Measuring and allocating systemic risk

Prof. Patrick CHERIDITO
Princeton University

Abstract

This paper develops a framework for measuring, allocating and managing systemic risk. SystRisk, our measure of total systemic risk captures the a priori cost to society for providing tail-risk insurance to the financial system. Our allocation principle distributes the total systemic risk among individual institutions according to their size-shifted marginal contributions. To describe economic shocks and systemic feedback effects we propose a reduced form stochastic model that can be calibrated to historical data. We also discuss systemic risk limits, systemic risk charges and a cap and trade system for systemic risk.

Friday, February 14, 2014, 10:30-12:00
Room 126, Extranef building at the University of Lausanne
Measuring and allocating systemic risk

Markus K. Brunnermeier and Patrick Cheridito*
Princeton University
December 2013

Abstract

This paper develops a framework for measuring, allocating and managing systemic risk. SystRisk, our measure of total systemic risk captures the a priori cost to society for providing tail-risk insurance to the financial system. Our allocation principle distributes the total systemic risk among individual institutions according to their size-shifted marginal contributions. To describe economic shocks and systemic feedback effects we propose a reduced form stochastic model that can be calibrated to historical data. We also discuss systemic risk limits, systemic risk charges and a cap and trade system for systemic risk.

Keywords: Systemic risk measure, systemic risk allocation, feedback effects, shadow prices, systemic risk limits, systemic risk charges, cap and trade.

1 Introduction

The purpose of this paper is to develop a framework for measuring, allocating and managing systemic risk. Financial services play an important role in modern free market economies. Therefore, governments often do not have a choice but to provide support to failing financial institutions to protect the broader economy. We address this issue by studying the following two questions: (i) how to measure the total systemic risk generated by the financial sector and (ii) how to allocate the total systemic risk to individual financial institutions?

We view possible government support of financial institutions as an externality and measure total systemic risk by determining its a priori cost to society. To allocate the total systemic risk among individual institutions we propose to use their marginal contributions. To describe economic shocks, spillover, amplification and adverse feedback effects we develop a reduced form stochastic model that can be calibrated to historical data.

The most important features of our approach are:

*We thank Anton Korinek, Hyun Shin as well as participants of the 7th Oxford-Princeton Workshop on Financial Mathematics & Stochastic Analysis and the 9th Cambridge-Princeton Workshop for helpful comments. Financial support from NSF Grant DMS-0642361 is gratefully acknowledged.
• It views financial institutions as parts of the financial system. For instance, a bank that behaves as part of a herd is allocated more systemic risk than a bank that acts independently of the rest of the financial sector.

• It relates the financial industry to the real economy. As a consequence, costs of externalities grow disproportionally if they become large compared to a country’s GDP. Moreover, negative externalities that happen in states of the world where the overall economy is strong cost less than those that happen when the economy is weak.

• It satisfies the so called clone property. That is, if a financial institution is split into \( n \) equal parts, the total systemic risk does not change and the risk attributed to the institution also splits into \( n \) equal parts.

• It can detect risks that might not be fully reflected in market quotes of traded securities since it considers scenarios in which taxpayers provide support to financial institutions resulting in distorted market prices.

• It can detect systemic risk in low volatility environments in which the risk of large economic shocks is low but the financial system is prone to negative feedback spirals in case a crisis erupts.

• It is based on a tolerance parameter that can be adjusted over time so as to implement countercyclical regulation.

During financial crises governments might have to support certain parts of the financial system to maintain a properly functioning economy. Whether this happens through loans, restructuring or nationalization, it all costs the taxpayer money. SystRisk, our measure of total systemic risk, quantifies the a priori cost of this externality to society. It has the appealing feature that it grows faster than linearly as the exposures become large compared to the real economy. It also gives more weight to losses occurring in states of the world in which the overall economy is depressed. We distribute the total systemic risk among the components of the system based on their marginal contributions and provide a formula that expresses them as expectations with respect to a shadow pricing measure. In order to ensure that the risk contributions add up to total systemic risk, we shift the marginal contributions by parameters characterizing an institution’s size such as the number of its employees, the amount of outstanding credit, or the total corporate tax payed in the previous year. Our allocation principle attributes systemic risk to individual entities according to the role they play in the financial sector. If a financial firm is part of a herd or exposed to spillover effects, it is assigned more systemic risk than one that is more independent and better prepared to withstand a financial crisis.

Systemic risk can build up during economic booms but only materializes when a crisis erupts. If not taken into account by regulation, this can lead to the situation that a financial system is more vulnerable when observed volatility is low, a phenomenon coined as “volatility paradox” in Brunnermeier and Sannikov (2012). The reason for this is that in a low volatility environment financial institutions feel emboldened to lever up and increase the liquidity mismatch of their
balance sheets, that is, they are holding assets of low market liquidity which in a crisis can only be sold at a high discount, and a large part of their funding is short-term and hence has to be rolled over frequently.\footnote{See Brunnermeier et al. (2013) for a discussion of different types of liquidity and a definition of the liquidity mismatch index.} We account for this by developing a simple two-stage reduced form model that can be calibrated to historical data. In a first step, individual firms receive shocks to their balance sheets. They can be the consequence of changes of macroeconomic variables or can also originate within individual firms. In a second step, feedback effects play out between the institutions. While past crises have been triggered differently, systemic feedback mechanisms have exhibited similar patterns.\footnote{We refer to Brunnermeier and Oehmke (2012) for a survey of financial crises and feedback effects that played an important role.} Our model differentiates between direct and indirect feedback effects. Direct effects happen when there exist contractual connections between companies. A defaulting firm directly damages the balance sheets of its counterparties possibly triggering further defaults. Indirect feedback effects are caused by fire sales of illiquid assets and dry ups of funding liquidity. They are more pronounced if the liquidity mismatch of financial firms is high since this makes them more vulnerable to sudden declines in market liquidity and increases in short-term interest rates. By explicitly including systemic feedback effects, our approach addresses the volatility paradox. SystRisk is also less procyclical than standard risk measures, which typically do not pick up systemic risk in boom times but skyrocket when a crisis erupts, forcing institutions to delever, which in turn increases the risk measure further and worsens the crisis. In addition, we allow for a tolerance parameter that describes how much systemic risk society is willing to bear. It can be adjusted so that more risk is tolerated during economic downturns to provide conditions for the economy to stabilize and return to a path of growth.

Any systemic risk measure naturally raises the question how it can be used to manage and regulate systemic risk. SystRisk and our allocation principle show where systemic risk is building up and provide guidance in which direction financial regulation is most effective in reducing it. However, financial institutions have their own objectives and will react to new rules. To find a socially optimal level of regulation, one can attempt to model the behavioral response of all market participants and develop a full equilibrium model. But doing this realistically is a daunting task. The financial sector consists of many heterogeneous institutions. For example, banks will react differently than insurance companies, which in turn will have a different response from pension funds, sovereign wealth funds and hedge funds. In addition, many firms will try to circumvent regulatory measures. We adopt a more cautious tatonnement approach and view financial regulation as an ongoing process rather than the establishment of ultimate rules. In other words, regulation is introduced knowing that it might have to be reoptimized based on observed responses by financial institutions and updates of systemic risk measures. In this procedure the tolerance parameter has a second important role to play. It can be adjusted iteratively to find an optimal tradeoff between financial stability and enough leeway for financial firms to be able to provide the services a modern society depends on.

We discuss three different implementations of financial regulation: (i) setting individual risk limits, (ii) imposing risk charges, and (iii) implementing a cap and trade system. Setting individual
risk limits is closest to current financial regulation in most developed countries. But in contrast to classical banking regulation, which views each firm as a separate entity, our method leads to risk limits for individual institutions that depend on the positions of other institutions and how the financial sector is related to the rest of the economy. Alternatively, in the spirit of Pigouvian taxes, one can impose systemic risk charges depending on the positions the institutions are taking. This can be viewed as an extension of deposit insurance premiums. But again, the risk charges proposed here depend on a financial institution’s contribution to systemic risk. Systemic risk limits can be complemented with a cap and trade scheme. For instance, an auction for systemic risk permits can be held, which then can be traded during a certain period of time. This has two advantages. First, it might allow the financial sector to mitigate systemic risk more efficiently, and second, it produces a market price for systemic risk.

Over the last few years, the literature on systemic risk has been growing fast. For a recent survey of different systemic risk measures we refer to Bisias et al. (2012). A large part of the literature aims at developing statistical measures with which systemic risk can be inferred directly from market quotes of financial securities; see e.g. Chan-Lau and Gravelle (2005), Avesani (2005), Avesani et al. (2006), Duan and Wei (2009), Adrian and Brunnermeier (2009), Lehar (2011), Giesecke and Kim (2011), Huang et al. (2011), Billio et al. (2011), Giglio (2012), Acharya et al. (2012). But market prices have to be treated with care since they can be artificially inflated if investors expect that the government will provide support in the event of a crisis. Since our approach is model based, it allows us to identify the risk of scenarios in which taxpayers have to bail out financial institutions. Early models of systemic risk put the main emphasis on networks describing direct domino effects caused by defaulting interbank loans; see e.g. Eisenberg and Noe (2001) or Elsinger et al. (2006). However, such models do not take all channels of contagion into account and may make the financial sector appear more robust than it is; see Upper (2007). In the financial crisis of 2007–2009 indirect spillover effects played an important role. In particular, informational contagion and common exposure (virtual links) amplified by fire sales and liquidity spirals exacerbated the crisis; see e.g. Brunnermeier and Pedersen (2009). One way to capture adverse feedback loops is to simulate scenarios, such as the RAMSI model developed by the Bank of England (Burrows et al., 2012). Acharya et al. (2010), Brunnermeier and Sannikov (2012) as well as He and Krishnamurthy (2013) embed the financial sector in an equilibrium model and describe the behavioral response of financial institutions explicitly. But they assume all of them to be of the same type and therefore cannot capture the heterogeneity of the financial industry. Chen et al. (2012) put forward an axiomatic approach to measuring systemic risk. Our approach is different since it models externalities of the financial sector and relates them to the real economy.

The rest of the paper is organized as follows: In Section 2 we define SystRisk, our measure of systemic risk of the financial sector. Section 3 introduces our systemic risk allocation principle. In Section 4 we propose a stochastic model for initial random shocks to the financial system and ensuing amplification mechanisms playing out within it. Section 5 discusses three different ways for the regulator to manage systemic risk. Setting individual risk limits, imposing systemic risk charges and implementing a cap and trade scheme for systemic risk. The proofs are collected in the appendix.
2 Measuring total systemic risk

In this section we introduce our approach to measuring systemic risk in a general setting in which financial institutions cause externalities on society. A more specific model of financial losses and resulting externalities is discussed in Section 4 below.

We work with a finite set Ω of different states of the world. Then the space $L = \{ X: \Omega \to \mathbb{R} \}$ can be identified with $\mathbb{R}^d$, where $d$ is the number of elements in $\Omega$. We write

- $X \geq X'$ if $X(\omega) \geq X'(\omega)$ for all $\omega \in \Omega$
- $X > X'$ if $X \geq X'$ and $X(\omega) > X'(\omega)$ for at least one $\omega \in \Omega$
- $X \gg X'$ if $X(\omega) > X'(\omega)$ for all $\omega \in \Omega$
- $L_+ := \{ X \in L : X \geq 0 \}$, $L_{++} := \{ X \in L : X \gg 0 \}$
- $\|X\|_\infty := \max_{\omega \in \Omega} |X(\omega)|$, $\|X\|_1 := \sum_{\omega \in \Omega} |X(\omega)|$
- $B_\infty^\epsilon(X) := \{ X' \in L : \|X - X'\|_\infty \leq \epsilon \}$
- $\mathcal{P} :=$ the set of all probability measures on $\Omega$
- $\mathcal{P}^f :=$ the set of all probability measures on $\Omega$ with full support.

2.1 The regulator

We assume there is a regulator overseeing a country’s financial sector. It represents taxpayers, who in the event of a financial crisis, might have to prop up financial firms to prevent a collapse of the real economy. We fix a time period (e.g. a year or a quarter of a year) and model the country’s real GDP generated over that period with a random variable $Y \in L$. We think of $Y$ as the output of the real economy when it receives the financial services it needs to function properly. If essential parts of the financial system break down, the rest of the economy suffers severe consequences. To prevent this, the government is assumed to step in and provide the support necessary to keep the most important financial functions intact. This represents a negative externality caused by the financial sector. On the other hand, financial institutions can also have positive externalities if they perform better than expected.

We assume that the aggregate utility society is deriving from $Y$ is given by $U(Y)$ for a preference functional $U: L \to \mathbb{R} \cup \{-\infty\}$. The effective domain of $U$,

$\text{dom } U := \{ X \in L : U(X) \in \mathbb{R} \},$

consists of all elements $X \in L$ for which $U(X)$ is real-valued. The interior of $\text{dom } U$ is given by

$\text{int } \text{dom } U := \{ X \in L : B_\infty^\epsilon(X) \subseteq \text{dom } U \text{ for some } \epsilon > 0 \}.$

We always assume that $U$ satisfies the following two conditions:
(D) Differentiability \hspace{1cm} \text{int dom} \, U \text{ is non-empty and } U \text{ is differentiable on } \text{int dom} \, U

(S) Strict monotonicity \hspace{1cm} U(X) > U(X') \text{ for all } X, X' \in \text{dom} \, U \text{ such that } X > X'.

If one identifies \( L \) with \( \mathbb{R}^d \), \( \text{int dom} \, U \) is an open subset of \( \mathbb{R}^d \) on which \( U \) is differentiable. We denote the gradient of \( U \) at \( X \) by \( \nabla U(X) \).

Examples 2.1

1. Expected utility

Let \( \mathbb{P} \in \mathcal{P}^f \) be a probability measure with full support and \( u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\} \) a function with non-empty effective domain \( \text{dom} \, u := \{x \in \mathbb{R} : u(x) \in \mathbb{R}\} \). If \( u \) is differentiable and strictly increasing on the interior of \( \text{dom} \, u \), the expected utility \( U(X) = \mathbb{E}_\mathbb{P} [u(X)] \) satisfies (D) and (S). For \( X \in \text{int dom} \, U \), the components of the gradient \( \nabla U(X) \) are given by \( u'(X(\omega))\mathbb{P}[\omega], \omega \in \Omega \). So one can write
\[
\nabla U(X) \cdot X' = \mathbb{E}_\mathbb{P} [u'(X)X'] \quad \text{for all } X' \in L,
\]
where \( \cdot \) denotes the scalar product between vectors in \( \mathbb{R}^d \).

2. Variational preferences

Model uncertainty can be incorporated by introducing a non-empty closed subset \( Q \subseteq \mathcal{P} \) together with a lower semicontinuous mapping \( c : Q \to \mathbb{R} \) and setting
\[
U(X) = \min_{Q \in Q} \{\mathbb{E}_Q [u(X)] + c(Q)\}
\]
for a function \( u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\} \). Preferences of this type are called variational preferences and were axiomatized by Maccheroni et al. (2006). If \( u \) is strictly increasing on \( \text{int dom} \, u \) and \( Q \) is contained in \( \mathcal{P}^f \), \( U \) satisfies (S). Moreover, if \( \text{dom} \, u \) is non-empty, \( u \) is differentiable on \( \text{int dom} \, u \) and \( c \) is chosen appropriately, \( U \) also has the differentiability property (D). For example, it is well-known (see e.g. Cheridito and Kupper (2009) for a simple proof) that for \( Q = \mathcal{P} \), \( \lambda > 0 \), \( \mathbb{P} \in \mathcal{P}^f \) and \( c(Q) = \frac{1}{\gamma} \mathbb{E}_Q \left[ \frac{dQ}{d\mathbb{P}} \log \frac{dQ}{d\mathbb{P}} \right] \), (2.1) becomes
\[
U(X) = -\frac{1}{\gamma} \log \mathbb{E}_\mathbb{P} [\exp(-\gamma u(X))],
\]
which satisfies (D) and (S) if \( u \) is strictly increasing and differentiable on \( \text{int dom} \, u \).

2.2 The financial system

We suppose that the financial system consists of \( I \in \mathbb{N} \) institutions, each of which causes an externality of the form
\[
E_i = -\alpha_i V_i^- + \beta_i (V_i - v_i)^+,
\]
where
- $V_i \in L$ is the net worth of institution $i$ at the end of the measuring period, calculated e.g., as market value of assets minus book value of liabilities. $V_i$ is assumed to already include non-linearities generated by negative feedback effects occurring during a crisis. A reduced form stochastic model for $V_i$ is proposed in Section 4 below. $V_i^-$ denotes the negative part $\max\{0, -V_i\}$ and $(V_i - v_i)^+$ the positive part $\max\{V_i - v_i, 0\}$.

- $\alpha_i \geq 0$ is a constant, and the term $\alpha_i V_i^-$ describes a potential cost to society if institution $i$'s net worth falls below zero. We assume $\alpha_i$ to be a number between 0 and 1. To avoid moral hazard, it is important that there be no bailout guarantee. But we assume that in case of a crisis, the government decides on a case by case basis which institutions have to be supported to prevent negative effects on the real economy. Companies could receive a mix of bailout payments and government loans at preferred conditions, or they could be nationalized. In all these cases some of the losses have to be borne by the taxpayer. We allow $\alpha_i$ to depend on the type of institution $i$ since the cost caused by a bankruptcy depends on the structure of a financial company. For instance, a financial institution with several different business lines is more difficult to save or liquidate than one which concentrates only on a few activities. Furthermore, a typical bank relies heavily on short-term debt financing. So to protect the economy from an impending collapse, a relatively high fraction of its losses will have to be covered. Other components of the financial system, such as insurance companies or pension plans have more long-term liabilities and can continue operating even if their net worth falls below zero. As a consequence it might be possible to help them through a crisis with a minimal amount of funding.

- $\beta_i, v_i \geq 0$ are constants, and $\beta_i(V_i - v_i)^+$ models a possible positive externality. For instance, if institution $i$’s net worth exceeds the level $v_i$, the government may benefit from additional tax revenues of the form $\beta_i(V_i - v_i)^+$.

If the aggregate externality $E := \sum_{i \in I} E_i$ is absorbed by society, the real GDP changes from $Y$ to $Y + E$. Our approach only needs the pair $(Y, E)$ as input to determine the total systemic risk. To solve the systemic risk allocation problem we will have to specify the members of the financial system further. This will be done in Section 3 below.

### 2.3 Total systemic risk

We assume the regulator determines a tolerance level $e \in \mathbb{R}$ satisfying $U(Y + e) > -\infty$ and deems an externality $E$ acceptable if $U(Y + E) \geq U(Y + e)$. For a given externality $E \in L$, we are asking how much money the regulator would have to receive to be compensated for absorbing $E$. This leads to the following definition of systemic risk.

**Definition 2.2** We define $\text{SystRisk}$ of $E \in L$ by

$$\rho(E) := \inf \{m \in \mathbb{R} : U(Y + E + m) \geq U(Y + e)\}.$$  \hspace{1cm} (2.2)
Note that $\rho(E)$ is the cost of the externality $E$ measured in currency units at the end of the measuring period. If used to determine systemic risk charges to be collected beforehand, it has to be discounted at the risk-free rate. In the benchmark case $e = 0$, the regulator does not accept any externality $E$ with an adverse effect on $Y$. But if $e < 0$, negative externalities are tolerated as long as their impact is not too severe. By adjusting the tolerance level $e$ to the economic situation, the regulator can fine tune the stringency of financial regulation and implement countercyclical policies. For instance, in a recession financial regulation can be relaxed to permit the financial sector to operate with fewer constraints.

Under our assumptions on $U$, $\rho$ has the following properties:

**Lemma 2.3** $\rho$ is a function from $L$ to $\mathbb{R}$ satisfying

(N) Normalization at the tolerance level $\rho(e) = 0$

(M) Monotonicity $\rho(E) \geq \rho(E')$ for all $E, E' \in L$ such that $E \leq E'$

(T) Translation property $\rho(E + m) = \rho(E) - m$ for all $E \in L$ and $m \in \mathbb{R}$

(L) Lipschitz-continuity $|\rho(E) - \rho(E')| \leq \|E - E'\|_\infty$ for all $E, E' \in L$.

In particular, $\rho$ is a monetary risk measure on $L$ in the sense of Föllmer and Schied (2004). But since it is of the particular form (2.2), it has additional properties. The next result shows that $\rho$ is differentiable and strictly decreasing at all $E \in L$ satisfying the condition

(I) $Y + E + \rho(E) \in \text{int dom } U$.

This will be important for the risk allocation method proposed in the next subsection. Note that if $E$ satisfies (I), the gradient $\nabla U(Y + E + \rho(E))$ exists, and since $U$ has the strict monotonicity property (S), all its components are strictly positive. So

$$Q^E := \frac{\nabla U(Y + E + \rho(E))}{\|\nabla U(Y + E + \rho(E))\|_1}$$

defines a probability measure with full support.

**Theorem 2.4** Assume $E \in L$ satisfies condition (I). Then $\rho$ is differentiable at $E$ with gradient

$$\nabla \rho(E) = - \frac{\nabla U(Y + E + \rho(E))}{\|\nabla U(Y + E + \rho(E))\|_1}. \quad (2.3)$$

In particular, all components of $\nabla \rho(E)$ are strictly negative,

$$\lim_{\varepsilon \to 0} \frac{\rho(E + \varepsilon E') - \rho(E)}{\varepsilon} = \mathbb{E}_{Q^E}[-E'] \quad \text{for all } E' \in L, \quad (2.4)$$

and

$$\rho(E + E') < \rho(E) < \rho(E - E') \quad \text{for all } E' \in L \text{ satisfying } E' > 0. \quad (2.5)$$
\( \mathbf{Q}^E \) can be viewed as a shadow pricing measure for systemic risk, as \( \mathbb{E}_{\mathbf{Q}^E}[-E'] \) gives the marginal change in systemic risk stemming from a small deviation \( E' \) from the aggregate externality \( E \).

The following lemma shows that \( \rho \) is convex if the preference functional \( U \) is concave and strictly convex modulo constants if \( U \) is strictly concave on \( \text{int dom} \ U \).

**Proposition 2.5** If \( U \) is concave, then \( \rho \) is convex. If, in addition, \( U \) is strictly concave on \( \text{int dom} \ U \), then

\[
\rho(\lambda E + (1 - \lambda)E') < \lambda \rho(E) + (1 - \lambda)\rho(E')
\]

for all \( 0 < \lambda < 1 \) and \( E, E' \in L \) satisfying (I) such that \( E - E' \) is not constant.

As a consequence of Proposition 2.5 one obtains that in case \( U \) is concave and strictly concave on \( \text{int dom} \ U \), the systemic risk measure \( \rho \) scales superlinearly. More precisely, one has the following

**Corollary 2.6** Assume \( U \) is concave on \( L \) and strictly concave on \( \text{int dom} \ U \). Let \( E \in L \) be a non-constant externality satisfying condition (I). Then

\[
\rho(\lambda E) > \lambda(\rho(E) - e) + e \quad \text{for all } \lambda > 1.
\]

In particular, in the case \( e = 0 \), one has

\[
\rho(\lambda E) > \lambda \rho(E) \quad \text{for } \lambda > 1.
\]

**Example 2.7** As in Example 2.1.1, choose a probability measure with full support \( \mathbb{P} \in \mathcal{P}^f \) and let \( u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\} \) be a function not identically equal to \( -\infty \) that is strictly increasing and differentiable on the interior of \( \text{dom} u := \{x : u(x) \in \mathbb{R}\} \). Then the expected utility \( \mathbb{E}_\mathbb{P}[u(x)] \) has the differentiability property (D) and the strict monotonicity property (S). In the following we discuss three different specifications of the function \( u \):

1. **CRRA**

For our purposes a CRRA expected utility is a natural choice since it is the preference functional of a representative agent resulting from aggregating individual CRRA expected utility maximizers. Consider a function \( u \) of the form

\[
 u(x) = \begin{cases} 
 \frac{1}{1-\gamma}x^{1-\gamma} & \text{if } x > 0 \\
 -\infty & \text{if } x \leq 0 
\end{cases} \quad \text{for some } \gamma > 1.
\]

Then \( U \) is strictly concave on \( \text{int dom} \ U = L^{++} \) and for all \( X \in \text{int dom} \ U \), one has

\[
 \nabla U(X) \cdot X' = \mathbb{E}_\mathbb{P} \left[ X^{-\gamma}X' \right], \quad X' \in L.
\]

By Theorem 2.4, \( \rho \) is differentiable at every \( E \in L \) satisfying condition (I), and \( \nabla \rho(E) \) acts on \( E' \in L \) like

\[
 \nabla \rho(E) \cdot E' = \mathbb{E}_{\mathbf{Q}^E}[-E'],
\]
where the shadow price measure $Q^E$ is given by
\[
\frac{dQ^E}{dP} = \frac{(Y + E + \rho(E))^{-\gamma}}{\mathbb{E}_P[(Y + E + \rho(E))^{-\gamma}]}.
\]

2. CARA
If the regulator is modeled with a CARA utility function of the form
\[
u(x) = -\exp(-\gamma x) \quad \text{for some } \gamma > 0,
\]
then $\text{dom } U = L$ and $u$ is strictly concave on $L$. Moreover,
\[
\nabla U(X) \cdot X' = \gamma \mathbb{E}_P \left[ e^{-\gamma X X'} \right], \quad \text{for all } X, X' \in L,
\]
and one obtains from Theorem 2.4 that $\rho$ is differentiable on $L$ with
\[
\nabla \rho(E) \cdot E' = \mathbb{E}_{Q^E}[-E'],
\]
for
\[
\frac{dQ^E}{dP} = \frac{\exp(-\gamma \{Y + E + \rho(E)\})}{\mathbb{E}_P[\exp(-\gamma \{Y + E + \rho(E)\})]}.
\]

3. Risk-neutral
As a limit case, one can consider a risk-neutral regulator, that is, $u(x) = x$. Then $\text{dom } U = L$ and $\rho$ is just the shifted negative expectation of the aggregate externality: $\rho(E) = e - \mathbb{E}_P[E]$.

3 Systemic risk allocation
It is important to know how much each component of the financial system contributes to the overall systemic risk. We call a vector $k \in \mathbb{R}^I$ a systemic risk allocation if it satisfies the full allocation principle
\[
(FA) \quad \sum_{i=1}^I k_i = \rho(E).
\]
If $E \in L$ satisfies condition (I), it follows from Theorem 2.4 that the marginal contributions
\[
MC_i := \lim_{\varepsilon \to 0} \frac{\rho(E + \varepsilon E_i) - \rho(E)}{\varepsilon} = \mathbb{E}_{Q^E}[{-E_i}]
\]
exist. However, in general they do not satisfy (FA). We correct for that by introducing size parameters $s_i \in \mathbb{R}$ corresponding to the members of the system. For instance, they could be the number of employees of institution $i$, its total outstanding credit, or the amount of corporate taxes it paid last year. We think of $s_i$ as being positive numbers. But for the following allocation rule to be well-defined, one only needs $\sum_{i=1}^I s_i \neq 0$.
Definition 3.1 Assume $E \in L$ satisfies condition (I) and $\sum_{i=1}^{I} s_i \neq 0$. Then the size-shifted marginal contributions are given by

$$SMC_i := MC_i + \lambda s_i,$$

where $s_i$ are the exogenous size parameters and $\lambda \in \mathbb{R}$ is chosen so that the full allocation principle (FA) holds.

The size-shifted marginal contributions have the following additivity property:

Proposition 3.2 Suppose $E \in L$ satisfies condition (I) and $\sum_{i=1}^{I} s_i \neq 0$. Assume two companies $i$ and $j$ are combined in such a way that the new externality and size parameter are $E_i + E_j$ and $s_i + s_j$ respectively. Then the total systemic risk $\rho(E)$ and the size-shifted marginal contributions of the components different from $i$ and $j$ stay the same, while the size-shifted marginal contribution of the new company equals $SMC_i + SMC_j$.

Proposition 3.2 shows that our allocation rule is additive in the externalities. However, while it is natural to assume that the size parameters of merging companies add up, their externalities can behave in a number of different ways. The precise behavior of externalities under mergers depends on the relation between the future net worths $V_i$. This issue is discussed in more detail in Subsection 4.4 below.

4 Modeling losses in the financial sector

In this section we propose a reduced form model for losses in the financial sector and resulting externalities. The suggested model has the advantage that its coefficients can be estimated from historical data. But our approach also works with a different underlying model, like for instance, a simulation model such as the RAMSI model developed at the Bank of England; see Burrows et al. (2012).

We assume that losses in the financial system are the result of initial random shocks and negative feedback effects in the system.

4.1 Macroeconomic shocks and initial losses

Initial shocks can be caused by macroeconomic factors or firm specific events. We suppose that at the beginning of the measuring period every institution of the financial system starts with a deterministic net worth $x_i \in \mathbb{R}$. Then all of them receive a random shock of the form $z_i(F, W_i)$, where $F = (F_1, \ldots, F_n) \in L^n$ is a vector of macroeconomic variables, $W_i \in L$ is an idiosyncratic shock and $z_i : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ a deterministic function. After the initial shocks the institutions’ net worths are

$$X_i = x_i + z_i(F, W_i).$$
4.2 Feedback mechanisms

In a second step spillover effects between financial institutions take place, potentially amplifying the crisis. We divide them into direct and indirect spillover effects.

- Direct spillovers are caused by contractual connections between firms.

- Indirect spillover effects that played an important role in the subprime mortgage crisis starting in 2007 were asset fire sales and the dry up of funding liquidity. Asset fire sales occur when firms sell illiquid assets. This creates downward pressure on their prices and affects other institutions that are holding them. Funding liquidity dries up when, due to distress in some parts of the financial sector, lenders become more risk averse and suddenly demand higher interest. This is especially problematic for firms with a lot of short term debt since they have to refinance more frequently.

To capture vulnerabilities to direct spillover effects, we introduce an $I \times I$-matrix $c = (c_{ij})_{i \neq j}$ of interconnectedness (it does not need diagonal elements, or they can be set equal to zero) and a vector $l = (l_1, \ldots, l_I)$ of liquidity mismatch indexes. Component $c_{ij}$ of the interconnectedness matrix $c$ gives a measure of future cash flows to be paid by institution $i$ to institution $j$. If the former is in distress, it might not be able to honor its commitments, creating a problem for the latter. The liquidity mismatch indexes measure the firms’ debt structure compared to the market liquidity of their assets. If the majority of an institution’s debt is long term and all its assets can easily be sold in a crisis, it will not be affected by indirect spillover effects. Consequently, its liquidity mismatch index is low. On the other hand, if it has a lot of short term debt and many illiquid assets, it is more prone to indirect spillovers and has a high liquidity mismatch index; see Brunnermeier et al. (2013).

To capture negative feedback effects, we assume that after the arrival of the initial shocks, amplification effects move institution $i$’s net worth from $X_i$ to

$$V_i = a_i(F, X, s, l),$$

where $a_i : \mathbb{R}^{n+1+I^2} \to \mathbb{R}$ is a deterministic function. The 2007–2009 crisis was triggered by the bursting of the real estate bubble. But amplification mechanisms inside the financial system turned out to be more severe than initial declines in real estate and related financial securities. The functions $a_i$ are meant to be designed such that firms with a lot of exposure to other financial companies and high liquidity mismatch parameters will be affected more severely by negative feedback effects.

4.3 Form of the regulator

We assume that the GDP of the real economy is of the form $Y = y(F)$, where the components of $F = (F_1, \ldots, F_n)$ are the macroeconomic variables introduced in Subsection 4.1 and $y : \mathbb{R}^n \to \mathbb{R}$
is a deterministic function. Suppose that $E$ satisfies condition (I). Then we obtain from Theorem 2.4 that the gradient of $\rho$ at $E$ acts on a random variable $E' \in L$ like

$$\nabla \rho(E) \cdot E' = \mathbb{E}_{\mathbb{Q}^E}[-E'],$$

where

$$\mathbb{Q}^E = \frac{\nabla U(Y + E + \rho(E))}{\|\nabla U(Y + E + \rho(E))\|_1}.$$ 

For instance, if $U$ is a CRRA expected utility as in Example 2.7.1, then $U(X) = \mathbb{E}_\mathbb{P}[u(x)]$ for a probability measure $\mathbb{P} \in \mathcal{P}^f$ and a function $u$ of the form

$$u(x) = \begin{cases} \frac{1}{1-\gamma}x^{1-\gamma} & \text{if } x > 0 \\ -\infty & \text{if } x \leq 0 \end{cases} \text{ for some } \gamma > 1.$$ 

In particular, $U$ is strictly concave on int dom $U$, and

$$\frac{d\mathbb{Q}^E}{d\mathbb{P}} = \frac{(Y + E + \rho(E))^{-\gamma}}{\mathbb{E}_\mathbb{P}[(Y + E + \rho(E))^{-\gamma}]}.$$ 

It follows from Corollary 2.6 that

$$\rho(\lambda E) > \lambda(\rho(E) - e) + e \quad \text{for all } \lambda > 1.$$ 

This means that if the financial sector decides to behave in way so that the aggregate externality multiplies by a factor $\lambda > 1$, then total systemic risk increases more than proportionally. The size-shifted marginal contributions of Definition 3.1 take the form

$$SMC_i = MC_i + \lambda s_i,$$

where $MC_i = \mathbb{E}_{\mathbb{Q}^E}[-E_i]$ and $\lambda \in \mathbb{R}$ is chosen so that the full allocation principle (FA) holds.

### 4.4 Mergers and spinoffs

We now discuss how systemic risk is affected by mergers and spinoffs. We consider the case where just two firms merge. A merger of more than two is equivalent to iteratively merging two, and a spinoff can be viewed as a merger in reverse. We assume that if two firms $i$ and $j$ merge, they are not restructured. So that the size parameter and net worth of the combined unit at the end of the measuring period are $s_{i+j} = s_i + s_j$ and $V_{i+j} = V_i + V_j$, respectively. However, since the externalities $E_i$ are non-linear in the net worths $V_i$, they do not necessarily add up. We recall that the externalities of firms $i$ and $j$ are

$$E_i = -\alpha_i V_i^- + \beta_i(V_i - v_i)^+ \quad \text{and} \quad E_j = -\alpha_j V_j^- + \beta_j(V_j - v_j)^+$$

for parameters $\alpha_i, \alpha_j, \beta_i, \beta_j, v_i, v_j \in \mathbb{R}_+$. The externality of the new firm is

$$E_{i+j} = -\alpha_{i+j} V_{i+j}^- + \beta_{i+j}(V_{i+j} - v_{i+j})^+$$

for parameters $\alpha_{i+j}, \beta_{i+j}, v_{i+j} \in \mathbb{R}_+$. Depending on the circumstances, a merger can have different effects on total systemic risk. In the following we discuss three different situations:
1. If $\alpha_i = \alpha_j = \alpha_{i+j} = \alpha \geq \beta_i = \beta_j = \beta_{i+j} = \beta$ and $v_i = v_j = v_{i+j} = 0$, the function $\varphi(x) = -\alpha x^- + \beta x^+$ is superlinear, and one has

$$E_{i+j} = \varphi(V_i + V_j) \geq \varphi(V_i) + \varphi(V_j) = E_i + E_j.$$  

That is, mergers have a positive effect on externalities and reduce total systemic risk. For instance, if $\alpha > 0$ and $\beta = 0$, only negative externalities are taken into account. If each firm makes profits in states of the world where the other one has losses, there will be netting effects if the two are combined. Profits from one firm can be used to cover losses of the other.

2. Spinoffs can reduce total systemic risk if the benefits of positive net worths are given more weight than the costs of negative net worths. For instance, if $\alpha_i = \alpha_j = \alpha_{i+j} = \alpha \leq \beta_i = \beta_j = \beta_{i+j} = \beta$ and $v_i = v_j = v_{i+j} = 0$, the function $\varphi(x) = -\alpha x^- + \beta x^+$ is sublinear, and one has

$$E_{i+j} = \varphi(V_i + V_j) \leq \varphi(V_i) + \varphi(V_j) = E_i + E_j.$$  

In such a situation a merger hampers the positive effects each firm is having on society if it is doing well, because as a combined unit, each is liable for the losses of the other. However, in reality the adverse effects of negative net worths are more significant than the benefits from positive net worths. That is, one has $\alpha_i > \beta_i$ for all firms $i$. But it can still have a negative effect to merge two financial institutions if they engage in different activities such that the combined entity is more complex. If this is reflected in a cost coefficient $\alpha_{i+j}$ for the combined bank that is larger than $\alpha_i$ and $\alpha_j$, this can lead to a situation where one has $E_{i+j} < E_i + E_j$, and a merger increases systemic risk.

3. Finally, if firms $i$ and $j$ are of the same size and have identical exposures, their size parameters $s_i$ and $s_j$ as well as their externalities $E_i$ and $E_j$ are equal. Then, in a realistic model, the externality of a new institution resulting from combining firms $i$ and $j$ is $E_i + E_j = 2E_i$, and one obtains from Proposition 3.2 that the total systemic risk $\rho(E)$ as well as the size-shifted marginal contributions of the firms different from $i$ and $j$ stay the same, while the size-shifted marginal contribution of the new firm is $SMC_i + SMC_j$. So the total systemic risk measure and our allocation principle satisfy the clone property.

5 Managing systemic risk: A discussion

There are different ways of managing systemic risk. Generally, the regulator regards the system as safe if $\rho(E) \leq 0$. But if $\rho(E) > 0$, it poses an unacceptable risk that has to be mitigated. $-\nabla \rho(E)$ points in the direction in which total systemic risk is reduced most effectively.

Financial institutions have their own aims and will respond to new regulation. But the financial system is heterogenous. A bank functions differently from an insurance company, pension plan, sovereign wealth fund or hedge fund. We do not model their behavior. This has the advantage that our approach does not depend on behavioral assumptions. But if new regulation is implemented, responses in the financial sector and their ramifications for the real economy have to be monitored.
The stricter the regulation, the tighter the constraints under which the financial sector has to operate. Our systemic risk measure and allocation principle show where the risk is coming from. By adjusting the tolerance parameter, the regulator can decide how much overall systemic risk to accept depending on current economic conditions.

5.1 Systemic risk limits

One way to manage systemic risk is to force each bank’s systemic risk contribution towards zero. The regulator can guarantee \( \rho(E) \leq 0 \) by requiring that all size-shifted marginal contributions satisfy \( SMC_i \leq 0, \ i = 1, \ldots, I \). If \( SMC_i > 0 \) for some \( i \), then firm \( i \) most efficiently changes its externality in the direction \(-\nabla \rho(E)\). Alternatively, one can determine the systemic capital surcharge for member \( i \) as the minimal amount of additional capital that makes \( SMC_i \) fall to zero. This amounts to setting systemic risk limits and is close to current regulation in most modern economies. But standard financial regulation views financial firms as separate entities and ignores their systemic interactions.

5.2 Systemic risk charges

Instead of setting risk limits, the regulator can let financial firms act freely and charge them a premium for their contribution to systemic risk. If collected at the beginning of the measuring period, they have to be in the amount

\[
\tau_i = \frac{SMC_i^+}{1 + r},
\]

where \( r \) is the risk-free interest rate, so that total systemic risk is compensated.

Compared to risk limits, systemic risk charges allow for the pooling of capital buffers. In case of a crisis they can be used to support failing components of the system. Moreover, if the abatement costs are only known to the financial institutions but not the regulator, imposing systemic risk charges has the advantage that firms can decide themselves how much risk is optimal for them to take on. Firms with low abatement costs will reduce their risk exposures to lower their charge. Others find it optimal to keep their positions and pay a higher premium. Systemic risk charges act like Pigouvian taxes. In contrast to systemic risk limits, they do not allow control of the aggregate risk in the financial system, but they require financial firms to pay compensation for it. To decide if it is preferable to regulate with systemic risk limits, charges or a mixture of the two is related to the classical question whether it is better to control an economic system through quantities or prices; see Weitzman (1974).

5.3 Cap and trade

Similarly to cap and trade schemes for the emission of pollutants, the regulator can distribute (for free or through an auction) a number \( P \in \mathbb{R}_+ \) of systemic risk permits and require that each member of the system holds them in an amount \( P_i \) such that \( SMC_i - P_i \leq 0 \). A possible implementation is as follows. At the beginning of each measuring period (e.g. quarterly) permits
are auctioned. Then they can be traded for a few days. A cap and trade system allows the regulator to control the overall level of systemic risk but lets the market decide how systemic risk is reduced most efficiently. Those financial institutions with the lowest abatement costs can reduce their contributions and sell off extra permits to others for which a reduction of systemic risk is more expensive. Importantly, the market price of systemic risk permits reveals to the regulator the marginal abatement costs, which is important information for the implementation of socially optimal regulation\(^3\).

6 Conclusion

This paper proposes a tractable and implementable way of measuring and allocating systemic risk. It quantifies each financial institution's contribution to total systemic risk and points out the direction in which it can be reduced most effectively. The method is based on an underlying stochastic model of shocks to the financial system and resulting feedback mechanisms. Total systemic risk is measured as the a priori cost of future government interventions that might become necessary in a financial crisis to protect the real economy from negative consequences. To allocate the aggregate systemic risk to the members of the system we compute their marginal contributions. To enforce the full allocation principle we shift them by parameters reflecting the firms' sizes, such as the number of their employees, their outstanding credit or the amount of corporate taxes they paid in the previous year.

In contrast to standard financial regulation, our approach views individual firms as parts of the financial system and takes direct as well as indirect feedback effects into account. It also relates the financial industry to the real economy. Therefore, costs of externalities grow faster than linearly if they become large compared to the output of the real economy. It satisfies the so called clone property, which means that if a financial institution is split into \(n\) equal clones, the total systemic risk does not change, and the institution's risk contribution also splits into \(n\) equal parts. Since it explicitly models scenarios in which the financial sector receives government support, it can detect risks that might not be fully reflected in market prices of traded instruments such as equity shares, bonds, options or credit default swaps. It also addresses the volatility paradox, which refers to the phenomenon that financial systems tend to be more vulnerable to systemic feedback effects if volatility is low. This is related to the procyclical problem. Standard risk measures typically do not pick up systemic risk in economic booms but spike up sharply when a crisis erupts. If this is immediately translated into tighter regulation, financial institutions are forced to decrease their leverage at the beginning of economic downturns, which exacerbates the problem. By incorporating liquidity mismatch and systemic spillover mechanisms, our approach is able to identify systemic risk in low volatility environments. Moreover, it includes a tolerance parameter that can be adjusted over time to implement countercyclical regulation.

The paper outlines three different ways how our method of measuring and allocating systemic risk can be translated into financial regulation: setting systemic risk limits, imposing systemic

\(^3\)Stein (2012) compares reserve requirements to a cap and trade system in which the central bank governs the overall supply of reserves, while the fed funds market determines the allocation of reserves within the banking system.
risk charges, and a cap and trade system for systemic risk. Implementation of financial regulation that is fully optimal for the whole society would require an equilibrium model that can predict the behavioral response of financial institutions to regulatory requirements and the consequences for the real economy. This paper takes a reduced form approach in that it focuses on quantifying the cost of financial crises to society. Our method detects where systemic risk is coming from. Generally, stricter regulation means that the financial sector is under tighter constraints and therefore, can offer fewer services. By adjusting the tolerance parameter, the regulator can find the right level of stringency for an optimal balance between financial stability and enough flexibility for the financial sector to operate profitably. This has to be achieved in an iterative process in which the regulator fine tunes the tolerance parameter and monitors the response from the financial sector and the effects on the rest of the economy.

A Proofs

Proof of Lemma 2.3
It is easy to check that if $U$ satisfies (S) and $U(Y + e) > -\infty$, $\rho$ is a function from $L$ to $\mathbb{R}$ with the properties (N), (M) and (T). So for $E, E' \in L$, one has

$$\rho(E) - \rho(E') \leq \rho(E) - \rho(E + \|E - E'\|_{\infty}) = \|E - E'\|_{\infty},$$

and by symmetry,

$$\rho(E') - \rho(E) \leq \|E - E'\|_{\infty}.$$  

This shows that $\rho$ satisfies (L). □

Proof of Theorem 2.4
First note that it follows from condition (I) that $U(Y + E + \rho(E)) = U(Y + e) \in \mathbb{R}$. Now let $E_n, n \in \mathbb{N}$, be a sequence in $L$ converging to $E$. Since $\rho$ satisfies the Lipschitz condition (L), $Y + E + E_n + \rho(E + E_n)$ is in int dom $U$ for $n$ large enough, and therefore,

$$U(Y + E + \rho(E)) = U(Y + e) = U(Y + E + E_n + \rho(E + E_n))$$

$$= U(Y + E + \rho(E)) + \nabla U(Y + E + \rho(E)) \cdot (E_n + \rho(E + E_n) - \rho(E)) + o(\|E_n\|_{\infty})$$

as $n \to \infty$. It follows that

$$\rho(E + E_n) - \rho(E) = - \frac{\nabla U(Y + E + \rho(E))}{\|U(Y + E + \rho(E))\|_1} \cdot E_n + o(\|E_n\|_{\infty})$$

for $n \to \infty$,

which shows that $\rho$ is differentiable at $E$ with gradient (2.3). In particular, one has (2.4), and all components of $\nabla \rho$ are negative. The latter implies the strict monotonicity (2.5) around $E$. □

Proof of Proposition 2.5
First assume that $U$ is concave. Let $E, E' \in L$ and $0 \leq \lambda \leq 1$. Choose $m, m' \in \mathbb{R}$ such that $\rho(Y + E + m) \geq U(Y + e)$ and $\rho(Y + E' + m') \geq U(Y + e)$. Then $U(Y + \lambda E + (1 - \lambda) E' + \lambda m + (1 -
\( \lambda m') \geq U(Y + e) \). So \( \rho(\lambda E + (1 - \lambda)E') \leq \lambda m + (1 - \lambda)m' \) and therefore, \( \rho(\lambda E + (1 - \lambda)E') \leq \lambda \rho(E) + (1 - \lambda)\rho(E') \).

Now assume \( U \) is concave and strictly concave on \( \text{dom} \, U \). If \( E, E' \) satisfy (I) and \( E + E' \) is not constant, then \( Y + E + \rho(E) \) and \( Y + E' + \rho(E') \) are both in \( \text{int} \, \text{dom} \, U \). Therefore,

\[
U(Y + E + \rho(E)) = U(Y + E' + \rho(E')) = U(Y + e).
\]

Moreover, \( E + \rho(E) \neq E' + \rho(E') \). So one obtains from the strict concavity of \( U \) that

\[
U(Y + \lambda E + (1 - \lambda)E' + \lambda \rho(E) + (1 - \lambda)\rho(E')) > U(Y + e) \quad \text{for} \quad 0 < \lambda < 1. \quad (A.1)
\]

Since \( \text{int} \, \text{dom} \, U \) is convex, one has

\[
Y + \lambda E + (1 - \lambda)E' + \lambda \rho(E) + (1 - \lambda)\rho(E') \in \text{int} \, \text{dom} \, U,
\]

which together with (A.1), implies \( \rho(\lambda E + (1 - \lambda)E') < \lambda \rho(E) + (1 - \lambda)\rho(E') \).

\[\square\]

**Proof of Corollary 2.6**

Assume \( U \) is concave and strictly concave on \( \text{dom} \, U \). Let \( E \in L \) be a non-constant externality that satisfies (I). Choose \( \lambda > 1 \) and set \( E' = \lambda E \). Since \( U \) is concave, one obtains from Proposition 2.5 that

\[
\rho(\mu E') \leq \mu \rho(E') + (1 - \mu)\rho(0) = \mu \rho(E') + (1 - \mu)e \quad \text{for all} \quad 0 \leq \mu \leq 1.
\]

Now let us assume the inequality is an equality for \( \mu = 1/\lambda \). Then it has to be an equality for all \( 0 \leq \mu \leq 1 \). Choose \( \mu_1, \mu_2 \) such that \( 0 < \mu_1 < 1 < \mu_2 < \lambda \) and set \( F = \mu_1 E, \, G = \mu_2 E \). Then \( F - G \) is not constant and

\[
\rho(\nu F + (1 - \nu)G) = \nu \rho(F) + (1 - \nu)\rho(G) \quad \text{for all} \quad 0 \leq \nu \leq 1. \quad (A.2)
\]

However, by Lemma 2.3, \( \rho \) is Lipschitz-continuous, So if \( \mu_1 \) and \( \mu_2 \) are chosen sufficiently close to 1, then \( F \) and \( G \) satisfy condition (I). But then (A.2) contradicts Proposition 2.5. So one must have

\[
\rho(E'/\lambda) < \rho(E')/\lambda + (1 - 1/\lambda)e,
\]

which is equivalent to \( \rho(\lambda E) > \lambda \rho(E) - (\lambda - 1)e \).

\[\square\]

**Proof of Proposition 3.2**

If components \( i \) and \( j \) form a new entity with externality \( E_i + E_j \) and size parameter \( s_i + s_j \), the aggregate externality \( E \) stays the same. So the total systemic risk \( \rho(E) \) and the size-shifted marginal contributions of the components different from \( i \) and \( j \) remain unchanged. According to (3.1), the size-shifted marginal contribution of the combined component is

\[
\mathbb{E}_{Q^E}[-E_i - E_j] + \lambda(s_i + s_j) = \mathbb{E}_{Q^E}[-E_i] + \lambda s_i + \mathbb{E}_{Q^E}[-E_j] + \lambda s_j,
\]

where \( \lambda \) is the unique number such that \( \sum_{i=1}^{l} \left\{ \mathbb{E}_{Q^E}[-E_i] + \lambda s_i \right\} = \rho(E) \). This proves the proposition.

\[\square\]
References


