of risk and the interest rate as functions of the drift and diffusion terms in the consumption share and the dividend dynamics,
\[
\theta_t = \gamma \left( \sigma \delta - \frac{s_{2t} \sigma_{s_{2t}, t}}{1 - s_{2t}} \right), \tag{53}
\]
\[
r_t = \rho + \gamma \mu \delta - \frac{1}{2} (1 + \gamma) \gamma \sigma^2 \delta - \gamma \frac{s_{2t} \mu_{s_{2t}, t}}{1 - s_{2t}} + \frac{2 \gamma^2 \sigma \delta (1 - s_{2t}) s_{2t} \sigma_{s_{2t}, t} - (1 + \gamma) \gamma s_{2t}^2 \sigma_{s_{2t}, t}^2}{2 (1 - s_{2t})^2}. \tag{54}
\]

Using equations \( [52] \) and \( [53] \) and the fact that the process \( \kappa_t \) is either constant or depends only on \( \theta_t \), we obtain a nonlinear equation for \( \theta \),
\[
\theta_t = \gamma \left( \sigma \delta - \frac{s_{2t} (\kappa_t \theta_t - \sigma \delta)}{1 - s_{2t}} \right) \tag{55}
\]

The solution of this problem is then used to express \( (r(\cdot), \mu_{s_{2t}(\cdot)}, \sigma_{s_{2t}(\cdot)}) \) as function of the consumption share only and correspond to equations in \( [18], [23] \) and \( [25] \), respectively.
The wealth processes are given by

\[ W_{1t} = E \left[ \int_t^\infty \frac{\xi_1(s, s_{2s}, \delta_s)}{\xi_1(t, s_{2t}, \delta_t)} (1 - s_{2s}) \delta_s ds \bigg| \mathcal{F}_t \right], \quad W_{2t} = \rho^{-1} s_{2t} \delta_t \]

and add up to the stock price due to market clearing,

\[ S_t = W_{1t} + W_{2t} = E \left[ \int_t^\infty \frac{\xi_1(s, s_{2s}, \delta_s)}{\xi_1(t, s_{2t}, \delta_t)} s_{2s} \delta_s ds \bigg| \mathcal{F}_t \right] + \rho^{-1} s_{2t} \delta_t - E \left[ \int_t^\infty \frac{\xi_1(s, s_{2s}, \delta_s)}{\xi_1(t, s_{2t}, \delta_t)} s_{2s} \delta_s ds \bigg| \mathcal{F}_t \right]. \tag{56} \]

The equation for the starting point of the consumption share process in (26) follows from using the value of the stock price in (56) and plugging the result in the definition of agent 2’s endowment, in (4). Finally, the lagrange multiplier of the unconstrained agent is set to \( y_1 = (1 - s_{20})^{-\gamma} \delta_0^{-\gamma} > 0 \).

**Proof of Proposition 3.** From the first order conditions of the representative agent’s problem

\[ u(c_t, \lambda_t) = \max_{c_{1t} + c_{2t} = c_t} \left\{ \frac{c_{1t}^{1-\gamma} - 1}{1 - \gamma} + \lambda_t \log c_{2t} \right\} \]

we obtain \( \xi_{1t} = e^{-\rho t} \frac{u_c(\delta_t, \lambda_t)}{u_c(\delta_0, \lambda_0)} \), where \( \lambda_t \) is an endogenous strictly positive process which represents the time varying weight of the constrained agent and is identified by the ratio of marginal utilities, \( \lambda_t = c_{2t}/c_{1t}^{\gamma} = s_{2t}(1 - s_{2t})^{-\gamma} \delta_t^{1-\gamma} \).

The bubble component follows from subtracting from the stock price its fundamental value and Fubini’s theorem,

\[ b_t = \rho^{-1} s_{2t} \delta_t - E \left[ \int_t^\infty \frac{\xi_1(s, s_{2s}, \delta_s)}{\xi_1(t, s_{2t}, \delta_t)} s_{2s} \delta_s ds \bigg| \mathcal{F}_t \right] \]

\[ = s_{2t} \delta_t E \left[ \int_t^\infty e^{-\rho(s-t)} \left( 1 - \frac{s_{2s}(1 - s_{2s})^{-\gamma} \delta_s^{1-\gamma}}{s_{2t}(1 - s_{2t})^{-\gamma} \delta_t^{1-\gamma}} \right) ds \bigg| \mathcal{F}_t \right] \]

\[ = \delta_t^{\gamma}(1 - s_{2t})^{-\gamma} \int_t^\infty e^{-\rho(s-t)} \left[ \lambda(s_{2t}, \delta_t) - E[\lambda(s_{2s}, \delta_s) | \mathcal{F}_t] \right] ds. \tag{59} \]

The agent’s optimal consumption plans must solve the representative agent’s utility maximization problem and it follows that \( c_{1t} = u_c(\delta_t, \lambda_t)^{-\gamma} = (1 - s_{2t}) \delta_t \), and \( c_{2t} = \frac{\lambda_t}{u_c(\delta_t, \lambda_t)} = s_{2t} \delta_t \), and the lagrange multipliers implied by the equilibrium are given by \( y_1 = u_c(\delta_0, \lambda_0), y_2 = \frac{u(\delta_0, \lambda_0)}{\lambda_0} \).

The equilibrium weighting process is thus given by \( \lambda_t = \lambda_0 \xi_{1t}/\xi_{2t} \), where \( \xi_{2t} \) is given in (49). Applying Ito’s lemma to \( \lambda \) gives

\[ \frac{d\lambda_t}{\lambda_t} = [\beta_t(\nu) + \theta_2t(\theta_2t - \theta_t)] dt + (\theta_2t - \theta_t) dB_t, \tag{60} \]
and from Proposition \(1\) \(\sigma_t \pi_t = \theta_{2t}, \theta_{2t} = \kappa_t \theta_t, \nu_t = -(1 - \kappa_t) \sigma_t \theta_t.\)

Replacing in the drift of equation (60)

\[
\beta_t (\nu) + \theta_{2t} (\theta_{2t} - \theta_t) = \kappa_t (1 - \kappa_t) (\sigma_t^{-1} \theta_t) \sigma_t \theta_t - \kappa_t (1 - \kappa_t) \theta \theta_t = 0
\]

which implies that \(\lambda\) is a nonnegative local martingale.

**Proof of Propositions 4 and 6.** Set \(I = (0, 1)\). (i) the drift and diffusion functions have continuous derivatives in \(I\) and (ii) \((s_2 \sigma_{s_2})^2 > 0\) in \(I\). We also verify a (iii) local integrability condition, for all \(x \in I\), there exists \(\epsilon > 0\) such that \(\int_{x-\epsilon}^{x+\epsilon} \frac{1+|\mu_{s_2}(y)|}{\sigma_{s_2}(y)^2} dy < \infty\). It is known that (i) implies that the coefficient are local Lipschitz, a sufficient condition for pathwise uniqueness of the solution (see Karatzas and Shreve (1988), Theorem 5.2.5). Also, conditions (i), (ii) and (iii) guarantee the existence of a weak solution (see Karatzas and Shreve (1988, Theorem 5.5.15) up possibly to an explosion time. The existence of a weak solution combined with pathwise uniqueness imply that equation (22) admits a strong solution up possibly to an explosion time, i.e., when \(s_{2t}\) hits one of the boundaries (the endpoints of \(I\)).

Define the stopping times \(T_\Delta = \inf\{t \geq 0 : s_{2t} \geq \Delta\}\), with \(\Delta \in I\), and let \(T_1 = \lim_{\Delta \uparrow 1} T_\Delta, T_0 = \lim_{\Delta \downarrow 0} T_\Delta\). To rule out explosions, we proceed as follows.

From equation (60), the weighting process is a nonnegative supermartingale under \(\mathbb{P}\),

\[
E[\lambda_T] \leq \lambda_0 > 0, \quad \forall T \in [0, \infty),
\]

consequently, it is a.s. finite under the objective probability measure, which implies

\[
\mathbb{P}[T_1 < T] = 0, \quad \forall T \in [0, \infty).
\]

Let \(\bar{s}_{2t} = s_{2,t \wedge T_\Delta}\). Using a comparison argument (see Proposition 5.2.18 in Karatzas and Shreve (1988)), we bound the stopped process \(s_{2t}\) from below by a process \(s_\ell\), with dynamics

\[
ds_{\ell t} = \mu_{s_\ell}(s_{\ell t}) s_{\ell t} dt + \sigma_{s_2}(s_{\ell t}) s_{\ell t} dB_t.
\]

This process, by construction, never reaches the left boundary and its diffusion is given by \(x \sigma_{s_2}(x)\). We fix \(s_{0\ell} = s_{20}\) and set the drift of \(s_\ell\) such that \(\mu_{s_{\ell}}(x) \leq \mu_{s_2}(x)\), which implies that \(\forall \Delta \in (0, 1)\),

\[
\mathbb{P}[T_0 < T_\Delta] = E[1\{T_0 < T_\Delta\}] = 0,
\]

that is, the probability of the consumption share process of hitting 0 before it reaches an arbitrary \(\Delta\) in \((0, 1)\) is zero.
In order to show that \( P[T_0 < T] = 0, \forall T \in [0, \infty) \), it suffices to note that
\[
E[\lim_{\Delta \to 1} 1_{\{T_0 < T_\Delta\}}] \leq \lim_{\Delta \to 1} E[1_{\{T_0 < T_\Delta\}}] = 0,
\]
which follows from Fatou’s lemma, and (61), implying \( P[T_0 < T] = 0, \forall T \in [0, \infty) \), since the probability of reaching 1 in finite time is zero a.s..

In order to close the proof, we need to find candidate processes \( s_{\ell} \) for each type of constraint.

(i) Mean variance constraint. The natural candidate for \( s_{\ell} \) corresponds to the consumption share process of a log agent in an unconstrained economy with risk aversion distribution \( \{\gamma, 1\} \), time discount rate \( \rho > 0 \) and a dividend process with parameters \( (\mu_{\ell}, \sigma_{\delta}) \). If parameters are such that utilities and price processes are finite, the consumption share process takes values in the set \((0, 1)\) if started in \((0, 1)\). One simple way to show it is by noticing that the process \( s_{\ell t}(1 - s_{\ell t})^{1 - \gamma} = \lambda_0 \delta_{\ell}^{\gamma - 1} \) takes values in the set \((0, \infty)\) a.s.. The dynamics of \( s_{\ell} \) are given by
\[
\frac{ds_{\ell \delta}}{s_{\ell \delta}} = \mu_{s_\ell}(s_{\ell \delta})s_{\ell \delta}dt + \left[ \frac{\gamma \delta}{1 + (\gamma - 1)s_{\ell \delta}} \sigma_{\delta} - \sigma_{\delta} \right] s_{\ell \delta}dB_t,
\]
with
\[
\mu_{s_\ell}(x) = \mu_{s_2}(x) = \frac{\gamma(1 - x)(1 - \kappa)}{[1 + (\gamma - 1)x][1 + (\gamma - 1)x]} \left[ \mu_{\ell}\delta - \frac{\sigma_{\delta}^2}{2[1 + (\gamma - 1)x]} \left( 1 + \gamma - \frac{\gamma \delta}{1 + (\gamma - 1)x} \right) \right].
\]
The processes \((s_{\ell}, s_2)\) share the same functional form in the diffusion term, and have different drifts. The dynamics of \( s_{\ell} \) do not depend on the time discount parameter \( \rho_{\ell} \), thus we can choose an arbitrary pair \((\mu_{\ell}\delta, \rho_{\ell})\) such that it represents a viable unconstrained economy, and \( \mu_{s_2}(x) \geq \mu_{s_\ell}(x) \).

(ii) Volatility constraint. The process \( s_{\ell} \) is a geometric Brownian motion with dynamics
\[
\frac{ds_{\ell \delta}}{s_{\ell \delta}} = \mu_{s_\ell}(s_{\ell \delta})s_{\ell \delta}dt - (1 - \varepsilon)\sigma_{\delta}s_{\ell \delta}dB_t,
\]
such that \( \mu_{s_2}(x) \geq \mu_{s_\ell} \).

\( \square \)

Proof of Proposition 5. Since Lemma 1 shows that the ratio of marginal unities, \( \lambda_t = c_{2t}/c_{1t} = s_{2t}(1 - s_{2t})^{-\gamma}\delta_{t}^{1 - \gamma} \) is a true martingale when the agent is subject to a mean variance constraint, the bubble term in (29) is zero.

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This implies that the price dividend ratio can be written as
\[
p(s_2t) = E \left[ \int_t^\infty e^{-\int_u^s (r_u + \theta_u \sigma_\delta - \mu_\delta - \frac{1}{2}(\theta_u - \sigma_\delta)^2) du - \int_u^s (\theta_u - \sigma_\delta) dB_u} ds \right] \bigg| \mathcal{F}_t
\]
\[
= E \left[ \int_t^\infty e^{-\int_u^s (r_u + \theta_u \sigma_\delta - \mu_\delta) du} \frac{M_s}{M_t} ds \right] \bigg| \mathcal{F}_t
\]
\[
= \bar{E} \left[ \int_t^\infty e^{-\int_u^s (r_u + \theta_u \sigma_\delta - \mu_\delta) du} ds \right] \bigg| \mathcal{F}_t
\]
(64)
\[
= \bar{E} \left[ \int_t^\infty e^{-\int_u^s (r_u + \theta_u \sigma_\delta)} du \right] \bigg| \mathcal{F}_t
\]
(65)

where the density of the measure \( \bar{P} \) is defined by the exponential martingale
\[
\frac{d\bar{P}}{dP} \bigg| \mathcal{F}_t = \bar{M}_t = e^{-\frac{1}{2} \int_0^t (\theta_s - \sigma_\delta)^2 ds - \int_0^t (\theta_s - \sigma_\delta) dB_s}.
\]

**Proof of Proposition 7.** Using the portfolio choice of the constrained agent in (13) in the market price of risk in equation (17) gives,
\[
\theta = \left[ 1 - \left( \frac{1}{1 + (|\theta|/\varepsilon \sigma_\delta - 1)^+} \right) R(s_2) s_2 \right]^{-1} R(s_2) \sigma_\delta
\]
(66)
which is uniquely solved by the positive, continuous and piecewise differentiable function
\[
\theta(s_2t) = \gamma \sigma_\delta \frac{1 - \varepsilon s_2t}{1 - s_2t} 1\{R(s_2t) \geq \varepsilon\} + \frac{\gamma \sigma_\delta}{1 - (\gamma - 1) s_2t} 1\{R(s_2t) < \varepsilon\}.
\]

**Proof of Proposition 8.** The proof follows from Lemma 2 and Novikov’s condition.

**Proof of Proposition 9.** The optimal consumption plan and trading strategy of the constrained agent are given by
\[
c_{2t} = e^{-\rho t} \frac{\eta_t}{y_2 \xi_{2t}}, \quad \sigma_{t\pi_{2t}} = \theta_{2t} - \bar{\mu}_t.
\]

(67)

Optimality of the constrained agent portfolio choice follows from solving
\[
\sup_{\pi \in C_t} \left\{ \pi \sigma_t (\theta_t - \bar{\mu}_t) - \frac{1}{2} (\sigma_t \pi)^2 \right\} ,
\]
where \( C_t = \{ \pi : (\sigma_t \pi)^2 \leq L^2 \} \).

Applying Ito’s lemma to the definition of the stochastic weight in equilibrium, \( \lambda_t = \lambda_0 \eta_t \xi_{1t}/\xi_{2t} \), gives
\[
\frac{d\lambda_t}{\lambda_t} = [(\theta_{2t} - \bar{\mu}_t) (\theta_{2t} - \theta_t)] dt + (\theta_{2t} - \bar{\mu}_t - \theta_t) dB_t,
\]
(68)

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with \( \beta_t(\nu) + (\theta_{2t} - \bar{\mu}_t)(\theta_{2t} - \theta_t) = 0 \), and hence, the process \( \lambda_t \) is a local martingale.

An application of Itô’s lemma to the state price density \( \xi_{1t} = e^{-\rho t} \frac{u_t(\delta_t, \lambda_t)}{u_t(0, \lambda_0)} \) identifies the market price of risk as \( \theta_t = R_t(\sigma_s - s_{2t} \Phi_t) \) and the interest rate as in (18), whereas the dynamics of the weighting process shows that \( \Phi_t = \theta_{2t} - \bar{\mu}_t - \theta_t \).

On the other hand, from the portfolio policy of agent 2 one gets

\[
\sigma_t \pi_{2t} = \Pi [\theta_t - \bar{\mu}_{2t} | \sigma_t C_t] = \frac{1}{1 + (|\theta_t - \bar{\mu}/\varepsilon\sigma_\delta| - 1)^+} (\theta_t - \bar{\mu}_t)
\]

where we have set \( L = \varepsilon\sigma_\delta \), with \( \varepsilon \geq 0 \), and therefore, putting all together gives a nonlinear equation for the market price of risk process,

\[
\theta = R(s_2)\sigma_\delta - R(s_2)s_2 \left[ \frac{1}{1 + (|\theta - \bar{\mu}|/\varepsilon\sigma_\delta - 1)^+} (\theta - \bar{\mu}) - \theta \right].
\]

The solution to this equation is uniquely given by the continuous function

\[
\theta(s_{2t}) = R(s_{2t})(\sigma_\delta + s_{2t}\bar{\mu})1_{\{s_{2t}\in S_u\}} + \gamma\sigma_\delta \frac{1 - \varepsilon s_{2t}}{1 - s_{2t}} 1_{\{s_{2t}\in S_1\}} + \gamma\sigma_\delta \frac{1 + \varepsilon s_{2t}}{1 - s_{2t}} 1_{\{s_{2t}\in S_2\}},
\]

where \( S_u = \{ x \in (0, 1) : |R(x)(\sigma_\delta + x\bar{\mu}) - \bar{\mu}| \leq \varepsilon\sigma_\delta \} \), \( S_1 = \{ x \in (0, 1) : R(x)(\sigma_\delta + x\bar{\mu}) - \bar{\mu} > \varepsilon\sigma_\delta \} \), and \( S_2 = \{ x \in (0, 1) : -R(x)(\sigma_\delta + x\bar{\mu}) + \bar{\mu} > \varepsilon\sigma_\delta \} \).

The dynamics of the consumption share process obeys

\[
ds_{2t} = s_{2t}\mu_{s_2}(s_{2t})dt + s_{2t}\sigma_{s_2}(s_{2t})dB_t
\]

where

\[
\mu_{s_2}(s_{2t}) = f(s_{2t}) - \rho + (\Phi(s_{2t}) + \theta(s_{2t}) + \sigma_\delta)(\theta(s_{2t}) - \sigma_\delta),
\]

\[
\Phi(s_{2t}) = -\bar{\mu} 1_{\{s_{2t}\in S_u\}} + \left[ \varepsilon\sigma_\delta - \gamma\sigma_\delta \frac{1 - \varepsilon s_{2t}}{1 - s_{2t}} \right] 1_{\{s_{2t}\in S_1\}} - \left[ \varepsilon\sigma_\delta + \gamma\sigma_\delta \frac{1 + \varepsilon s_{2t}}{1 - s_{2t}} \right] 1_{\{s_{2t}\in S_2\}},
\]

\[
\sigma_{s_2}(s_{2t}) = \Phi(s_{2t}) + \theta(s_{2t}) - \sigma_\delta
\]

\[
= -\left( 1 - s_{2t} \right) \bar{\mu} + \left( 1 - \gamma \right) \sigma_\delta 1_{\{s_{2t}\in S_u\}} - \left( 1 - \varepsilon \right) \sigma_\delta 1_{\{s_{2t}\in S_1\}} - \left( 1 + \varepsilon \right) \sigma_\delta 1_{\{s_{2t}\in S_2\}}.
\]

Following an argument similar to the one in the proof of propositions 4 and 5, it can be shown that this process, if started in \((0, 1)\), does not reach either zero or one in finite time. \( \square \)

**Proof of Proposition 10.** We explicitly solve the individual agent’s problem for \( n \) risky assets in the proof of Proposition 1.

Following an argument similar to the one used in the proof of Proposition 2 and 3, we construct
a suitable consumption sharing rule and identify the state price density from agent 1’s optimality condition. We assume that the volatility matrix is invertible at all times, and hence, market are dynamically complete for the unconstrained agent.

The market prices of risk are the solution to a system of nonlinear equations of the form

\[ \theta_t = R(s_{2t}) \left( \sigma_\delta + s_{2t} (1 - \kappa_t) \theta_t \right), \tag{69} \]

which once solved, allows us to identify all equilibrium quantities as functions of the consumption share. Note that the solution to the above equation is particularly simple for the mean-variance constraint.

The ratio of marginal utilities is given by \( \lambda_t = c_{2t}/c_{1t}^\gamma \), applying Ito’s lemma to \( \lambda \) gives

\[ \frac{d\lambda_t}{\lambda_t} = - (1 - \kappa(s_{2t})) \theta(s_{2t})^\top dB_t, \tag{70} \]

which implies that \( \lambda \) is a local martingale, and thus, the test for the presence of bubbles in the market portfolio is performed exactly as in the one risky asset case. The starting point and dynamics of the consumption share of the constraint agent is obtained using the same procedure of Proposition 2. We omit the details.

B Testing for the presence of bubbles

Lemma 1. (Martingality of \( \lambda_t \) with constant risk reduction) The equation

\[ \lambda_t = \lambda_0 - \int_0^t \lambda_s \frac{(1 - \kappa(s_{2t}))\gamma \sigma_\delta}{1 + (\gamma \kappa - 1)s_{2s}} dB_s, \tag{71} \]

with \( \kappa \in (0, 1] \), admits a unique and strong solution and is a strictly positive martingale.

Proof. The volatility of the logarithm of the weighting process given by function

\[ \Phi(x) = - \frac{(1 - \kappa)\gamma \sigma_\delta}{1 + (\gamma \kappa - 1)x} \tag{72} \]

which is continuous and bounded in \([0, 1]\), for \( \kappa > 0 \). The existence of a unique strong solution follows from Theorem V.6 in Protter (2004), in particular, note that the diffusion term \( \lambda \Phi(\cdot) \) is (random) Lipschitz. From Proposition 2 the process \( \lambda \) is a local martingale. To establish that \( \lambda \) is a strictly positive martingale, it suffices to show the process as an exponential martingale with initial datum \( \lambda_0 = s_{20}(1 - s_{20})^{-\gamma} \delta_0^{1-\gamma} > 0 \) and verify the Novikov condition, which follows from the boundedness of \( \Phi(\cdot) \).
Lemma 2. (Martingality of $\lambda_t$ with time varying risk reduction when constraints are always binding) The equation

$$\lambda_t = \lambda_0 + \int_0^t \lambda_s \left[ \varepsilon \sigma_\delta - \gamma \sigma_\delta \frac{1 - \varepsilon s_2}{1 - s_2} \right] dB_s,$$  \hspace{1cm} (73)

with $\varepsilon \in [0,1)$ and $\sigma_\delta > 0$, admits a unique and strong solution, and is a positive local martingale but fails to be a true martingale when $\gamma \geq 1$. If $\gamma < 1$ and $(1 - \gamma) \left( \mu_\delta - \frac{1}{2} \gamma \sigma_\delta^2 \right) - \frac{1}{2} (1 - \varepsilon)^2 \sigma_\delta^2 > 0$, the process is a true martingale.

Proof. We use the exponential local martingale

$$M^\lambda_t = \frac{\lambda_t}{\lambda_0} = e^{-\int_0^t \frac{1}{2} \Phi(s_2^t) ds + \int_0^t \Phi(s_2^t) dB_t},$$

where

$$\Phi(x) = \varepsilon \sigma_\delta - \gamma \sigma_\delta \frac{1 - \varepsilon x}{1 - x}$$  \hspace{1cm} (74)

as the density of a candidate equivalent change of measure $P^\lambda$. We verify the properties of the consumption share process, whose dynamics under $P^\lambda$ follows

$$ds_2^t = \mu^\lambda_{s_2}(s_2^t) s_2^t dt - (1 - \varepsilon) \sigma_\delta s_2^t dB^\lambda_t$$  \hspace{1cm} (75)

with

$$\mu^\lambda_{s_2}(x) = (\gamma - 1) \frac{1 - x}{1 + (\gamma - 1)x} \mu_\delta + (\gamma - 1) \gamma \frac{2 - \varepsilon(2 - \varepsilon)x}{2(1 - x)[1 + (\gamma - 1)x]} \sigma_\delta^2 + (1 - \varepsilon)^2 \sigma_\delta^2,$$  \hspace{1cm} (76)

where $dB^\lambda_t = dB_t - \Phi(s_2^t) dt$ is a $P^\lambda$- Brownian motion. In equilibrium, the consumption share lives in $(0, 1)$, therefore, if under $P^\lambda$ the process hits one of the boundaries with positive probability, $P^\lambda$ could not be equivalent to $P$ because this behavior is not possible under the objective measure. As shown by Heston, Loewenstein and Willard (2007, Theorem A.1.), this is equivalent to testing for the martingality of $\lambda$.

When $\gamma = 1$, the process under $P^\lambda$ corresponds to a geometric Brownian motion, and hence, it reaches 1 in finite time with positive probability, $P^\lambda[T_1 < T] > 0$. This contradicts the equivalence between $P$ and $P^\lambda$, and thus, the solution in equation (73) is a strictly positive local martingale but fails to be martingale.

When $\gamma > 1$, the drift diverges to plus infinity when $s_2 = 1$ (note that the numerator in the second term is positive as $s_2 \rightarrow 1$), therefore by using a standard comparison argument (the drift can be bounded from below using a linear function), we get $P^\lambda[T_1 < T] > 0$. This contradicts the equivalence between $P$ and $P^\lambda$, and thus, the solution in equation (73) is a strictly positive local martingale but fails to be martingale.

When $\gamma < 1$, unlike the previous cases, the behavior of the process under $P^\lambda$ resembles its
behavior under $\mathbb{P}$, that is, its drift diverges to negative infinity when $s_2 \to 1$.

In order to obtain information about the behavior of the consumption share as it approaches 1, we construct its scale function, $S(x)$,

$$S(x) = \int_c^x \exp \left[ -2 \int_c^y \frac{z \mu_2(z)}{z^2 \sigma_2^2(z)} \, dz \right] \, dy$$

$$= \frac{(1-x)x}{b_3 - 1} \left[ 1 - x \right]^{(1+\gamma)} \frac{x^{b_3-1}}{c^b} \left[ \frac{1 + (\gamma - 1)x}{1 + (\gamma - 1)c} \right]^{-b_2} F_1\left[ 1, b_1, b_2, b_3, x, \frac{\gamma x}{1 + (\gamma - 1)x} \right]$$

$$- \frac{(1-c)c}{b_3 - 1} F_1\left[ 1, b_1, b_2, b_3, c, \frac{\gamma c}{1 + (\gamma - 1)c} \right]$$

where $c$ is an arbitrary constant in $I$ and $F_1(\cdot)$ denotes the (Appell) hypergeometric function of two variables$^{31}$ with $b_1 = \frac{2 \mu_3 - \gamma [2 - \epsilon(2 - \epsilon)] \sigma_2^2}{(1-\epsilon) \sigma_2^2}$, $b_2 = -2 \gamma \frac{\mu_3 - \frac{1}{2} \gamma \sigma_2^2}{(1-\epsilon) \sigma_2^2} - 1$, $b_3 = 2(1 - \gamma) \frac{\mu_3 - \frac{1}{2} \gamma \sigma_2^2}{(1-\epsilon) \sigma_2^2}$.

When $b_3 - 1 > 0$ or equivalently$^{32}$

$$(1 - \gamma) \left( \mu_3 - \frac{1}{2} \gamma \sigma_2^2 \right) - \frac{1}{2} (1 - \epsilon)^2 \sigma_2^2 > 0, \quad (77)$$

we obtain $\lim_{\Delta \downarrow 1} S(\Delta) = \infty$, which is a sufficient condition that guarantees that starting from any point in $I$, the right boundary cannot be reached in finite time, i.e., $\mathbb{P}^\lambda[T_1 < T] = 0$ (see Karatzas and Shreve (1988), p. 348).

In order to ensure that the share process does not reach 0 given that it doesn’t reach 1 a.s., we use a comparison argument similar to that of Proposition$^5$ we omit the details.

\section*{C Examples}

\subsection*{C.1 One risky asset: boundary value problems with homogeneous beliefs}

When the constraints are always binding, the coefficients in equation$^{32}$ are given by:

(i) Mean variance constraint.

$$f(x) = \rho + (\gamma - 1) \frac{1 - x}{1 + (\gamma - 1)x} \mu_3 + \frac{\gamma}{1 + (\gamma - 1)x} \sigma_2^2 - \frac{\gamma [1 + \gamma + (\gamma - 1)(1 + 2\gamma)x]}{2[1 + (\gamma - 1)x]^3} \sigma_2^2$$

$$+ (\kappa - 1)(\gamma - 1)^2 \frac{(1 - x)x^2[\gamma + \gamma \kappa - 2 + 2(\gamma - 1)(\gamma \kappa - 1)x]}{2[1 + (\gamma - 1)x]^3[1 + (\gamma \kappa - 1)x]^2} \sigma_2^2,$$

$$\sigma_{s_2}(x) = \left[ \frac{\gamma \kappa}{1 + (\gamma \kappa - 1)x} - 1 \right] \sigma_2.$$

$^{31}$see Whittaker and Watson (1990), Ex. 22, p. 300. and http://functions.wolfram.com/HypergeometricFunctions/AppellF1/

$^{32}$This parametric condition also allows us to write $F_1$ with an integral representation on the real axis.
(ii) Volatility constraint. 

\[
f(x) = \rho + (\gamma - 1) \frac{1 - x}{1 + (\gamma - 1) x} \mu - (\gamma - 1) \frac{2 - \varepsilon (2 - \varepsilon) x}{2 (1 - x) [1 + (\gamma - 1) x]} \sigma^2,
\]

\[
\sigma_s(x) = -(1 - \varepsilon) \sigma_x.
\]

C.2 n risky assets: cash flow model and approximation method

Cash flow model. The share \( x_{it} \) follows the mean reverting process \(^{33}\)

\[
dx_{it} = \eta (\bar{x}_i - x_{it}) dt + x_{it} \sigma_x^i (x_t) dB_t,
\]

where \( \bar{x}_i \in (0, 1) \) represents the long run mean, \( \eta > 0 \) is the speed of mean reversion, and the term \( \sigma_x^i (x_t) \) in the volatility is determined by

\[
\sigma_x^i (x_t) = v_i - x^T_t v = v^i - \sum_{j=1}^n x_{jt} v^j,
\]

with \( x_t = \begin{bmatrix} x_{1t} \\ \vdots \\ x_{nt} \end{bmatrix}, v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}, v_i = [v^i_1 \ldots v^i_n] \).

The term \( \sigma_x^i (x_t) \) ensures that \( \sum_{i=1}^n x_{it} = 1 \), while the mean reverting drift implies that no asset will dominate the economy. Note that the process \( x \) lies on the unit simplex at all times. By applying Ito’s lemma to \( \delta^i = \delta t x_{it} \), the dynamics of the dividends of asset \( i \) follow

\[
d\delta^i_t = \left[ \mu - \eta \left( \frac{x_i}{\bar{x}_i} - 1 \right) + \theta^C_F - x^T_t \theta^C_F \right] dt + [\sigma_x^i + \sigma_x^i (x_t)] dB_t,
\]

where \( \theta^C_F = v^i \sigma_x^i \).

Approximation method. One equilibrium that can be easily characterized is the no bubble equilibrium with mean-variance constraints and \( \kappa = 1/\gamma \), with \( \gamma > 1 \). Using the fact that the consumption share process is deterministic and applying Fubini’s theorem, stock prices can written as

\[
S^i_t = s_2 t \delta_t \int_t^\infty e^{-\rho(s-t)} s_2^{-1} E^\lambda \left[ x_{is} \mid F_t \right] ds \tag{78}
\]

where \( E^\lambda \) is the expected value operator under the probability measure whose density is defined by the exponential martingale \( \lambda \theta_0 = e^{-\frac{1}{2} \| \Phi \|^2 + \Phi^T B_t} \) with \( \Phi = (1 - \gamma) \sigma_x \in \mathbb{R}^n \).

The value of the consumption share at time \( t \) can be obtained by inverting the price dividend ratio function of the baseline case model. The expected value in (78) solves a linear PDE that can be approached numerically, however, we describe a polynomial approximation that performs well when compared to Monte Carlo simulations. We describe the method for two assets, which can be generalized to \( n > 2 \). In the example, the matrix \( v \), which contains the row vectors \( v^i \), is a diagonal matrix with \( ii \)-element given by \( v^i \).

The term \( E^\lambda \left[ x_{t+\tau} \mid F_t \right] \) is computed by adapting a method suggested in Gabaix (2008) to approximate arbitrary processes with linearity-generating processes, which fits here nicely as powers

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\(^{33}\)The use of share process to distribute the aggregate endowment across firms was first introduced in Menzly, Santos and Veronesi (2004).
of $x$ can be ordered decreasingly up to arbitrary order. We use a third order polynomial to approximate the expected value. Let $Y = (y_1, y_2, y_3)^\top$, where $y_\ell = x^\ell$. Setting $\gamma = 2$, $y_5 = y_6 = 0$ and $\psi = \theta_{CF}^1 - \theta_{CF}^2$, and using the slaving principle (see Gabaix (2008)), an application of Ito's lemma shows that $Y$ follows approximately $dY_t = (A_0 + A_1 Y_t) dt + \Sigma(Y_t) dB^\lambda_t$, with

$$A_0 = \begin{bmatrix} \eta \bar{x} & 0 & 0 \end{bmatrix}^\top,$$

$$A_1 = \begin{bmatrix} - (\eta + \psi) & \psi & 0 \\ 2 \eta \bar{x} & -2 \left( \eta + \psi - \frac{1}{2} \|v\|^2 \right) & 2 \left( \psi - \|v\|^2 \right) + \frac{\eta \bar{x} \|v\|^2}{(\eta + \psi - \frac{1}{2} \|v\|^2)} \\ 0 & 3 \eta \bar{x} & -3 \left( \eta + \psi - \|v\|^2 \right) - \left( \psi - 2 \|v\|^2 \right) \frac{\eta \bar{x}}{(\eta + \psi - \frac{1}{2} \|v\|^2)} \end{bmatrix}.$$ 

The approximate dynamics of $Y$ imply $E^\lambda [Y_{t+\tau} | \mathcal{F}_t] = \Psi (\tau) Y_t + \int_0^\tau \Psi (\tau - s) A_0 ds$, where $\Psi (\tau)$, a $3 \times 3$ matrix, solves the homogeneous linear system with constant coefficients $\frac{d\Psi (\tau)}{d\tau} = A_1 \Psi (\tau)$, $\Psi (0) = I_{3 \times 3}$. If $A_1$ has real and distinct eigenvalues $\varpi_i$, then $\Psi (\tau) = U e^{\varpi_i \tau} U^{-1}$, where $\varpi$ is the vector of eigenvalues, $[e^{\varpi_i \tau}]$ is a diagonal matrix with the $ii-$element given by $e^{\varpi_i \tau}$ and $U$ is the matrix of associated eigenvectors (the three linearly independent eigenvectors of $A_1$ are set up as columns).

The expected value is computed as $E^\lambda [x_{t+\tau} | \mathcal{F}_t] = E^\lambda [e_1 Y_{t+\tau} | \mathcal{F}_t]$ where $e_1 = (1, 0, 0)$. 

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References


Figure 1: The figure plots the market price of risk and the interest rate in an economy with mean-variance constraints. Parameters are set to $\mu = 0.02$ and $\sigma = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The solid line corresponds to an unconstrained economy.

Figure 2: The figure plots the drift and diffusion of the consumption share of the constrained agent, $s_2$, in an economy with mean-variance constraint. Parameters are set to $\mu = 0.02$ and $\sigma = 0.05$, left panels $\gamma = 2$, right panels $\gamma = 1/2$, and $\rho = 0.03$. The consumption share is a deterministic process when $\kappa = \gamma^{-1}$. The solid line corresponds to an unconstrained economy.
Figure 3: Levels in economies with heterogeneous agents. The equilibrium quantities represented with dashed lines share the same ‘effective’ risk aversion distribution. The constrained economy corresponds to the dashed black line with constant risk reduction set at $\kappa = 1/4$ and risk aversion distribution $\{2, 1\}$. The solid line corresponds to an unconstrained economy. The risk aversion distribution in the different (unconstrained) economies is $\{2, 4\}$ (dashed green), $\{1/2, 1\}$ (dashed blue). Parameters are set at $\mu_\delta = 0.05$, $\sigma_\delta = 0.1$, and $\rho = 0.03$.

Figure 4: The figure plots the portfolio policy of the constrained agent, $\pi_2$, in an economy with mean-variance constraints. Parameters are set to $\mu_\delta = 0.02$ and $\sigma_\delta = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The fraction of wealth invested in the risky asset decreases with the severity of the constraint. The solid line corresponds to an unconstrained economy. When $\kappa = 1/2$, agents find optimal to trade only in the stock and the optimal consumption policy is a deterministic proportion of the aggregate endowment. The market price of risk is given by the constant $\theta = \gamma \sigma_\delta$ and the volatility of the stock price equals its fundamental term.
Figure 5: The figure plots the volatility in an economy with mean-variance constraints. The left panel shows how the volatility decreases as the dispersion in ‘effective’ risk aversion is narrowed. The right panel shows that the volatility increases as the dispersion in ‘effective’ risk aversion increases. The volatility attains its maximum when the second agent has a larger share of aggregate consumption. Parameters are set to $\mu = 0.02$ and $\sigma = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The solid line corresponds to an unconstrained economy.

Figure 6: The figure plots the market leverage ratio in an economy with mean-variance constraints. The right panel shows that the market leverage increases as the dispersion in ‘effective’ risk aversion increases. An increase in the market leverage ratio is coupled with an increase in the interest rate volatility. Parameters are set to $\mu = 0.02$ and $\sigma = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The solid line corresponds to an unconstrained economy.
This limiting behavior also generates a large asymmetry in the discount rate and the interest rate diverges to minus infinity as the consumption share approaches one if the constraint is active.

The figures resemble figures 1 and 2 in Basak and Cuoco (1998), as the market price of risk diverges to plus infinity.

Figure 8: The figure plots the market price of risk and the interest rate in an economy with volatility constraints. The figures resemble figures 1 and 2 in Basak and Cuoco (1998), as the market price of risk diverges to plus infinity and the interest rate diverges to minus infinity as the consumption share approaches one if the constraint is active. This limiting behavior also generates a large asymmetry in the discount rate \( f(\cdot) \). Parameters are set to \( \mu_s = 0.02 \) and \( \sigma_s = 0.05 \), left panel \( \gamma = 2 \), right panel \( \gamma = 1/2 \), and \( \rho = 0.03 \). The solid line corresponds to an unconstrained economy.
Figure 9: The figure plots the drift and diffusion of the consumption share of the constrained agent, $s_2$, in an economy with volatility constraints. The drift diverges to minus infinity as the consumption share approaches one if the constraint is active. Parameters are set to $\mu_\delta = 0.02$ and $\sigma_\delta = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The solid line corresponds to an unconstrained economy.

Figure 10: The figure plots the portfolio policy of the constrained agent, $\pi_2$, in an economy with volatility constraints. The fraction of wealth invested in the risky asset decreases with the severity of the constraint. Parameters are set to $\mu_\delta = 0.02$ and $\sigma_\delta = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The solid line corresponds to an unconstrained economy.
Figure 11: The figure plots the volatility of returns in an economy with volatility constraints. The left panel shows how the volatility decreases as the dispersion in ‘effective’ risk aversion is narrowed. Parameters are set to $\mu_\delta = 0.02$ and $\sigma_\delta = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The solid line corresponds to an unconstrained economy.

Figure 12: The figure plots the market leverage ratio in an economy with volatility constraints. The constraint generates an increase in market leverage that is coupled with an increase in the volatility of the interest rate. Parameters are set to $\mu_\delta = 0.02$ and $\sigma_\delta = 0.05$, left panel $\gamma = 2$, right panel $\gamma = 1/2$, and $\rho = 0.03$. The solid line corresponds to an unconstrained economy.
Figure 13: Price dividend ratio, volatility of returns and risk premium in Basak and Cuoco (1998) restricted market participation model ($\varepsilon = 0$). Parameters are set to $\mu_s = 0.02$ and $\sigma_s = 0.05$, and $\rho = 0.03$. 