Correlation Risk

and Optimal Portfolio Choice

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ABSTRACT

We develop a new framework for intertemporal portfolio choice when the covariance matrix of returns is stochastic. An important contribution of this framework is that it allows to derive optimal portfolio implications for economies in which the degree of correlation across different industries, countries, and asset classes is time-varying and stochastic. In this setting, markets are incomplete and optimal portfolios include distinct hedging components against both stochastic volatility and correlation risk. The model gives rise to simple optimal portfolio solutions that are available in closed-form. We use these solutions to investigate, in several concrete applications, the properties of the optimal portfolios. We find that the hedging demand is typically four to five times larger than in univariate models and it includes an economically significant correlation hedging component, which tends to increase with the persistence of variance covariance shocks, the strength of leverage effects and the dimension of the investment opportunity set. These findings persist also in the discrete-time portfolio problem with short-selling or VaR constraints.

JEL classification: D9, E3, E4, G12

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This paper develops a new multivariate modeling framework for intertemporal portfolio choice under a stochastic variance covariance matrix. We consider an incomplete market economy, in which stochastic volatilities and stochastic correlations follow a multivariate diffusion process. In contrast to previous GARCH-type specifications, in this setting volatilities and correlations can be conditionally correlated with returns and optimal portfolio strategies include distinct hedging components against volatility and correlation risk. We solve the optimal portfolio problem and provide simple closed-form solutions that allow us to study the volatility and correlation hedging demands in several realistic asset allocation settings. We document the importance of modeling the multivariate nature of second moments especially in the context of optimal asset allocation and find that the optimal hedging demand can be significantly different from the one implied by more common models with constant correlations or single-factor stochastic volatility.

The importance of solving portfolio choice models taking into account the time-variation in volatilities and correlations is highlighted by Ball and Torous (2000), who study empirically the comovement of a number of international stock market indices. They find that the estimated correlation structure is changing over time depending on economic policies, the level of capital market integration, and relative business cycle conditions. They conclude that ignoring the stochastic component of the correlation can easily imply erroneous portfolio choices and risk management decisions. An important thread within the asset pricing literature has explored and documented the characteristics of this time-variation. See Longin and Solnik (1995), Bekaert and Harvey (1995, 2000), Erb, Harvey, and Viskanta (1994), Ang and Chen (2002), Ledoit, Santa-Clara, and Wolf (2003), Moskowitz (2003), Barndorff-Nielsen and Shephard (2004), among others.

A revealing example of the importance, both theoretical and practical, of modeling time-varying correlations in optimal portfolio choice is offered by the comovement of financial markets during the recent 2007-2008 financial markets crisis. During the period between April 2004 and April 2008, the sample average correlation of U.S. and Nikkei (FTSE) weekly stock market returns has been less than 0.20. However, its time-variation has been very big and since summer 2007 international equity correlations increased dramatically, with correlations between the S&P500 and FTSE reaching a value close to 0.80 for the quarter ending in April 2008 (see Figure 1).

\footnote{Longin and Solnik (1995) reject the null hypothesis of constant international stock market correlations and find that these increase in periods of high volatility. Ledoit, Santa-Clara, and Wolf (2003) show that the level of correlation depends on the phase of the business cycle. Erb, Harvey, and Viskanta (1994) find that international markets tend to be more correlated when countries are simultaneously in a recessionary state. Moskowitz (2003) documents that covariances across portfolio returns are highly correlated with NBER recessions and that average correlations are highly time-varying. Ang and Chen (2002) show that the correlation between US stocks and the aggregate US market is much higher during extreme downside movements than during upside movements. Barndorff-Nielsen and Shephard (2004) find similar results. Bekaert and Harvey (1995, 2000) provide direct evidence that market integration and financial liberalization change the correlation of emerging markets’ stock returns with the global stock market index.}
A second feature highlighted by the data is that the two correlation processes seem far from being independent: As correlation with the FTSE has increased, the correlation with the Nikkei also increases, reaching its highest value of 0.60 in the same month. It has been often reported in the press that several asset managers have found (ex-post) their portfolios much less diversified than originally planned (or hoped) and many breached, or run close to, their Value-at-Risk limits, thus inducing forced de-leveraging. A third feature, which is particularly evident during this period, is the correlation leverage effect: correlations of stock returns tend to be higher in phases of market downturn (see Figure 2). From an optimal portfolio choice perspective, this is very important since correlations reach their highest levels exactly when marginal utilities are high. While some of these empirical facts have already been documented in the literature (see, e.g., Harvey and Siddique (2000), Roll (1988) and Ang and Chen (2002)) little is known about (a) the solution of the optimal portfolio choice problem when correlations are stochastic and (b) the extent to which stochastic correlations affect, in practice, the characteristics of optimal portfolios if one considers realistic economic settings.

These questions are of broad interest in financial economics as time-varying correlations are playing an increasingly important role in some explanations of empirical asset pricing anomalies. Pastor and Veronesi (2008) explain the behavior of asset prices during technological revolutions by modeling the change in the nature of the risk associated with new technologies. Initially, this risk is mostly idiosyncratic, due to the small scale of production and the low probability of adoption. However, for the technologies that are ultimately adopted the risk gradually changes from idiosyncratic to systematic as the correlation between cash flow shocks to the new-economy technology and representative agent’s wealth increases. The behavior of correlations plays an important role also in Moskowitz (2003), who argues that some pricing anomalies such as momentum and size effect can be explained by stochastic correlations. Driessen, Maenhout, and Vilkov (2006) document that the implied volatility smile is flatter for individual stock options than for index options and attribute the difference to a priced correlation risk factor.

An extensive literature has explored the implications of stochastic volatility for intertemporal portfolio choice. However, the implications of stochastic correlations in a multivariate setting are still not well known. In part, this is due to the difficulty in formulating a flexible and tractable model satisfying the tight nonlinear constraints implied by a well defined correlation process: Correlations need to be bounded between -1 and +1 and the covariance matrix must be symmetric and positive definite. In this paper, we follow a new approach in modeling stochastic variance covariance risk and directly specify the covariance matrix process as a Wishart diffusion process, following Bru (1991). This process can reproduce several of the empirical features of returns covariance matrices highlighted by Figure 1 and
Figure 2. At the same time, it is sufficiently tractable to grant closed-form solutions to the optimal portfolio problem, which we can easily interpret economically. Since volatilities and correlations are stochastic, we consider an incomplete markets economy in which a constant relative risk aversion agent maximizes his utility of terminal wealth. This setting allows us to investigate the effect of the investment horizon on the optimal holdings in risky assets. We use this model to address a number of questions on the role of correlation hedging for intertemporal portfolio choice:

(a) What is the economic importance of stochastic variance covariance risk for intertemporal portfolio choice? We estimate the model using a dataset of international stock and US bond returns and find that, even for a moderate number of assets, the hedging demand can be about five times larger than in univariate stochastic volatility models. This has two reasons. First, correlation hedging can count for a substantial part of the total hedging demand. Its importance tends to increase with the strength of leverage effects and the dimension of the investment opportunity set. Second, our findings show not only that joint features of volatility and correlation dynamics are better described by a multivariate model with nonlinear dependence and leverage, but also that these features play an important role in the implied optimal portfolios. For instance, in a univariate stochastic volatility model we find that the estimated total hedging demand for S&P500 Futures of investors with relative risk aversion 8 and investment horizon 10 years is only about 4.8% of the myopic portfolio. This finding is consistent with the results in Chako and Viceira (2005). However, in a multivariate (three risky assets) model, the total hedging demand for S&P500 Futures is 28% and correlation hedging demand is 16.9% of the myopic portfolio.

(b) How do both the optimal investment in risky assets and the correlation hedging demand vary with respect to the investment horizon? This question plays an important role for optimal life-cycle decisions as well as for pension fund managers. We find that the absolute correlation hedging demand increases with the investment horizon. If the correlation hedging demand is positive (negative), this feature implies an optimal investment in risky assets that increases (decreases) in the investment horizon. For instance, in a multivariate model with three risky assets the estimated total hedging (correlation hedging) demand for S&P500 Futures of investors with relative risk aversion 8 is only about 6.3% (4.5%) of the myopic portfolio at horizons of three months. For horizons of 10 years the total hedging demand increases to 28%.

(c) What is the link between the persistence of correlation shocks and the demand for correlation hedging? The persistence of correlation shocks varies across markets. In highly liquid markets, like the Treasury and foreign-exchange markets, which are less affected by private information issues, correlation shocks are less persistent. In other markets, frictions such as asymmetric information and differences in beliefs about future cash-flows make price deviations from the equilibrium more difficult to be arbitrag ed away. Examples include both developed and emerging equity markets. Consistently with this intuition, we find that the optimal hedging demand against correlation risk increases with the degree of correlation shock persistence.
(d) What is the impact of discrete trading and portfolio constraints on correlation hedging demands? In the absence of derivative instruments to complete the market, we find that the correlation hedging demand in continuous time and discrete time settings are comparable. Simple short selling constraints tend to reduce the correlation hedging demand of risk tolerant investors, typically by a moderate amount, but Value-at-Risk constraints can even reinforce the correlation hedging motive. For instance, in the unconstrained discrete time model with two risky assets the estimated total hedging (correlation hedging) demand for S&P500 Futures of investors with relative risk aversion 2 is about 12.5% (4.6%) of the myopic portfolio at horizons of two years. In the VaR constrained setting, the total hedging (correlation hedging) demand increases to 16.7% (8.1%).

**Literature Review.** This paper draws upon a large amount of literature on optimal portfolio choice under a stochastic investment opportunity set. One set of papers studies optimal portfolio and consumption problems with a single risky asset and a riskless deposit account. Kim and Omberg (1996) solve the portfolio problem of an investor optimizing utility of terminal wealth, where the riskless rate is constant and the risky asset has a mean reverting Sharpe ratio and constant volatility. Wachter (2002) extends this setting to allow for intermediate consumption and derives closed-form solutions in a complete markets setting. Chacko and Viceira (2005) relax the assumption on both the preferences and the volatility. They consider an infinite horizon economy with Epstein-Zin preferences, in which the volatility of the risky asset follows a mean reverting square-root process. Liu, Longstaff, and Pan (2003) model events affecting market prices and volatility, using the double-jump framework in Duffie, Pan, and Singleton (2000). They show that the optimal policy is similar to that of an investor facing short-selling and borrowing constraints, even if none are imposed. Although their approach allows for a rather general model with stochastic volatility, they focus on a single risky asset economy. Our contribution to this literature is an investigation of an economy with multivariate risk factors, in which the correlation between factors is stochastic and acts as an independent source of risk. Moreover, we investigate the optimal portfolio implications when markets are incomplete. This aspect is especially important when volatilities and correlations are stochastic, in that it limits the ability of the portfolio manager to span the state-space using portfolios of marketed assets. However, in order to derive closed-form solutions we work with CRRA preferences. This assumption is more restrictive than Chacko and Viceira (2005), who consider a more general set of Epstein-Zin preferences.

Portfolio selection problems with multiple risky assets have been considered in a further series of papers, but the majority of these are based on the assumption that volatilities and correlations are constant. Examples include Brennan and Xia (2002), who study optimal asset allocations under inflation risk, and Sangvinatsos and Wachter (2005), who investigate

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\[2\text{In that the available instruments for investment are all made up of two assets, the resulting portfolio optimization problem is in fact univariate, because the budget constraint allows one portfolio weight to be eliminated.}\]
the portfolio problem of a long-run investor with both nominal bonds and stocks. A notable exception to the constant volatility assumption is Liu (2007), who shows that, under additional assumptions, the portfolio problem can be characterized by a sequence of differential equations in a model with quadratic returns. To solve in closed-form a concrete model with a riskless asset, a risky bond and a stock, independence is assumed between the state variable driving pure term structure risk and the additional risk factor influencing the volatility of the stock return. In that model, correlations are stochastic, but are restricted to being functions of stock and bond return volatilities. Therefore, optimal hedging portfolios do not allow volatility and correlation risk to have separate roles. Our setting avoids deterministic dependencies between volatilities and correlations. Moreover, it can be used to analyze portfolio choice problems of arbitrary dimensions.

We model the stochastic covariance matrix of returns using a single-regime mean-reverting diffusion process, in which the strength of the mean reversion can generate different degrees of persistence in volatilities, correlations and co-volatilities. To obtain closed-form portfolio solutions, we refrain from introducing an unpredictable jump component in the joint process for returns and correlations. This approach allows us to study the properties of the optimal hedging demand under a persistent correlation process. A completely different approach to modeling co-movement in portfolio choice relies on either a Markov switching-regime in correlations or on the introduction of a sequence of unpredictable joint Poisson shocks in asset returns. Ang and Bekaert (2002) consider a dynamic portfolio model with two i.i.d. switching regimes, one of which is characterized by higher correlations and volatilities. Using numerical methods, they find that when the international portfolio manager has access to a risk-free asset, the optimal portfolio is significantly sensitive to asymmetric correlations between the two regimes. Our model is different from theirs because we model an incomplete markets economy in which a single regime features persistent volatility and correlation shocks. Moreover, the analytical solutions of the optimal portfolio allow us to study the contribution of the different hedging demands for volatility and correlation risk to the overall portfolio. Since the solutions hold for an arbitrary number of assets, we can also investigate the behavior of correlation hedging as the number of risky assets increases. Das and Uppal (2004) study systemic risk, modeled as an unpredictable common Poisson shock, in a setting with a constant opportunity set and in the context of international equity diversification. They show that systemic jump risk reduces the gain from diversification and penalizes the investor from holding levered positions. In their model, due to the structure of systematic risk, the correlation between assets is unpredictable and transitory. In our model, in contrast, correlation and volatility shocks are persistent. Thus, they generate a motive for

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3I.e., the interest rate, the maximal squared Sharpe ratio, the hedging coefficient vector, and the unspanned covariance matrix are all quadratic functions of a state variables process with quadratic drift and diffusion coefficients.

4Persistence of second moments has also proven to be an important dimension to interpret traditional asset pricing puzzles: See, among others, Barsky and De Long (1990, 1993), Bansal and Yaron (2004) and Parker and Julliard (2005).
intertemporal hedging. These features have substantially different implications on portfolio choice.

This article is also related to both the multivariate GARCH literature and the more recent literature making use of Wishart processes to model multivariate stochastic volatility in finance. Pioneering models in the multivariate GARCH literature, as for instance Bollerslev (1987) and Bollerslev, Engle, and Wooldridge (1988), either restrict the correlation to be constant or do not necessarily imply a positive definite covariance matrix. Further important contributions include Harvey, Ruiz, and Shephard (1994), who specify a model with correlation dynamics that are driven by the same factors affecting volatility, and Barndorff-Nielsen and Shephard (2004). A key feature in our model is that correlations can have dynamics that are not fully correlated with factors affecting the volatility processes. Many recent multivariate GARCH-models ensure a positive definite covariance matrix that can be estimated by a computationally feasible estimation procedure. Engle (2002) proposes a Dynamic Conditional Correlation (DCC) specification with time-varying correlations and positive definite covariance matrices, which builds upon a set of univariate GARCH processes. However, the DCC-model and its extensions, which include specifications that account for volatility and correlation asymmetries, are analytically intractable for dynamic portfolio choice purposes. Moreover, due to the implicit assumption of a zero conditional correlation between innovations in correlations and returns, some important features related to the dynamics of the hedging policies are necessarily restricted. As in multivariate GARCH settings, our model incorporates persistence in volatilities and correlations. However, it preserves the tractability required to study the implied optimal portfolio strategies analytically. The convenient properties of Wishart processes for modeling multivariate stochastic volatility in finance have been exploited first by Gouri´eroux and Sufana (2004); Gouri´eroux, Jasiak and Sufana (2004) provide a thorough analysis of the properties of Wishart processes, both in discrete and continuous time.

The article is organized as follows: Section I describes the model, the theoretical properties of the implied correlation process, and the solution to the portfolio problem. In Section II, we estimate our model in a real data example and quantify the portfolio impact of correlation risk. Section III discusses model extensions that study the impact of discrete rebalancing and portfolio constraints on correlation hedging. Section IV concludes. All proofs are in the Appendix.

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5For a review, see, for example, Poon and Granger (2003).

6A well-known additional issue of these specifications is the “curse of dimensionality”. For n assets, one needs to model \( \frac{n(n+1)}{2} \) elements of the covariance matrix, which implies that parameters matrices A and B have \( \frac{1}{2}n^2(n + 1)^2 \) elements.
I. The Model

An investor with Constant Relative Risk Aversion utility over terminal wealth trades three assets, a riskless asset with instantaneous riskless return $r$, and two risky assets, in a continuous-time frictionless economy on a finite time horizon $[0, T]$. Our analysis extends to opportunity sets consisting of any number of risky assets and correlations, without affecting the existence of closed-form solutions and their general structure. We focus on a two-dimensional setting to keep our notation simple and focus on the key economic intuition and implications of the solution.

The dynamics of the price vector $S = (S_1, S_2)'$ of the risky assets is described by the bivariate stochastic differential equation:

$$dS(t) = I_S \left[ (r\bar{1}_2 + \Lambda(\Sigma, t))dt + \Sigma^{1/2}(t)dW(t) \right]; \quad I_S = \text{Diag}[S_1, S_2],$$  \hspace{1cm} (1)

where $r \in \mathbb{R}_+$, $\bar{1}_2 = (1, 1)'$, $\Lambda(\Sigma, t)$ is a vector of possibly state-dependent risk premia, $W$ is a standard two-dimensional Brownian motion and $\Sigma^{1/2}$ is the positive square root of the conditional covariance matrix $\Sigma$ of returns. The investment opportunity set is stochastic because of the time varying market price of risk $\Sigma^{-1/2}(t)\Lambda(\Sigma, t)$, which is a function of the stochastic covariance matrix $\Sigma(t)$. The constant interest rate assumption can easily be relaxed. Such an extension is investigated in Section I.D.5.

The diffusion process for $\Sigma$ is detailed below. Let $\pi(t) = (\pi_1(t), \pi_2(t))'$ denote the vector of shares of wealth $X(t)$ invested in the first and the second risky asset, respectively. Agent’s wealth evolves as:

$$dX(t) = X(t) \left[ r + \pi(t)'\Lambda(\Sigma, t) \right]dt + X(t)\pi(t)'\Sigma^{1/2}(t)dW(t).$$  \hspace{1cm} (2)

The agent selects the portfolio process $\pi$ that maximizes CRRA utility of terminal wealth, with RRA coefficient $\gamma$. If $X_0 = X(0)$ denotes the initial wealth and $\Sigma_0 = \Sigma(0)$ the initial covariance matrix, the investor’s optimization problem is:

$$J(X_0, \Sigma_0) = \sup_{\pi} \mathbb{E} \left[ \frac{X(T)^{1-\gamma} - 1}{1 - \gamma} \right],$$  \hspace{1cm} (3)

subject to the dynamic budget constraint (2). This setting allows us to investigate how the optimal portfolio allocation varies over the life-cycle of the agent.

A. The Stochastic Variance Covariance Process

To model stochastic covariance matrices in a convenient way, we use the continuous-time Wishart diffusion process introduced by Bru (1991) and studied by Gouriéroux and Sufana (2004). This process is a matrix-valued extension of the univariate square-root process that gained popularity in the term structure and stochastic volatility literature; see, for instance, Cox, Ingersoll, and Ross (1985) and Heston (1993). Let $B(t)$ be a $2 \times 2$ matrix-valued standard Brownian motion. The diffusion process for $\Sigma$ is defined as:

$$d\Sigma(t) = [\Omega\Omega' + M\Sigma(t) + \Sigma(t)M']dt + \Sigma^{1/2}(t)dB(t)Q + Q'dB(t)'\Sigma^{1/2}(t),$$  \hspace{1cm} (4)
where $\Omega, M, Q$ are $2 \times 2$ square matrices (with $\Omega$ invertible). Matrix $M$ drives the mean reversion of $\Sigma$ and is assumed negative semi-definite to ensure stationarity. Matrix $Q$ determines the co-volatility features of the stochastic variance covariance matrix of returns.

Process (4) satisfies several important properties that make it ideal to model stochastic correlation in finance. First, if $\Omega \Omega' >> Q'Q$ then $\Sigma$ is a well defined covariance matrix process. Under this condition, the implied correlation process is well behaved. Second, if $\Omega \Omega' = kQ'Q$ for some $k > n - 1$ then $\Sigma(t)$ follows a Wishart distribution; see Bru (1991). This distribution has been studied in Bayesian statistics to model priors on multivariate second moments, but it has not been used to study intertemporal optimal portfolio choice problems in continuous time. Third, the process (4) is affine in the sense of Duffie and Kan (1996) and Duffie, Filipovic, and Schachermayer (2003). This feature implies closed-form expressions for all conditional Laplace transforms. Fourth, if $d \ln S_t$ is a vector of returns with a Wishart covariance matrix $\Sigma(t)$, then the variance of the return of a portfolio $\pi$ is a Wishart process. This features is not shared by settings in which volatilities and correlations are modeled by multivariate GARCH processes, because GARCH models are not invariant under linear aggregation. Fifth, process (1), (4), can feature some important empirical properties of financial asset returns documented in the literature, such as leverage and co-leverage.

To model leverage effects conveniently in our multivariate portfolio setting, we introduce a nonzero correlation between innovations in stock returns and innovations in the variance-covariance process itself. Specifically, we define the standard Brownian motion $W(t)$ in the return dynamics as:

$$\begin{align*}
W(t) &= \sqrt{1 - \vec{\rho}'\vec{\rho}}Z(t) + B(t)\vec{\rho},
\end{align*}$$

(5)

where $Z$ is a two-dimensional standard Brownian motion independent of $B$ and $\vec{\rho} = (\rho_1, \rho_2)'$ is a vector of correlation parameters $\rho_i \in [-1, 1]$ such that $\vec{\rho}'\vec{\rho} \leq 1$. Parameters $\rho_1$ and $\rho_2$ allow for a flexible parameterization of leverage in volatilities and correlations of the multivariate return process (1). Importantly, since $n$ risky assets are available for investment and the covariance matrix dynamics depends on $n(n+1)/2$ independent Brownian shocks, the market is incomplete when $n \geq 2$.

**B. Specification of the Risk Premium**

Here it is important to notice that the investment opportunity set can be stochastic due to changes in expected returns or changes in conditional variances and covariances. It is well-known that to obtain closed-form solutions one needs to impose restrictions on the functional form of the squared Sharpe-ratio. In our setting, affine squared Sharpe-ratios imply affine solutions if the underlying state process is affine. Thus, to complete the specification of the return process (1), we consider risk premium specifications $\Lambda(\Sigma, t)$ that imply an affine dependence of squared Sharpe ratios on the given state process.

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7See Section I.C. for a more detailed discussion and interpretation of the leverage effects arising in the model.
In the first specification, we investigate a setting with a constant market price of variance covariance risk, \( \Lambda(\Sigma, t) = \Sigma(t) \lambda \) for \( \lambda = (\lambda_1, \lambda_2)' \in \mathbb{R}^2 \). This assumption implies squared Sharpe ratios that increase with volatilities, but which increase or decrease in the correlation level depending on the sign of the prices of risk. The assumption of a constant market price of variance covariance risk implies a positive risk return tradeoff and embeds naturally the univariate model studied, among others, in Heston (1993) and Liu (2001). In this setting, the model parameters have a clear interpretation since the relevant state variable is the covariance matrix of returns itself. Moreover, the total hedging demand can be naturally separated into a part due to volatility risk and another one due to correlation risk. We solve the dynamic portfolio problem implied by this specification in Section I \[D.2\].

In the second specification, we investigate a setting with a constant risk premium, \( \Lambda(\Sigma, t) = \mu^e \), for some \( \mu^e = (\mu^e_1, \mu^e_2)' \in \mathbb{R}^2 \), and an affine matrix diffusion of the type \[4\] for the precision process \( \Sigma^{-1} \); see also Chacko and Viceira (2005). This assumption implies squared Sharpe ratios that decrease with volatilities and correlations if risky assets pay positive risk premia. In this setting, the investment opportunity set is stochastic exclusively due to the stochastic covariance matrix. Therefore, we can directly identify the aggregate hedging demand for volatility and correlation risk. The disadvantage of this specification is that state variables are defined by means of \( \Sigma^{-1} \), which makes the interpretation of model parameters, e.g., in terms of volatility and correlation leverage effects, more difficult. We solve the dynamic portfolio problem for the constant risk premium specification in Section I \[D.5\].

C. Correlation Process and Leverage

An application of Itô’s Lemma presents immediately the correlation dynamics implied by the Wishart diffusion \[4\].

**Proposition 1** Let \( \rho \) be the correlation diffusion process implied by the covariance matrix dynamics \[4\]. The instantaneous drift and conditional variance of \( d\rho(t) \) are given by:

\[
\mathbb{E}_t[d\rho(t)] = \left[ E_1(t)\rho(t)^2 + E_2(t)\rho(t) + E_3(t) \right] dt, \tag{6}
\]

\[
\mathbb{E}_t[d\rho(t)^2] = \left[ (1 - \rho^2(t)) (E_4(t) + E_5(t)\rho(t)) \right] dt, \tag{7}
\]

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8Recent empirical evidence for a positive risk return tradeoff is presented in Ghysels, Santa-Clara and Valkanov (2005). The assumption of a constant market price of variance covariance risk can certainly be supported by a Breeden’s (1979) consumption-based model if aggregate consumption has a diffusion component given by \( \text{tr}(A\Sigma^{1/2}dB) \), for a fixed symmetric 2 \( \times \) 2 matrix \( A \) and denoting by \( \text{tr}(\cdot) \) the trace operator. In such a setting, the risk premium of asset \( i \) is given by \( \gamma \text{Cov}_t[dC/C, dS^i/S^i] = \gamma \left[ \text{tr}(A\Sigma^{1/2}\text{dB}(t)), e_i'\Sigma^{1/2}dW(t) \right] \), where \( e_i \) is the \( i \)-th unit vector and \( \gamma \) is the relative risk aversion coefficient of the representative investor. It is easy to show that the implied risk premium is affine in \( \Sigma(t) \), using the identity \( \text{Cov}_t[dB_a, dB_b] = a'b d\text{Id}t \), where \( I \) is the 2 \( \times \) 2 identity matrix and \( a, b \) are arbitrary vectors in \( \mathbb{R}^2 \). This last result is more general and holds for any stochastic discount factor with a diffusion term equal to \( \text{tr}(A\Sigma^{1/2}dB(t)) \).
where coefficients $E_1$, $E_2$, $E_3$, $E_4$, $E_5$ depend exclusively on $\Sigma_{11}$, $\Sigma_{22}$ and the model parameters $\Omega$, $M$ and $Q$. The explicit expression for coefficients $E_1, \ldots, E_n$ in the correlation dynamics is derived in Appendix A.

Since $\Sigma$ is a well-defined covariance matrix process, the correlation is a bounded process between -1 and +1. The correlation dynamics is not affine, because the correlation itself is a nonlinear function of variances and covariances. This yields a drift and a volatility of the correlation that are nonlinear functions of $\rho(t)$, where the drift is a quadratic and the volatility a cubic polynomial. The nonlinearity of the drift and volatility functions potentially imply nonlinear mean reversion and persistence properties of the correlation process, depending on the selected model parameters. It is important to note that the drift and volatility of the correlation have coefficients that are functions of the volatility of asset returns, implying that the correlation itself is not a univariate Markov process. This property emphasizes the intrinsic multivariate nature of our model for correlations, and is a clear distinction from other approaches that model the correlation by means of a scalar diffusion, as for instance in Driessen, Maenhout, and Vilkov (2006).

Black’s volatility ‘leverage’ effect, that is the negative dependence between returns and volatility, is an important empirical feature of stock returns, which has important implications for optimal portfolio choice. It has also been explicitly modeled by Heston (1993) to reproduce the option-implied volatility skew. Roll’s (1988) correlation ‘leverage’ effect, that is the negative dependence between returns and average correlation shocks, is also a well established stylized fact; see, e.g., Ang and Chen (2002). As in standard diffusion settings, leverage arises because the return dynamics can be instantaneously correlated with the variance-covariance process. This feature is not shared by multivariate GARCH-type models with dynamic correlations (see, e.g., Engle, 2002, Ledoit, Santa Clara, and Wolf, 2003, and Pelletier, 2006), in which volatilities and correlations are conditionally uncorrelated with asset returns.

The leverage in our model is controlled by the parameter vector $\overline{\rho}$ and the matrix $Q$. To see this explicitly, one can use the properties of the Wishart process to obtain:

\[
\text{corr}_t \left( \frac{dS_i}{S_i}, d\Sigma_{11} \right) = \frac{q_{11}\overline{\rho}_1 + q_{21}\overline{\rho}_2}{\sqrt{q_{11}^2 + q_{21}^2}}, \quad \text{corr}_t \left( \frac{dS_i}{S_i}, d\rho \right) = \frac{(q_{11}\overline{\rho}_1 + q_{21}\overline{\rho}_2)(1 - \rho^2(t))}{\sqrt{\left( \mathbb{E}_{t}[d\rho^2]/dt \right) \Sigma_{22}(t)}}, \tag{8}
\]

where for any $i, j = 1, 2$ parameters $q_{ij}$ denote the $ij$-th element of matrix $Q$ and the expression for $\mathbb{E}_{t}[d\rho^2]$ is given in equation (7). The expressions for the second asset are symmetric, with $q_{12}$ replacing $q_{11}$, and $q_{22}$ replacing $q_{21}$, both in the first and the second equality. $\Sigma_{11}$ replaces $\Sigma_{22}$ in the second equality. From these formulas, the parameter vector $Q\overline{\rho}$ controls the dependence between returns, volatility and correlation shocks: Volatility and correlation-leverage effects arise for all assets if both components of $Q\overline{\rho}$ are negative. The parameter $\overline{\rho}$ allows for a flexible parametrization of the leverage effects, in which the elements of the matrix $Q$ can be used to match the co-volatility of returns directly.
D. The Solution of the Investment Problem

The first challenge in solving the investment problem (3) subject to the covariance matrix dynamics (4) is that markets are incomplete. For \( n > 1 \) risky assets, we have \( n(n+1)/2 \) state variables and at least \( n(n+1)/2 > n \) Brownian innovations. This feature avoids a perfect correlation between returns and some variance covariance state-variables in the model. If we consider a market with only primary risky securities, then there is no (non-degenerate) specification of the model that allows the number of available risky assets to match the dimensionality of the Brownian motions. It follows that a multiplicity of equivalent martingale measures exists in our model.

D.1. Incomplete Market Solution Approach

To solve the portfolio problem, we consider the dual value function characterization implied by the minimax martingale measure. He and Pearson (1991) prove that this value function can be characterized in terms of the following static problem:

\[
J(X_0, \Sigma_0) = \inf_{\nu} \sup_{\pi} \mathbb{E}\left[\frac{X(T)^{1-\gamma} - 1}{1 - \gamma}\right],
\]

subject to

\[
\mathbb{E}[\xi_{\nu}(T)X(T)] \leq x,
\]

where \( \nu \) indexes the set of all equivalent martingale measures in the model and \( \xi_{\nu} \) is in the set of associated state price densities. The minimax equivalent martingale measure \( \nu^* \) is the one that achieves the infimum in equation (9).

In a constant covariance setting with complete markets, the market prices of risk associated with the Brownian innovations \( W \) are simply equal to \( \Theta = \Sigma^{1/2} \lambda \). When markets are incomplete, He and Pearson (1991) show that each admissible market price of risk can be written as the sum of two orthogonal components, one of which is spanned by the asset returns. Since in our setting there are no frictions, the first component prices the shocks to asset returns and is simply given by \( (\Sigma^{1/2} \lambda, 0_{i \times j}) \), where \( 0_{i \times j} \) is an \( i \times j \) matrix of zeros. The second component is \( (0_{2 \times 1}, \Sigma^{1/2} \nu) \), where \( \nu \) is the \( 2 \times 2 \)-dimensional matrix pricing the variance-covariance matrix innovations. Let \( \Theta_{\nu} \) be the \( 2 \times 3 \) matrix-valued extension of \( \Theta \).

\[\text{In order to hedge volatility and correlation risk, one may consider derivatives with a pay-off that depends on the variances of a portfolio of the primary assets, for instance variance swaps or some options on a “market” index; see for instance Leippold, Egloff, and Wu (2007) for a univariate dynamic portfolio choice problem with variance swaps. If these derivatives completely span the state space generated by variances and covariances, then they can be used to complete the market and to solve in closed-form the optimal portfolio choice problem. The extent to which volatility and correlations hedging demands in the basic securities will arise depends on the ability of these additional derivatives to span the variance covariance state space. Since variance swaps are available only in some specific markets, variance covariance risk is likely to be in many cases not completely hedgeable, which makes the incomplete market case of primary interest.}\]

\[\text{See also Pliska (1986) and Cox and Huang (1989) for the Markovian complete markets case.}\]
which prices the matrix of Brownian motions that generate the uncertainty in our economy, 
\([W, B]: \Theta_\nu = \Sigma^{1/2} [\lambda, \nu]\). Given \(\Theta_\nu\), the associated martingale measure \(\xi_\nu(T)\) takes the form:

\[
\xi_\nu(T) = \exp \left( - \int_0^T \left( r(s) + \frac{1}{2} tr(\Theta'_\nu(s) \Theta_\nu(s)) \right) ds - \int_0^T tr(\Theta_\nu(s)'d[W(s), B(s)]) \right),
\]

where \(tr(\cdot)\) is the trace operator. In addition, it is well known that the optimality condition for the optimization over \(\pi\) in problem (9) is

\[
X(T) = (\psi_\nu(T))^{-1/\gamma},
\]

where \(\psi\) is the multiplier of the constraint (10). Therefore, problem (9) can be written as:

\[
J(X_0, \Sigma_0) = \inf_\nu \mathbb{E}\left[ (\psi_\nu(T))^{(\gamma-1)/\gamma} \right] - \frac{1}{1-\gamma} = X_0^{\gamma} \inf_\nu \frac{1}{\gamma} \mathbb{E}\left[ \xi_\nu(T)^{(\gamma-1)/\gamma} \right]^{\gamma} - \frac{1}{1-\gamma},
\]

and we can focus without loss of generality on the solution of the problem: \(^{11}\)

\[
\hat{J}(0, \Sigma_0) = \inf_\nu \mathbb{E}\left[ \xi_\nu(T)^{(\gamma-1)/\gamma} \right]. \tag{11}
\]

To characterize the portfolio choice implications of process (11), we need to solve a corresponding Hamilton-Jacobi-Bellman equation. Therefore, it is convenient to introduce the infinitesimal generator \(A\) of the process \(\Sigma\). Since the joint process \((\Sigma_{11}, \Sigma_{22}, \Sigma_{12})\) can be written as a trivariate diffusion process, \(A\) is defined in the standard way, as in Merton (1969), for functions \(\phi = \phi(\Sigma)\). Using the particular structure of the dynamics (11) one can additionally show that \(A\) can be written in a very compact and simple matrix form. More precisely, let \(\phi = \phi(\Sigma)\) be a smooth function. Then, the generator \(A\) associated with the diffusion process (11) takes the form:

\[
A\phi = tr \left\{ (\Omega\Omega' + M\Sigma + \Sigma M') D\phi + 2\Sigma D(Q'QD\phi) \right\}, \tag{12}
\]

where \(D\) is a matrix of differential operators defined by \(D := \left( \frac{\partial}{\partial \Sigma_{ij}} \right)_{1 \leq i,j \leq 2}\). In this form, it is clear that this operator is affine in \(\Sigma\), because the argument of the trace is affine in \(\Sigma\); see also Bru (1991).

We characterize the value function of the static problem (9)–(10) by solving problem (11). The Bellman equation characterizing the minimax martingale measure for problem (11) reads:

\[
0 = \frac{\partial \hat{J}}{\partial t} + \inf_\nu \left\{ A^\nu \hat{J} + \hat{J} \left[ -\frac{\gamma - 1}{\gamma} r + \frac{1 - \gamma}{2\gamma^2} tr(\Theta'_\nu \Theta_\nu) \right] \right\},
\]

\(^{11}\)Results in Schroder and Skiadas (2003) imply that if the original optimization problem has a solution, the value function of the static problem coincides with the value function of the original problem. The above equality holds for all times, and not just at time 0. Cvitanic and Karatzas (1992) have shown that the solution to the original problem exists under additional restrictions on the utility function, most importantly that the relative risk aversion does not exceed one. Cuoco (1997) proves a more general existence result, imposing minimal restrictions on the utility function.
subject to the terminal condition \( \hat{J}(T, \Sigma) = 1 \). In this equation, the infinitesimal generator \( \mathcal{A}^\nu \) of the covariance matrix dynamics under the equivalent martingale measure indexed by \( \nu \) is:

\[
\mathcal{A}^\nu \phi = \mathcal{A} \phi - \frac{\gamma}{\gamma - 1} tr \{(Q'(\sigma' + \nu')\Sigma + \Sigma(\sigma' + \nu)Q)D\phi\},
\]

To obtain in closed form the value function to this problem, we take advantage of the fact that process (4) follows an affine Wishart dynamics also under the minimax martingale measure that characterizes the solution of the static incomplete-market problem. This important property is linked to the affine structure of the infinitesimal generator \( \mathcal{A}^\nu \). More detailed explanations and proofs can be found in Appendix A.

### D.2. Exponentially Affine Value Function and Optimal Portfolios

The affine structure of the above generator is preserved and implies that the solution of the dynamic portfolio problem is exponentially affine in \( \Sigma \), with coefficients obtained as solutions of a system of matrix Riccati differential equations.\footnote{See Reid (1972) for a review of Riccati differential equations.} These equations can be solved in closed form.

**Proposition 2** Given the covariance matrix dynamics (4), the value function of problem (3) takes the form:

\[
J(X_0, \Sigma_0) = \frac{X_0^{1-\gamma} \hat{J}(0, \Sigma_0)\gamma - 1}{1 - \gamma},
\]

where the function \( \hat{J}(t, \Sigma) \) is given by:

\[
\hat{J}(t, \Sigma) = \exp (B(t, T) + tr (A(t, T) \Sigma)), \tag{13}
\]

with \( B(t, T) \) and the symmetric matrix-valued function \( A(t, T) \) solving the system of matrix Riccati differential equations:

\[
0 = \frac{dB}{dt} + tr[A\Omega\Omega'] - \frac{\gamma - 1}{\gamma} r, \tag{14}
\]

\[
0 = \frac{dA}{dt} + \Gamma' A + A\Gamma + 2A\Lambda A + C, \tag{15}
\]

under the terminal conditions \( B(T, T) = 0 \) and \( A(T, T) = 0 \). Constant matrices \( \Gamma, \Lambda \) and \( C \), as well as the closed-form solution of the system of matrix Riccati differential equations (14)-(15), are reported in Appendix A.

**Remark.** In the literature on affine term structure models, it is well known that modeling correlated stochastic factors is not straightforward. Duffie and Kan (1996) show that, for a well-defined affine process to exist, parametric restrictions on the drift matrix of the factor dynamics have to be satisfied. In particular, its out-of-diagonal elements must have the
same sign. This feature restricts the correlation structures that these models can fit (see, e.g., Duffee, 2002). In the Dai and Singleton (2000) classification for affine $A_m(n)$ models, specific restrictions need to be imposed for the model to be solved in closed form: the Gaussian factors are allowed to be correlated, but the correlation between Gaussian and square-root factors must be zero. This issue is well acknowledged also in the portfolio choice literature. For instance, Liu (2007) addresses it by assuming a triangular factor structure in an affine portfolio problem with two risky assets.

Using the Wishart specification (4) for the variance covariance dynamics, we are able to obtain a simple affine solution for problem (3), which does not imply excessive restrictions on the interdependency of variance covariance factors. The Wishart specification of the state space is also a powerful tool more generally, e.g., for term structure modeling. Buraschi, Cieslak and Trojani (2007) develop a completely affine model with Wishart state dynamics, in which the flexibility of the stochastic volatilities and correlations of factors helps to explain several empirical regularities of the term structure at the same time. □

One advantage of the exponentially affine form of function $\hat{J}$ in Proposition 2 is that it allows for a simple description of the partial derivatives of the marginal indirect utilities of wealth with respect to the individual variance and covariance factors $\Sigma_{ij}(t)$, $1 \leq i, j \leq 2$. This property provides us with a simple and easily interpretable solution to the incomplete-market multivariate portfolio choice problem.

**Proposition 3** Let $\pi$ be the optimal portfolio obtained under the assumptions of Proposition 2. It then follows,

$$\pi = \frac{\lambda}{\gamma} + 2 \left[ (q_{11}\bar{p}_1 + q_{21}\bar{p}_2)A_{11} + (q_{12}\bar{p}_1 + q_{22}\bar{p}_2)A_{12} \right]$$

$$\pi = \frac{\lambda}{\gamma} + 2 \left[ (q_{11}\bar{p}_1 + q_{21}\bar{p}_2)A_{11} + (q_{12}\bar{p}_1 + q_{22}\bar{p}_2)A_{12} \right],$$

(16)

where $A_{ij}$ denotes the $ij$-th component of the matrix $A$, which characterizes the function $\hat{J}(t, \Sigma)$ in Proposition 2 and the coefficients $q_{ij}$ are the entries of the matrix $Q$ appearing in the Wishart dynamics (4).

The portfolio policy $\pi = (\pi_1, \pi_2)'$ is the sum of a myopic demand and a hedging demand. The interpretation is simple and can easily be linked to the Merton’s (1969) solution. The myopic portfolio is the optimal portfolio that would prevail in an economy with a constant opportunity set, i.e., a constant covariance matrix. When the opportunity set is stochastic, the optimal portfolio also consists of an intertemporal hedging demand. This portfolio component reduces the impact of shocks to the indirect utility of wealth. The size of intertemporal hedging depends on two components: (a) the extent to which investors marginal utility of wealth is indeed affected by shocks in the state variables and (b) the extent to which these state variables are correlated with returns. Using Merton’s notation, the optimal hedging demand in our model, denoted $\pi_h$, can be written as:

$$\pi_h = - \sum_{i,j} J_{XX}^{-1} \Sigma_{ij}^{-1} \text{Cov}_t(I_S^{-1}dS, d\Sigma_{ij}) \frac{dt}{J_{XX}}.$$

(17)
In this expression, the term \(-\frac{J_{X\Sigma_{ij}}}{XJ_{XX}} = -\frac{J_{X}}{XJ_{XX}} \cdot \frac{J_{X\Sigma_{ij}}}{J_{X}} = A_{ij}\) is a risk-tolerance weighted sensitivity of the log indirect marginal utility of wealth with respect to the state variable \(\Sigma_{ij}\). The regression coefficient \(\Sigma^{-1}Cov_t(I_S^{-1}dS, d\Sigma_{ij})\) captures the ability of asset returns to hedge unexpected changes in this state variable. The hedging portfolio is zero if and only if either \(J_{X\Sigma_{ij}} = 0\) for all \(i\) and \(j\) (e.g., log utility investors) or \(\Sigma^{-1}Cov_t(I_S^{-1}dS, d\Sigma_{ij}) = 0\). Using the properties of the Wishart process, the hedging portfolio then follows in explicit form:

\[
\pi = \Sigma^{-1}Cov_t(I_S^{-1}dS, \sum_{i,j} A_{ij}d\Sigma_{ij}) = 2 \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} \begin{pmatrix} q_{11}\rho_1 + q_{12}\rho_2 \\ q_{12}\rho_1 + q_{22}\rho_2 \end{pmatrix},
\]

(18)

This is the second term in the sum on the right hand side of formula (16). Hedging demands are generated by the willingness to hedge unexpected changes in the portfolio total utility due to shocks in the state variables \(\Sigma_{ij}\): Hedging demands proportional to \(A_{ij}\) are hedging demands against unexpected changes in \(\Sigma_{ij}\). Therefore, hedging portfolios proportional to \(A_{11}\) and \(A_{22}\) are volatility hedging portfolios. Similarly, hedging portfolios proportional to \(A_{12}\) are covariance hedging portfolios. We can clarify the direct role of parameters \(Q\) and \(\bar{\rho}\) in the hedging demand by writing equation (18) in the equivalent form:

\[
\pi = 2A_{11} \begin{pmatrix} q_{11}\rho_1 + q_{21}\rho_2 \\ 0 \end{pmatrix} + 2A_{22} \begin{pmatrix} 0 \\ q_{12}\rho_1 + q_{22}\rho_2 \end{pmatrix} + 2A_{12} \begin{pmatrix} q_{12}\rho_1 + q_{22}\rho_2 \\ q_{11}\rho_1 + q_{21}\rho_2 \end{pmatrix}.
\]

(19)

Parameters \(Q\) and \(\bar{\rho}\) determine the ability of asset returns to span shocks in latent risk factors, because they completely determine the regression coefficients \(\Sigma^{-1}Cov_t(I_S^{-1}dS, d\Sigma_{ij})\) in equation (17):

\[
\Sigma^{-1}Cov_t(I_S^{-1}dS, d\Sigma_{11}) = 2 \begin{pmatrix} q_{11}\rho_1 + q_{21}\rho_2 \\ 0 \end{pmatrix},
\]

(20)

\[
\Sigma^{-1}Cov_t(I_S^{-1}dS, d\Sigma_{22}) = 2 \begin{pmatrix} 0 \\ q_{12}\rho_1 + q_{22}\rho_2 \end{pmatrix},
\]

(21)

\[
\Sigma^{-1}Cov_t(I_S^{-1}dS, d\Sigma_{12}) = 2 \begin{pmatrix} q_{12}\rho_1 + q_{22}\rho_2 \\ q_{11}\rho_1 + q_{21}\rho_2 \end{pmatrix}.
\]

(22)

By comparing equations (20)-(22) with the leverage expressions (8), we see that the sign of each component of \(\Sigma^{-1}Cov_t(I_S^{-1}dS, d\Sigma_{ij})\) is equal to the sign of the co-movement between returns, variances, and correlations. Thus, the role of parameters \(\bar{\rho}\) and \(Q\) for the optimal hedging demand is simply interpreted. The first and second column of directly \(Q\) impact the volatility and covariance hedging demand for the first and second asset, respectively, via the coefficient vectors \((q_{11}, q_{21})'\) and \((q_{12}, q_{22})'\). Instead, the parameter \(\bar{\rho}\) directly impacts on all hedging portfolios. Overall, risky assets are better at spanning the risk generated by variance covariance shocks when \(q_{11}\rho_1 + q_{21}\rho_2\) and \(q_{12}\rho_1 + q_{22}\rho_2\) are greater in absolute value. Moreover, note that a risky asset \(i\) is a better hedging instrument against its stochastic volatility \(\Sigma_{ii}\), and less so against shocks in the covariance \(\Sigma_{ij}\), when the coefficient \(q_{1i}\rho_1 + q_{2i}\rho_2\) of the
given asset $i$ is the largest one. Despite the simple form of the hedging portfolio, a rich variety of other hedging implications can arise. For instance, when $q_{11} \tilde{\rho}_1 + q_{21} \tilde{\rho}_2$ and $q_{12} \tilde{\rho}_1 + q_{22} \tilde{\rho}_2$ are both negative, volatility and correlation leverage effects arise for all returns. However, if parameters $q_{11} \tilde{\rho}_1 + q_{21} \tilde{\rho}_2$ and $q_{12} \tilde{\rho}_1 + q_{22} \tilde{\rho}_2$ have mixed signs some returns will feature volatility or correlation leverage effects, but at the same time other returns will not.

D.3. Sensitivity of the Marginal Utility of Wealth to the State Variables

The second determinant of the hedging demand is related to the sensitivity of the marginal utility of wealth to the state variables $\Sigma_{ij}$. This effect is summarized by the components $A_{ij}$. Therefore, it is useful to gain some intuition on the dependence of $A_{ij}$ on the structural parameters. For brevity, we focus on investors with risk aversion above 1 and on an excess return parameter vector $\lambda$ such that $\lambda_1 \lambda_2 \geq 0$. This setting includes as a special case the choice of parameters implied by the model estimation results in Section II.

**Proposition 4** Consider an investor with risk aversion parameter $\gamma > 1$. (i) The following inequalities, describing the properties of the sensitivity of the indirect marginal utility of wealth with respect to changes in the state variables $\Sigma_{ij}$ hold true: $A_{11}, A_{22} \leq 0$ and $|A_{12}| \leq \frac{|A_{11} + A_{22}|}{2}$. (ii) Furthermore, if it is additionally assumed that either $\lambda_1 \geq \lambda_2 \geq 0$ or $\lambda_1 \leq \lambda_2 \leq 0$, then $A_{12} \leq 0$ and $|A_{22}| \leq |A_{12}| \leq |A_{11}|$.

This result is important because it describes the link between the indirect marginal utility of wealth and the state variables $\Sigma_{ij}$: $A_{ij} = -\frac{J_X \Sigma_{ij}}{X J_X X}$. This sensitivity is increasing with the sensitivity of the stochastic opportunity set, i.e. the squared Sharpe ratio, to unexpected changes in $\Sigma_{ij}$. For a constant market price of variance covariance risk, the squared Sharpe ratio is given by $\lambda_1^2 \Sigma_{11} + \lambda_2^2 \Sigma_{22} + 2 \lambda_1 \lambda_2 \Sigma_{12}$. Its sensitivity to the variance risk factor $\Sigma_{11}$ is highest when $|\lambda_1| \geq |\lambda_2|$, and vice versa. The sensitivity to the covariance risk factor is bounded by the absolute average sensitivity to the variance factors, because squared Sharpe ratios depend on $\Sigma_{12}$ via a loading that is twice the product of $\lambda_1$ and $\lambda_2$. To understand the sign of $A_{ij}$, recall that investors with risk aversion above 1 have a negative utility function bounded from above. Wealth homogeneity of the solution in Proposition 2 implies $J_X(t) = X(t)^{\gamma-1} J(t, \Sigma(t))^{1-\gamma}$, so that $J_{\Sigma_{ij}}$ and $J_{XX\Sigma_{ij}}$ have the same sign. An increase in the variance $\Sigma_{ii}$ of one risky asset increases the squared Sharpe ratio of the optimal portfolio, but at the same time it increases the squared Sharpe ratio variance. Investors with risk aversion above 1 dislike this effect, because ex-ante they profit less from higher future Sharpe ratios than they suffer from higher future Sharpe ratio variances. These features imply the negative sign of $A_{ii}$. The sign of $J_{\Sigma_{ij}}$ depends on how squared Sharpe ratios depend on $\Sigma_{ij}$. If $\lambda_1 \lambda_2 \geq 0$, $\Sigma_{12}$ affects the squared Sharpe ratios positively and using the same arguments as those for the volatility factors imply $A_{ij} \leq 0$.

13This variance equals $4 \lambda^2 \Sigma \lambda Q^\prime Q \lambda$, using the properties of Wishart processes.
D.4. Volatility and Correlation Hedging

A second way to gain economic intuition on the implications of the model is to separate the total hedging demand into two distinct hedging components: A volatility hedging part dealing with changes in returns covariance due to changes in volatility and a correlation hedging demand.

**Proposition 5** The hedging demand of the optimal portfolio in equation (16) for asset $i$ is the sum of three components $\pi_{i}^{\text{vol}}$, $\pi_{i}^{\text{vol/cov}}$, $\pi_{i}^{\rho}$, which hedge, respectively, pure volatility risk, covariance risk due to volatility, and correlation risk. The explicit expressions for these hedging demands are as follows:

1. **Pure Volatility hedging:**
   \[
   \pi_{i}^{\text{vol}} = 2(q_{11}\bar{p}_{1} + q_{21}\bar{p}_{2})A_{11},
   \pi_{i}^{\text{vol}} = 2(q_{12}\bar{p}_{1} + q_{22}\bar{p}_{2})A_{22}.
   \] (23)

2. **Covariance hedging due to volatility:**
   \[
   \pi_{i}^{\text{vol/cov}} = 2(q_{11}\bar{p}_{1} + q_{21}\bar{p}_{2})A_{12}\rho\sqrt{\frac{\Sigma_{22}}{\Sigma_{11}}},
   \pi_{i}^{\text{vol/cov}} = 2(q_{12}\bar{p}_{1} + q_{22}\bar{p}_{2})A_{12}\rho\sqrt{\frac{\Sigma_{11}}{\Sigma_{22}}}.
   \] (24)

3. **Correlation hedging:**
   \[
   \pi_{i}^{\rho} = 2A_{12}\left[ (q_{12}\bar{p}_{1} + q_{22}\bar{p}_{2}) - (q_{11}\bar{p}_{1} + q_{21}\bar{p}_{2})\rho\sqrt{\frac{\Sigma_{22}}{\Sigma_{11}}} \right],
   \pi_{i}^{\rho} = 2A_{12}\left[ (q_{11}\bar{p}_{1} + q_{21}\bar{p}_{2}) - (q_{12}\bar{p}_{1} + q_{22}\bar{p}_{2})\rho\sqrt{\frac{\Sigma_{11}}{\Sigma_{22}}} \right].
   \] (25)

Proposition 5 essentially rephrases Merton (1969) results with respect to the state variables $\Sigma_{11}$, $\Sigma_{22}$ and $\rho$. The hedging demands are proportional to the sensitivity of the marginal utility of wealth to the risk factors, where each sensitivity is weighted by the coefficient of a conditional linear regression of the risk factor on asset returns.

The pure volatility hedging demands in equation (23) remain unchanged and reflect previous results. The covariance hedging demands due to volatility in equation (24) are proportional to the level of correlation and $A_{12}$. The intuition behind that is that stronger correlations imply a greater effect of a change in volatility on covariances, as well as a larger risk of an adverse covariance movement. The sign of $A_{12}$ depends on $\lambda$, and it is negative when $\lambda_{1}\lambda_{2} \geq 0$ and the relative risk aversion is above 1; see Proposition 4, (ii). Assuming that volatilities and the correlation co-vary negatively with both risky returns, this implies a positive (negative) covariance hedging demand due to volatility if and only if $\rho(t)$ is negative.
The correlation hedging demand in equation (25) is also proportional to $A_{12}$. This is the intuition: The greater $A_{12}$, the greater the sensitivity of the marginal utility of wealth to correlation shocks, and the stronger the correlation hedging motive. Contrary to the covariance hedging component, correlation hedging demands might be greater than volatility hedging demands. According to equations (23)-(25), volatility and correlation leverage effects on both assets are more likely to exist when the average correlation between returns is negative.

D.5. Constant Risk Premia and Stochastic Interest Rate

When risk premia are stochastic, it is more difficult to interpret the hedging demand as a hedging portfolio against variance covariance risk, since a greater (lesser) covariance matrix may imply, at the same time, a greater (lesser) risk premium for some risky asset. However, even in this setting it is possible to isolate theoretically the part of hedging demands generated directly by shocks in the returns covariance matrix from the part due to the stochastic risk premium component. The results in the Appendix (Proposition 7) demonstrate that in absolute value the total hedging demand of Proposition 3 is equal to the part of hedging demand for hedging exclusively unexpected changes in the stochastic variance covariance matrix. This result implies that in order to quantify the absolute size of volatility and correlation hedging demands we can use, without loss of generality, the absolute value of the hedging demand components $\pi^\rho$, $\pi^{\text{vol}}$ and $\pi^{\text{vol/cov}}$ in Proposition 5.

A direct way to study pure variance covariance hedging demands is by assuming a constant risk premium. For analytical purposes, this comes at the cost of specifying a Wishart state process for the precision matrix $\Sigma^{-1}$, which implies a less transparent interpretation of some model parameters. This can be quite easily achieved even in a setting with a stochastic interest rate, where the interest rate can depend also on some of the risk factors driving the covariance matrix of asset returns.

Assumption 1 Let the process $Y$ satisfy the following Wishart dynamics:

$$dY(t) = [\Omega Y + MY(t) + Y(t)M']dt + Y^{1/2}(t)dBQ + Q'dB'Y^{1/2}(t),$$

where matrices $\Omega$, $M$ and $Q$ are now of dimension $3 \times 3$ and where $B$ is a three-dimensional square matrix of independent Brownian motions. We model $\Sigma^{-1}$ as a projection of matrix $Y$:

$$\Sigma^{-1} = SYS',$$

In this way, local asymmetries in the covariance matrix dynamics can be introduced in the model. To model asymmetric correlations across regimes, Ang and Bekaert (2002) use an i.i.d. switching regime setting, in which one of the regimes is characterized by greater correlations and volatilities.
where the $2 \times 3$ matrix $S$ is such that $SS' = id_{2 \times 2}$. The stochastic riskless rate $r(t)$ is defined by:

$$r(t) = r_0 + tr(Y(t)D),$$

(27)

where $r_0 > 0$ and $D$ is a $3 \times 3$ matrix.

Notice that the non-negativity of $r(t)$ can be easily ensured simply by assuming that matrix $D$ is positive definite. Since $\Sigma = SY^{-1}S'$, it is apparent that we can define $\Sigma^{1/2}$ as the $2 \times 3$ matrix $SY^{-1/2}$ and introduce the following process for asset returns:

$$dS(t) = IS\left[\begin{pmatrix} r(t) + \mu_1 \\ r(t) + \mu_2 \end{pmatrix} dt + \Sigma^{1/2}(t)dW(t)\right],$$

(28)

where the excess return vector $\mu^e = (\mu_1', \mu_2')' \in \mathbb{R}^2$ is constant and $r(t)$ is given by equation (27). To model leverage effects, we define the standard Brownian motion $W$ as:

$$W(t) = \sqrt{1 - \rho^T\rho}Z(t) + B(t)\bar{\rho},$$

(29)

where $Z$ is a three-dimensional standard Brownian motion and $\bar{\rho} = (\bar{\rho}_1, \bar{\rho}_2, \bar{\rho}_3)'$ is a vector of instantaneous correlations with $\bar{\rho}_i \in [-1, 1]$ and $\bar{\rho}^T\bar{\rho} \leq 1$.

This setting is effectively a six-factor model with some interest rate risk factors that might be linked to the covariance matrix of stock returns, depending on the form of the matrix $D$ in equation (27). The squared Sharpe ratio in this model is affine in $Y$. Therefore, we can solve in closed-form the dynamic portfolio choice problem in this extended dynamic setting as well.

**Proposition 6** The solution of the portfolio problem (3) for the returns dynamics (26)-(28) and under a stochastic interest rate (27) is:

$$J(X_0, Y_0) = \frac{X_0^{1-\gamma} \tilde{J}(0, Y_0)^\gamma - 1}{1 - \gamma},$$

where

$$\tilde{J}(t, Y) = \exp(B(t, T) + tr(A(t, T)Y)),$$

with $B(t, T)$ and the symmetric matrix-valued function $A(t, T)$ solving in closed-form the following system of matrix Riccati differential equations:

$$-\frac{dB}{dt} = -\gamma - 1\gamma r_0 + tr(A\Omega\Omega'),$$

(30)

$$-\frac{dA}{dt} = \Gamma' A + A \Gamma + 2A'\Lambda A + C,$$

(31)

15 A possible choice for $S$ is a $2 \times 3$ selection matrix, e.g.:

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In this case, $SS' = id_{2 \times 2}$ and $SYS'$ is the $2 \times 2$ upper diagonal sub-block of $Y$.

16 A proof of this statement is presented in Appendix A.
subject to $B(T,T) = A(T,T) = 0$. In these equations, the coefficients $\Gamma$, $\Lambda$ and $C$ are given by:

$$\Gamma = M - \frac{\gamma}{\gamma - 1}Q'\mu'\mu' S$$
$$\Lambda = Q'(\gamma I_3 + (1 - \gamma)\rho \rho')Q$$
$$C = \frac{1 - \gamma}{2\gamma^2}S'\mu'\mu' S - \frac{\gamma - 1}{\gamma} D.$$  

Finally, the optimal policy for this portfolio problem reads:

$$\pi = \frac{1}{\gamma} \Sigma^{-1} \mu' + 2 \Sigma^{-1} S A Q' \rho.$$  \hspace{1cm} (32)

The optimal policy (32) consists of a myopic and an intertemporal hedging portfolio, which are both proportional to the stochastic inverse covariance matrix. As noted by Chacko and Viceira (2005), in the univariate setting the relative size of the hedging and myopic demands is independent of the current level of volatility. This property also holds in the multivariate case, in the sense that both policies are proportional to the inverse covariance matrix $\Sigma^{-1}$. In what follows, we investigate the empirical implications of the previous two specifications in a set of realistic economic scenarios.

II. Hedging Stochastic Variance Covariance Risk

We quantify volatility and correlation hedging for a realistic stock-bond portfolio problem, in which a portfolio manager allocates wealth between the S&P500 Index Futures contract, traded at the Chicago Mercantile Exchange, the Treasury Bond Futures contract, traded at the Chicago Board of Trade, and a riskless asset.

A. Data and Estimation Results

The model can be estimated by GMM using the conditional moment conditions of the process, which are derived in closed-form\(^{17}\). The methodology is of simple implementation and provides asymptotic tests for overidentifying restrictions. As a first step, we use Andersen, Bollerslev, Diebold and Labys (2003) methodology to obtain model-free realized volatilities and covariances from daily quadratic variations and covariances of the log price processes\(^{18}\). The high-frequency dataset we use to compute realized volatilities and correlations in our

\(^{17}\) Detailed expressions are given in Appendix B. The closed form expression for the moments of the Wishart process can be found, e.g., in the Appendix of Buraschi, Cieslak and Trojani (2007).

\(^{18}\) Bandorf-Nielsen and Shephard (2002) use quasi-likelihood estimation based on a time-series of realized volatilities to estimate the parameters of continuous-time stochastic volatility models. Their estimators are characterized by negligible bias. Bollerslev and Zhou (2002) propose GMM estimation with high-frequency foreign exchange and equity index returns for stochastic volatility models. Monte Carlo evidence indicates that the estimation of the parameters is accurate and the statistical inference is reliable.
empirical exercise is from ‘Price-Data’ and ‘Tick-Data’ and includes tick-by-tick Futures returns for the S&P500 index and the 30-year Treasury bond from January 1990 to October 2003.

We use both weekly and monthly returns, realized volatilities and covariances, in order to investigate the impact of different exact discretizations of the model on the selected parameters. Let \( \theta := ( \text{vec}(M)' , \text{vec}(Q)' , \lambda', \rho')' \) be the vector of parameters in the model implying a positive risk return tradeoff. A GMM estimator of \( \theta \) is given by:

\[
\begin{align*}
\hat{\theta} = \arg \min_{\theta} & \ (\mu(\theta) - \mu_T)'V(\theta)(\mu(\theta) - \mu_T), \\
\text{subject to} \ & \mu(\theta) = \mu_T,
\end{align*}
\]

where \( \mu_T \) is the vector of empirical moments, implied by the historical returns and their realized variance-covariance matrices, and \( \mu(\theta) \) is the theoretical vector of moments in the model. \( V(\theta) \) is the GMM optimal weighting matrix in the sense of Hansen (1982), estimated using a Newey West estimator with 12 lags. We estimate \( \theta \) using moment conditions that provide information about returns, their realized volatilities and correlations, and the leverage effects. \( \mu_T \) consists of the following moment restrictions: Unconditional risk premia of log-returns, unconditional first and second moments of variances and covariances of log-returns, and unconditional covariances between returns and each element of the variance-covariance matrix of returns. This leaves us with 17 moment restrictions for a 13-dimensional parameter vector that has to be estimated, so that the model has 4 over-identifying restrictions.

Table I summarizes sample unconditional moments of returns, realized volatilities and realized correlations in our dataset.

Insert Table I about here

The monthly unconditional mean of S&P500 and 30-year Treasury Futures returns is about 0.042% and 0.015%, respectively. Stock index future returns feature a higher unconditional volatility and a higher volatility of volatility. The unconditional sample correlation is about 5% and the sample standard deviation of the correlation is about 33%.

A.1. Basic Estimation Results

Table II presents results of our GMM model estimation, both for weekly and monthly returns.

Insert Table II about here

Hansen’s test of over-identifying restrictions does not reject the model specification at the weekly and monthly frequency. The obtained parameter estimates strongly support the

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19 See the web pages www.grainmarketresearch.com and www.tickdata.com for details. When we estimate the model with three risky assets, in Section II.B.4, we also use returns for the Nikkei225 Index Futures contract.

20 We also estimated the model using daily returns, realized volatilities and covariances, but in this case the Hansen’s statistic rejected the model. We found that jumps in returns and realized conditional second
multivariate specification of the correlation process in our setting. First, the null hypothesis that the volatility of volatility matrix $Q$ is identically zero is clearly rejected, at the 5% significance level, which supports the hypothesis of a stochastic correlation process. Second, the parameter estimates for the components of matrix $M$ are also almost all significant, supporting a multivariate mean reversion and persistence structure in variances and covariances. In particular, by looking at the implied eigenvalues of matrix $M$ we find clear-cut evidence for two very different mean reversion frequencies, a high and low one, underlying the returns covariance matrix. All estimated eigenvalues are negative, which supports the stationarity hypothesis of the variance covariance process. The larger eigenvalues estimated with monthly returns imply a larger persistence of variance covariance shocks at monthly frequencies. The estimated components of vector $\bar{\rho}$ are all negative and significant. Together with the positive point estimates for the coefficients of the matrix $Q$, this feature implies the existence of volatility and correlation leverage features for all risky asset returns, which is consistent with the example discussed in the Introduction. The point estimates of $\bar{\rho}_1$ and $\bar{\rho}_2$ implied by monthly returns are not statistically different from each other. At the same time, the point estimates for matrix $Q$ highlight a large estimated parameter $q_{11}$. Therefore, the first leverage parameter $q_{11}\rho_1 + q_{21}\rho_2$ is about twice the size of the leverage parameter $q_{12}\rho_1 + q_{22}\rho_2$. This implies that S&P 500 returns are better hedging vehicles to hedge their volatility risk than 30-year Treasury returns. At the same time, 30-year Treasury returns are better hedging instruments for hedging the covariance risk of the two asset classes.

A.2. Estimated Correlation Process and Leverage Effects

Using the model parameter estimates, we can easily study the nonlinear dynamic properties of the implied correlation process. The correlation drift and correlation volatility implied by the GMM point estimates for monthly returns are illustrated in Figure 3.

Insert Figure 3 about here

The estimated correlation volatility peaks at a correlation level of approximately 0.3. The correlation drift is positive for correlations that range from $-1$ to approximately 0.2, at which level it crosses the zero line and becomes negative. The nonlinearity of the drift induces a nonlinear mean-reversion of the correlation to a level slightly higher than its unconditional mean of approximately 0.05.

A convenient measure of the mean reversion properties of a nonlinear diffusion process is given by its pull function - see Conley, Hansen, Luttmer, and Scheinkman (1997). The pull function $\phi(x)$ of a process $X$ is the conditional probability that $X_t$ reaches the value $x + \epsilon$ moments are mainly responsible for this rejection, suggesting the misspecification of a pure multivariate diffusion in this context. The extension of our setting to a matrix valued affine jump diffusion would be an interesting topic for future research.
before $x - \epsilon$, if initialized at $X_0 = x$. To first order in $\epsilon$, this probability is given by:

$$\varphi(x) = \frac{1}{2} + \frac{\mu_X(x)}{2\sigma_X^2(x)} \epsilon + o(\epsilon),$$ \hspace{1cm} (34)

where $\mu_X$ and $\sigma_X$ are the drift and the volatility function of $X$. Figure 4 presents nonparametric estimates of the pull function for the correlation and volatility processes of the S&P500 Futures and 30-year Treasury Futures returns, shifted by the factor $1/2$ in equation (34).

**Insert Figure 4 about here**

Each panel in the left column plots the estimated pull functions for volatilities and correlations for weekly and monthly data. The panels in the middle (right) column plot pull functions estimated from a long time series of observations simulated from our model. These pull functions are all inside a two-sided 95%—confidence interval around the empirical pull functions, which indicates that our model can capture adequately the nonlinear mean reversion properties of volatilities and correlations in our data. Consistently with the different persistence of the variance covariance process implied by the point estimates for matrix $M$, the pull functions for monthly data are on average lower than those estimated for weekly data (see Panels 7, 8 and 9). Estimated pull functions for the correlation are highly asymmetric and are typically smaller in absolute value for positive correlations above 0.3 than for negative correlations below -0.4. This feature indicates a higher persistence of correlation shocks when correlations are positive and large. The pull function for the volatility of S&P500 future returns in the first row of Figure 4 is almost flat and moderately positive for volatilities larger than 10%. On average, the pull function estimated for the volatility of 30-year Treasury future returns tends to be larger in absolute value, which indicates a lower persistence of shocks in the volatility of Treasury future returns. The dynamic properties of the correlation process have implications for the unconditional distribution of the correlation. For instance, a more asymmetric mean reversion typically yields a more asymmetric unconditional distribution. This intuition if confirmed by nonparametric kernel density estimates applied to model-implied and realized correlations

Given the apparent evidence of nonlinear mean reversion features of volatilities and correlations in the data, it is natural to ask to which extent univariate Markov continuous-time models can reproduce these features accurately. We estimate Heston (1993) square-root processes for volatility and an autonomous specification for the correlation process à la Driessen, Maenhout, and Vilkov (2006). When we compute the model implied pull functions, we find that they are often outside the 95% confidence band around the empirical pull function. Moreover, these univariate specifications imply pull functions that are almost linear in shape, which is difficult to reconcile with the data.

Additional evidence supporting the existence of a multifactor structure in the variance covariance structure of asset returns is provided by Da and Schaumburg (2006), who use

\[21\] More details on these results are available from the authors upon request.
Asymptotic Principal Component analysis applied to a panel of realized volatilities for US stock returns. They find that three to four factors explain no more than 60% of the variation in realized volatility measures. They also show that the forecasting power of multifactor volatility models is superior to univariate ones. Similar findings are obtained by Andersen and Benzoni (2007) using realized volatilities extracted from intra-day Treasury Bill data. Additional economic insight on these features is provided by Calvet and Fisher (2007) who develop an equilibrium model in which innovations in dividend volatility are affected by shocks that decay at different frequencies. They show that the different persistence of these volatility factors is crucial in terms of the forecasting performance of the model.

The GMM point estimates for parameters $Q$ and $\rho$ imply a volatility and correlation leverage effect for all asset returns, which are illustrated in the four panels of Figure 5.

The negative relation between returns and volatilities in the scatter plots of Panel 1 and Panel 2 highlights the volatility leverage for both returns. The sample correlation between returns and changes in volatility in these scatter plots is $-0.24$ and $-0.30$ for the S&P500 Futures and the 30-year Treasury Futures returns, respectively. The correlation leverage for both returns is summarized in the scatter plots of Panel 3 and Panel 4. The implied sample correlation between returns and correlation changes is $-0.38$ and $-0.31$ for the S&P500 and the Treasury Futures return, respectively.

**B. The Size of Correlation Hedging**

In what follows, we study the structure of the hedging demands based on our parameter estimates, and compute the optimal hedging demand components $\pi_{\text{vol}}^i$, $\pi_{\text{vol/cov}}^i$, and $\pi_{\rho}^i$ in Proposition 5 as a function of the relative risk aversion and the investment horizon. For computing the hedging demand components, we initialize $\Sigma(t)$ at its unconditional sample value.

**B.1. Basic Results**

Table III summarizes the estimated pure volatility and correlation hedging demands, as a percentage of the myopic portfolio allocations.

Overall, monthly estimates of the hedging demands are greater than the weekly estimates. The main reason for this difference is the lower estimated persistence of weekly variance-covariance shocks implied by our GMM estimates for matrix $M$ in the last section. A more persistent variance covariance process implies that shocks to the variance covariance have more persistent effects on future squared Sharpe ratios and their volatilities. This feature yields a higher absolute sensitivity of the marginal utility of wealth to variance covariance.
shocks and greater absolute hedging demands on average; see equation (19) and the following discussion.

Consider, for illustration purposes, the hedging demands estimated for monthly returns under an investment horizon of $T = 5$ years and a relative risk aversion parameter $\gamma = 6$. The estimated risk premium for the S&P500 Futures returns implies a loading of volatility higher than the one for the risk premium of 30-year Treasury Futures returns: $\lambda_1 > \lambda_2$. Thus, the estimated sensitivity of the marginal utility of wealth to the returns volatility is highest for S&P500 Futures returns: $|A_{11}| > |A_{22}|$. Moreover, at the GMM parameter estimates stocks are better instruments to hedge their volatility than bonds: $q_{11}\mu_1 + q_{21}\mu_2 \geq q_{12}\mu_1 + q_{22}\mu_2$. These features imply a higher volatility hedging demand for stocks (about 13% of the myopic portfolio) relative to the volatility hedging component for bonds (about 8% of the myopic portfolio). The total average volatility hedging demand is approximately 10.5% of the myopic portfolio, while the total average correlation hedging demand on the two risky assets is higher (about 12.5%). At the GMM point estimates, bonds are better hedging vehicles than stocks to hedge covariance risk: $q_{12}\rho_1 + q_{22}\rho_2 \geq q_{11}\rho_1 + q_{21}\rho_2$. This effect determines the higher correlation hedging demand for bonds (about 17%) than for stocks (about 7%).

Given the evidence in the previous section of a misspecification of univariate models with respect to the variance covariance dynamics in the data, we compare the portfolio implications of our setting with those of univariate portfolio choice models with stochastic volatility; see Heston (1993) and Liu (2001), among others. This is easy, since these models are nested in our setting in the special case in which the dimension of the investment opportunity set is set equal to 1. For each risky asset in our data set, we estimate these univariate stochastic volatility models by GMM. Table IV presents the resulting parameter estimates. For each asset, the table also presents estimated volatility hedging demands in percentage of the myopic portfolio.

Insert Table IV about here

For illustration purposes, consider a relative risk aversion coefficient $\gamma = 8$ and an investment horizon of $T = 5$ years. The volatility hedging demands estimated for the univariate models are 4.8% and 4%, respectively, for stocks and bonds. In the multivariate model, the corresponding pure volatility hedging demands are 13.6% and 8.8%, respectively. Moreover, the average total hedging demand is as large as 21.1% of the myopic portfolio. One explanation for these findings is the very different mean reversion and persistence properties of second moments in the data relative to those implied by univariate stochastic volatility models. This is emphasized by the different pull functions of the correlation process implied by the univariate and our multivariate model; see again in Section I. A second reason is the fact that univariate models cannot capture the correlation and co-volatility dynamics, which generate a good portion of the total hedging demand in the multivariate setting.
B.2. Comparative Statics

To get more detailed insight into the determinants of hedging motives in our model, it is useful to study comparative statics with respect to model parameters. To this end, we change the value of these parameters in an interval of one sample standard deviation around the true parameter estimate in the data, and compute the implied hedging demand components.

It is natural to focus on parameters that have an impact on the indirect marginal utility sensitivities $A_{ij}$, and the volatility and correlation leverage effects. Matrix $M$ drives the persistence on the variance covariance process, but leaves unaffected the leverage properties of asset returns. For brevity, we consider comparative statics with respect to the parameters $m_{12}$ and $m_{22}$. The matrix $Q$ and vector $\vec{\rho}$ affect primarily the ability of each asset to span unexpected shocks in the variance covariance process, by influencing the leverage properties of asset returns. They also influence the persistence of variance covariance shocks under the minimax measure, which is the relevant pricing measure to describe the expected utility implied in the incomplete market setting; see again He and Pearson (1991). To study the effects of these parameters, we consider for brevity comparative statics with respect to $q_{11}$ and both components of the vector $\vec{\rho}$. In doing so, we decompose the total effect on hedging demands in a part due to a modification of the leverage properties of returns and in a part due to the change in the marginal utility sensitivity coefficients $A_{ij}$. The investment horizon we consider is $T = 5$ years and the relative risk aversion is $\gamma = 6$.

In the upper plots of Figure 6, the comparative statics with respect to $m_{12}$ show that, coeteris paribus, correlation and volatility hedging demands increase with $m_{12}$: For a high parameter value of $m_{12}$ equal to 1.18, i.e. one standard deviation above the GMM estimate, the pure correlation (volatility) hedging component increases to 8.5% (14%) for stocks and 25% (14%) for bonds.

This effect is due exclusively to the higher persistence of the variance covariance process implied by an increase in $m_{12}$, which implies a greater absolute marginal utility sensitivity to all variance covariance risk factors.

The middle plots of Figure 6 show the comparative statics with respect to $m_{22}$. As $m_{22}$ increases, all volatility and correlation hedging demands also increase. The intuition is similar: As $m_{22}$ increases, the persistence of shocks in returns second moments increases, and it generates stronger hedging motives against both volatility and correlation risk for all assets.

The bottom plots of Figure 6 present comparative statics with respect to $q_{11}$. Note that as $q_{11}$ increases the first risky asset becomes a better hedging instrument against its volatility risk, but the second risky asset becomes a better hedging instrument against correlation risk. This follows from the form of the coefficients of the linear regression of state variables on returns, given in (19). Parameter $q_{11}$ also has an effect on the marginal utility sensitivities $A_{ij}$. We find that the higher variability of variance covariance shocks implied by a higher parameter $q_{11}$ lowers all absolute sensitivities $|A_{ij}|, 1 \leq i, j \leq 2$. However, this effect is...
considerably smaller than the one implied by the change in the leverage structure of asset returns, which is the dominating one. Consequently, as $q_{11}$ increases we obtain a decreasing (increasing) correlation hedging demand for the S&P 500 Futures (30-year Treasury Futures), but also an increasing (decreasing) volatility hedging component.

The comparative statics with respect to parameters $\overline{\rho}_1$ and $\overline{\rho}_2$ are given in Figure 7.

As $\overline{\rho}_1$ decreases, all assets become better hedging instruments against volatility and correlation risk; see again equation (19). At the same time, the variance covariance process under the minimax measure becomes more persistent, increasing each absolute sensitivity coefficient $|A_{ij}|$. These two effects go in the same direction, even if the effect on leverage is proportionally greater, and substantially increase the volatility and correlation hedging demands for all assets. Interestingly, as $\overline{\rho}_2$ decreases almost no variation in volatility and correlation hedging demands is seen. This follows from the fact that the leverage coefficients (19) and the minimax variance covariance dynamics depend on $\overline{\rho}_2$ with a weight that is proportional to the parameters $q_{12}$ and $q_{22}$ in the second column of $Q$. According to our GMM results, these two parameters are much smaller than coefficients $q_{11}$ and $q_{21}$ in the first column of $Q$. This feature implies both a small change in leverage and a small change in the absolute sensitivity coefficients $|A_{ij}|$ as $\overline{\rho}_2$ varies.

B.3. Time horizon

An important question addressed by the optimal portfolio choice literature is how the optimal allocation in risky assets varies with respect to the investment horizon. Brennan, Schwartz, and Lagnado (1997), Barberis (2000), Kim and Omberg (1996), and Wachter (2002) address this issue in the context of time-varying expected returns. When volatilities are constant, they find that the optimal investment in risky assets increases with the investment horizon. For instance, Kim and Omberg (1996) show that for the investor with utility over terminal wealth and for $\gamma > 1$ the optimal allocation increases with the investment horizon, as long as the risk premium is positive. Wachter (2002) extends this result to the case of utility over intertemporal consumption under the assumption of a constant correlation between asset returns. However, when correlations are stochastic it is reasonable to expect that the optimal demand for hedging correlation risk could mitigate, or strengthen, the speculative components. Our model offers a simple theoretical framework to investigate how the stochastic properties of the correlation can affect optimal hedging demands. Figure 8 reports intertemporal hedging demands for the S&P500 Index Futures and the 30–year Treasury Futures, as functions of the investment horizon, using GMM parameter estimates for the underlying opportunity set dynamics.

We find that the total hedging demand of an investor with risk aversion $\gamma > 1$ increases with investment horizons of up to 5 years. Optimal policies reach a steady-state level approxi-
mately at this horizon. The reason for such a convergence is the stationarity of the Wishart process (4) implied at our parameter estimates: Shocks in the variance-covariance matrix seem not to affect the transition density of the estimated variance covariance process over horizons longer than 5 years. At very short horizons, e.g., 3 months, all hedging demands are small. For investment horizons of 5 years and higher, the total hedging demand is approximately 25% and 20% of the myopic portfolio, for the S&P500 and the Treasury Futures contracts, respectively. The correlation hedging demand for the 30-year Treasury Futures increases quite quickly with the investment horizon and reaches a steady state level of approximately 18% of the myopic demand. The correlation hedging demand for the S&P500 Futures reaches a steady state of approximately 9% of the myopic portfolio as the investment horizon increases.

B.4. Higher-Dimensional Portfolio Choice Settings

For brevity and for simplicity of interpretation we have yet focused on a portfolio choice setting with only two risky assets. However, it is also interesting to gain some more intuition on the significance of volatility and correlation hedging when more than two risky assets are potentially available for investment. The complexity of the portfolio setting increases, as more volatility and correlation factors affect asset returns, which makes general statements and conclusions more difficult. On the one hand, given that the number of correlation factors increases quadratically with the dimension of the investment universe, but the number of volatility factors increases only linearly, one could expect correlation hedging to become proportionally more important than volatility hedging when the dimension of the investment universe rises. On the other hand, as the number of assets rises one could also argue that correlation risk can become less important than volatility risk, because the potential for portfolio diversification increases. The final effect on the hedging demand depends on the extent to which shocks to the different correlation and volatility processes are diversifiable across assets.

To study quantitatively these issues in a concrete portfolio setting, we include also the Nikkei225 Index Futures contract in the previous opportunity set consisting of the S&P500 Futures and the 30-year Treasury Futures contracts. We then estimate by GMM the three-dimensional version of model (1)-(4) using monthly time-series of returns, realized volatilities and realized covariances for these three risky assets. GMM moment restrictions are obtained in closed form as for the bivariate case above using the properties of the Wishart process. It is also straightforward to extend the proofs of Proposition 2 and 3 to cover the general setting with \( n \) risky assets. With these results, we compute the estimated optimal portfolios for the model with three risky assets. Table \( \text{V} \) Panel A, presents the results of our GMM model estimation. The implied hedging demands for correlation and pure volatility hedging on each asset are given in Panel B.

Insert Table \( \text{V} \) about here.
For illustration purposes, consider the optimal policies implied by a relative risk aversion coefficient $\gamma = 6$ and an investment horizon of $T = 5$ years. The correlation hedging demand for the S&P500 Futures is now approximately 14% of the myopic portfolio, almost twice the hedging demand of 7% estimated in the two risky assets setting. The inclusion of the Nikkei225 Futures in the portfolio sensibly lowers the correlation hedging demand for 30-year Treasury Futures, which drops from 17% in the two-assets case to 6.5% in the setting with three risky assets. The correlation hedging demand for the Nikkei225 Futures is approximately 14% of the myopic allocation. On average, these results imply a correlation hedging demand of about 11.5% of the myopic portfolio. The volatility hedging demands for these three assets are 9%, 6.1% and 7.6%, respectively, and imply an average volatility hedging of about 7.5%. In the model with two risky assets, the average correlation hedging demand is about 12.3% and the average volatility hedging demand is about 10.7%. These findings support the intuition that correlation hedging can become proportionally more important than volatility hedging as the dimension of the investment opportunity set rises.

The intuition is simple: As the dimension of the investment opportunity set increases, the relative importance of correlation shocks to the optimal portfolio squared Sharpe ratio increases. The Nikkei225 provides a good opportunity to diversify domestic equity risk, under the assumption that correlations do not change. At the parameter estimates, an increased weight of the equity investment becomes increasingly coupled with a greater demand for hedging potential changes in these correlations.

### B.5. Constant Risk Premia

To investigate the implication of a constant risk premium specification for the variance covariance hedging demand, we use the solutions for the model in Section I[D,5] and estimate the relevant state dynamics assuming for simplicity a constant interest rate ($D = 0$). This setting is the exact multivariate extension of the univariate model considered in Chacko and Viceira (2005). We use the same basic GMM estimation procedure and the same data used in for the estimation of model (I), (II), but we now apply it to the information matrix $\Sigma^{-1}$. The GMM moment restrictions for the variance-covariance matrix process are replaced by those for the precision process, which is assumed to follow a Wishart diffusion process. Table VI, Panel A, presents estimation results for the model with a constant risk premium. Panel B summarizes the estimated hedging demands.

The myopic portfolio is time varying, via the variation of the inverse covariance matrix $\Sigma^{-1}$. This time variation is also partly reflected in the time variation of hedging demands. All in all, the absolute size of total hedging demands is comparable to the one in the specification (I),

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22The total hedging demand for the Nikkei225 Futures, which is approximately 20.6%, is again several times larger than the volatility hedging demand in the univariate portfolio choice model with a Heston-type stochastic volatility process, which is less than 3.6%; see Panel B of Table IV.
for a constant market price of variance-covariance risk. For example, for a risk aversion parameter $\gamma = 6$ and an investment horizon of $T = 5$ years the average hedging demand is approximately 23% of the myopic portfolio. Similar demands are obtained for higher risk aversions and investment horizons. These hedging demands are again substantially higher than the volatility hedging demands of univariate stochastic volatility models with constant excess returns. We do not report these results for brevity of exposition.

III. Robustness and Extensions

In this section, we study the robustness of our findings, e.g., with respect to a discrete-time solution of the portfolio choice problem or the inclusion of different types of portfolio constraints.

A. Risk Aversion

Our findings emphasize the relevance of hedging demands for volatility and correlation risk when the returns covariance matrix is stochastic. These findings do not depend on the choice of the relative risk aversion parameter used to illustrate the main results. Figure 9 plots the total hedging portfolio weights and the hedging portfolio as a function of risk aversion.

Insert Figure 9 about here.

The total hedging portfolio weights for stocks and bonds peak at a relative risk aversion of about two (Panel 3 and 4). Hedging demands in percentage of the myopic portfolio are monotonically increasing in the relative risk aversion coefficient, although the increase is small for relative risk aversion parameters above 6 (Panel 1 and 2). Average correlation hedging demands in percentage of the myopic portfolio are typically higher than average volatility hedging demands (Panel 2). For instance, the average correlation (volatility) hedging demand for a relative risk aversion of 10 is approximately 14% (10.5%) of the myopic portfolio.

Thus, although we assume a constant relative risk aversion utility function to preserve closed-form optimal portfolios, our findings are likely to be even stronger in a setting with intertemporal consumption and Epstein-Zin recursive preferences, since the level of risk aversion can be calibrated at a higher level without generating undesirable properties for the elasticity of intertemporal substitution.

B. Discrete-Time Solution of the Portfolio Choice Problem

In our model, the optimal dynamic trading strategy is given by a portfolio that must be rebalanced continuously over time. In practice, this can at best be an approximation, because trading is only possible at discrete trading dates. Moreover, transaction costs, liquidity constraints, or policy disclosure considerations might further refrain investors from frequent portfolio rebalancing. Even if we do not model these frictions explicitly these frictions in
our setting, it is interesting to study the impact of discrete trading on the optimal hedging strategy in the context of our model.

Several studies have found that, as long as the investment opportunity set does not contain derivatives, the gains/losses of the optimal discrete-time portfolio policy with respect to a naively discretized continuous-time policy are small. See, for instance, Campbell, Chacko, Rodriguez and Viceira (2004) and Branger, Breuer and Schlag (2006). We study whether similar conclusions hold in our multivariate portfolio choice setting. We consider the exact discrete-time process implied by the continuous-time model (1), (4), in which observations are generated at fixed, evenly spaced, points in time. The parameters of the continuous time model (1), (4), have been estimated by GMM using the exact discrete time moments of this process. The moments are easily obtainable in closed form for each sampling frequency because the Wishart process (4) allows for aggregation over time. By construction, the estimated parameters are then consistent with the discrete time transition density of the process, which is the relevant one to study optimal portfolio choice in discrete time.

The discrete-time portfolio choice problem does not allow for closed-form solutions. Therefore, we rely on standard numerical methods to compute the optimal portfolio strategies. Table VII presents the total hedging demands in S&P500 Futures ($\pi_1$) and Treasury Futures ($\pi_2$), as fractions of the myopic demand. The transition density used for the discrete time portfolio optimization is the one implied by the estimated continuous time model with monthly returns, realized volatilities and realized correlations.

We focus on optimal portfolios that can be rebalanced monthly, but compute also optimal strategies using a weekly and daily rebalancing frequency, in order to verify the convergence of our numerical solution to the continuous time portfolio problem solution. At a daily frequency, the hedging demands in the discrete time model are virtually indistinguishable from the continuous time hedging demands computed in Table III. Consistent with the findings in the literature, the discrete time optimal hedging demands for the monthly frequency are close to the hedging demands computed from the continuous time model: The mean absolute difference between the hedging demands using daily and monthly rebalancing is less than 10% of the hedging demand implied by a monthly rebalancing frequency. These findings suggest that the main implications derived from the continuous time multivariate portfolio choice solutions are realistic even in the context of monthly rebalancing.

C. Portfolio constraints

Portfolio constraints are useful to avoid unrealistic portfolio weights, which can potentially come about due to some extreme assumptions on expected returns, volatilities and correlations, or from inaccurate point estimates of the model parameters. The empirical results of the previous sections can imply, for instance, levered portfolios in settings of low risk aversion. E.g., for a relative risk aversion $\gamma = 2$, the optimal portfolio of an investor with
horizon $T = 5$ years implies an investment of approximately 260% of the total wealth in stocks and 170% in bonds. Intuitively, constraints on short selling or on the portfolio VaR tend to restrain the investor from selecting optimal portfolios that are excessively levered. Therefore, it is interesting to study these types of portfolio constraints and their impact on the volatility and correlation hedging demands in our setting. We solve the discrete time portfolio choice problem in the last section and additionally impose, in two separate steps, short selling and VaR constraints. In order to quantify the correlation and volatility hedging components, we numerically compute the projection of the total hedging demand on the implied elasticity of the indirect marginal utility of wealth with respect to volatilities and correlations.

In the first exercise, we consider state-independent constraints on the optimal portfolio weights. For every fraction $\pi_i$ of total wealth invested in the risky asset $i$, we first enforce a short-selling constraint $\pi_i \geq 0$. In a second step, we also consider a less severe position limit $\pi_i \geq -1$. Table VIII presents the optimal volatility and correlation hedging demands implied by these two settings. Note that even in cases where the current constraint might not be binding, the optimal hedging strategy is different from the one implied by the unconstrained solution. This feature exists because the future opportunity set is restricted by the fact that the constraint might be binding, with some probability, in the future. The indirect marginal utility of wealth in the constrained problem depends on the strength of this effect. Therefore, the optimal intertemporal hedging demand is different.

Table VIII shows that the more severe the constraint is, the smaller the absolute demands for volatility and correlation hedging are as a percent of the myopic portfolio. However, the impact of the constraint is quite moderate, even in the short-selling case, and does not influence the relative size of the hedging demands much against volatility and correlation risk across assets. For instance, for an investment horizon of $T = 10$ years and a risk aversion $\gamma = 2$, the average correlation (volatility) hedging demand is 10.5% (7%) in the unconstrained case and 8.5% (6.5%) in the setting with short selling constraints. For a higher risk aversion $\gamma = 8$, the average correlation (volatility) hedging demand is 13.25% (10.25%) in the unconstrained case and 10.75% (9%) in the setting with short selling constraints. These findings are consistent with the state independent nature of the constraint used, which is not a function of the conditional covariance matrix of returns. The slightly larger percentage decrease in the hedging demands of low risk aversion investors in the constrained case is mainly due to their large myopic demands in the unconstrained portfolio problem.

The results are different when we study the effects of (state dependent) Value at Risk (VaR) constraints. At each trading date, we impose a constant upper bound on the VaR of the optimally invested wealth at the next trading date. We use a VaR at a confidence level of 99%. Since the VaR is computed for a monthly rebalancing frequency and investment horizons longer than one month, the VaR constraint is dynamically updated, as in Cuoco, He and Issaenko (2001). Table IX summarizes our findings for the optimal VaR constrained
portfolios. For computational tractability of our numerical solutions, we focus on investment horizons up to $T = 2$ years.

Insert Table IX about here

The VaR constraint has a more significantly effect on the optimal portfolios of investors with low risk aversion, which are those with the largest exposure to risky assets in the unconstrained setting. E.g., for a risk aversion coefficient $\gamma = 2$ and an investment horizon $T = 2$, the mean total allocation to stocks (bonds) shrinks from approximately 250% (160%) to about 175% (115%) of the total wealth. At the same time, the relative importance of the correlation hedging demand increases: Already for a moderate investment horizon of $T = 2$ years and a low risk aversion $\gamma = 2$, the correlation and volatility hedging demands are on average 11% and 7% of the myopic portfolio, respectively. With the same choice of parameters, the corresponding hedging demands in the unconstrained case are 7.7% and 10.7%, respectively. For a higher risk aversion $\gamma = 8$ and the same investment horizon, the correlation hedging demand is on average about 11% of the myopic portfolio both in the VaR constrained and VaR unconstrained cases.

The VaR constrained investor dislikes more volatile or extreme portfolio values than the unconstrained agent does, since (coeteris paribus) the VaR constraint becomes more restrictive when the volatility on the optimally invested portfolio increases. It follows that the investor is more concerned about the total volatility of the portfolio, which can cause the VaR constraint to be hit with a too large probability. Therefore, the VaR constrained investor reduces the size of the myopic demand. Furthermore, since changes in correlation have a first order impact on the VaR of the portfolio, the investor increases the correlation hedging demand, exploiting the spanning properties of the risky assets. Thus, in this setting, which is relevant for institutions subject to capital requirement or for asset managers with self-imposed risk management constraints, the impact of correlation risk is economically very significant.

IV. Discussion and Conclusions

We develop a new multivariate continuous-time framework for intertemporal portfolio choice, in which stochastic second moments of asset returns imply distinct motives for volatility and correlation hedging. The model is solved in closed-form and allows us to study the implications of volatility and correlation hedging in several realistic economic settings. The multivariate nature of second moments in our model has important consequences for optimal asset allocation: Hedging demands are significantly different from and typically four to five times larger than those of models with constant correlations or single-factor stochastic volatility. They include a substantial correlation hedging component, which tends to increase with the persistence of variance covariance shocks, the strength of leverage effects and the dimension of the investment opportunity set. These findings also exist when we consider exact discrete-time versions of our setting with short-selling or VaR constraints.
The hedging demands against variance covariance risk in our model are typically smaller than those found in the literature on intertemporal hedging with stock returns predictability. This finding follows mainly from the fact that in our applications returns span shocks in their covariance matrix much less than they typically do for shocks to the predictive variables used in the literature. In this respect, it is also interesting to recall that our model does not incorporate explicitly Bayesian learning about model parameters. In continuous time models it is difficult to motivate a learning behavior about returns second moment, because these quantities are observable from the quadratic variation of the process. In discrete time, Bayesian models with learning about both first and second moments can be more naturally considered. However, also in this case it is difficult to obtain tractable solutions for portfolio choice without introducing a simple structure for the variance covariance process. Barberis (2000), among others, studies the implications of estimation risk about the parameters of a predictive equation in a model with homoskedastic returns, and finds that parameter uncertainty can reduce dramatically the exposure to stocks over longer horizons. Our setting is very different from that one. However, one might be tempted to conclude by analogy that learning could substantially reduce hedging demands also in our model. Interesting evidence on this issue can be found in Brandt, Goyal, Santa-Clara and Stroud (2005). They develop a dynamic programming algorithm, which can be used to solve efficiently the portfolio problem with predictability and learning about first and second moments. When learning is considered, they find hedging demands that are comparable to those found in our paper. When learning is neglected, these policies are much higher than our ones, due to the presence of return predictability. Interestingly, the hedging demand reduction implied by Bayesian learning is almost entirely due to the learning about the predictability equation: Learning about the variance-covariance matrix has a small influence on the optimal portfolios. An interesting direction for future research could use the discrete time Wishart process to study more systematically the portfolio implications of learning about the covariance matrix of returns in a multivariate setting.

The proposed approach to model stochastic second moments is parsimonious, tractable, and proves useful in investigating a number of additional economic questions. For instance, an important strand of the empirical asset pricing literature has investigated the characteristics of hedge funds performance. Kosowski, Naik, and Teo (2006) document that, even after controlling for market risk, hedge fund alphas are significantly positive and persistent. Hedge funds in the top decile of the return distribution have alphas well in excess of 1% per month. A proposed interpretation of this evidence is that these funds have superior managerial ability. However, about 34% of the hedge funds are classified as long/short funds and 7% as fixed income arbitrage funds. A part of these excess returns may compensate for the exposure to correlation risk, which is key in any long/short strategies. In a general equilibrium framework, it is legitimate to expect that portfolio hedging demands are linked

to asset risk premia by a standard market clearing condition. Therefore, correlation risk might be priced in equilibrium and can affect expected excess returns when investors are not myopic. Since hedge fund alphas are typically obtained without explicitly controlling for exposure to correlation risk, it is natural to investigate whether there exists an empirical link between correlation risk and hedge fund alphas. This conjecture has recently found supporting empirical evidence in Buraschi, Kosowski, and Trojani (2008).

Correlation risk also plays a direct role in the pricing, hedging, and risk management of correlation derivatives, such as quantos. In these financial instruments, the underlying asset is denominated in one currency, while the instrument itself is settled in another currency at some fixed exchange rate. Such products are attractive for portfolio managers or hedge funds who wish to have exposure to a foreign asset, without carrying the corresponding exchange rate risk. Well-known examples include differential swaps (also known as quantity-adjusted swaps, guaranteed exchange rate swaps, Libor differential swaps), quanto options, quanto equity swaps, and quanto futures (such as the Nikkei Futures traded on the CME). In these cases, the pricing, hedging, and risk management of these instruments depend directly on the correlation between the risk factors (see Reiner, 1992 and Dravid, Richardson, and Sun, 1993). This is the reason that these instruments are also referred to by practitioners as “correlation products”. In differential swaps, for example, the dealer commits to paying a floating rate on a fixed US dollar theoretical amount, rather than on a fixed amount in the foreign currency, as with a typical cross-currency swap. This commitment exposes the dealer to changes in the correlation between the Libor and the exchange rates. Since static hedging strategies are generally not viable, the dealer must manage the residual correlation risk by using optimal portfolio techniques. An interesting avenue of future research is to investigate how the Wishart setting can be used in the pricing and risk management of correlation products.

Finally, the adequate modeling of correlation risk is a key issue for credit derivative markets, because the likelihood of a default of one credit may affect the likelihood of default of another credit. Typical examples of such correlation-based products are instruments written on baskets of credits, such as Collateralized Debt Obligations (CDOs) and first-to-default (FTD) swaps, and Credit Default Swaps (CDSs). Since they are defined on a portfolio of firm liabilities, the time-variation of the correlations in the portfolio is a primary source of pricing and risk management issues to the extent that traders managing these risks are often called “correlation traders” and structured credit products are quoted in terms of implied base correlations. Figure 10 illustrates these features by plotting the implied correlations of 7 year maturity mezzanine tranches of CDX and iTraxx CDO’s in the period
from September 2004 to April 2008. The average implied correlations are approximately 0.68 and 0.75 for iTraxx and CDX products, respectively, but they can vary rapidly in a short period of time. In particular, during the recent subprime mortgage crisis and the dramatic stock market downturn between November 2007 and April 2008, base correlations rapidly increased simultaneously to higher levels, with CDX base correlations being virtually one at several points in time. The joint empirical properties of these time series highlight very eloquently the importance of modeling co-movements in a multivariate context and the potential of the model presented in this article for studying a number of additional asset pricing questions in interesting economic settings.

24 The iTraxx Europe index is composed of the most liquid 125 CDS referencing European investment grade credits, while the CDX index is composed of CDS referencing North American and Emerging Market credits. Both of them are expressed directly in implied-correlation terms (similar to the implied-volatility in equity option markets).
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Appendix A: Proofs

Proof of Proposition 1

The dynamics of the correlation process implied by the Wishart covariance matrix diffusion is computed using Itô’s Lemma. Let

\[ \rho(t) = \frac{\Sigma_{12}(t)}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} \]  

be the instantaneous correlation between the returns of the first and the second risky assets and denote by \( \sigma_{ij}, q_{ij} \) and \( \omega_{ij} \) the \( ij \)-th component of the volatility matrix \( \Sigma^{1/2} \), the matrix \( Q \) and matrix \( \Omega'\Omega = kQ'Q \) in equation (4), respectively. Applying Itô’s Lemma to (A1) and using the dynamics for \( \Sigma_{11}, \Sigma_{22} \) and \( \Sigma_{12} \), implied by (4), it follows:

\[
\begin{align*}
    d\rho &= \left[ \frac{\omega_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} - \frac{\rho}{2\Sigma_{11}}\omega_{11} - \frac{\rho}{2\Sigma_{22}}\omega_{22} + \frac{\rho}{2} \left( \frac{q_{11}^2 + q_{21}^2 + q_{12}^2 + q_{22}^2}{\Sigma_{11}} \right) \right] dt \\
    &\quad + \left( \rho^2 - 2 \right) \frac{q_{11}q_{12} + q_{21}q_{22}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \left( 1 - \rho^2 \right) m_{21}\Sigma_{11} + m_{12}\Sigma_{22} \right] dt \\
    &\quad - \left[ \frac{\rho}{2\Sigma_{11}\Sigma_{22}} \left( \Sigma_{22}\sigma_{11}q_{11} + \Sigma_{11}\sigma_{12}q_{12} \right) - \frac{\sigma_{12}q_{11} + \sigma_{11}q_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dB_{11} \\
    &\quad - \left[ \frac{\rho}{2\Sigma_{11}\Sigma_{22}} \left( \Sigma_{11}\sigma_{22}q_{12} + \Sigma_{22}\sigma_{21}q_{11} \right) - \frac{\sigma_{22}q_{11} + \sigma_{21}q_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dB_{21} \\
    &\quad - \left[ \frac{\rho}{2\Sigma_{11}\Sigma_{22}} \left( \Sigma_{22}\sigma_{11}q_{21} + \Sigma_{11}\sigma_{12}q_{22} \right) - \frac{\sigma_{12}q_{21} + \sigma_{11}q_{22}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dB_{12} \\
    &\quad - \left[ \frac{\rho}{2\Sigma_{11}\Sigma_{22}} \left( \Sigma_{11}\sigma_{22}q_{22} + \Sigma_{22}\sigma_{21}q_{21} \right) - \frac{\sigma_{21}q_{22} + \sigma_{22}q_{21}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dB_{22}
\end{align*}
\]

\[ B_{ij}(t), i, j = 1, 2, \] are the entries of the \( 2 \times 2 \) matrix of Brownian motions in (4). Therefore, the instantaneous drift of the correlation process is a quadratic polynomial with state dependent coefficients:

\[ \mathbb{E} [d\rho(t)| \mathcal{F}_t] = [E_1(t) \rho(t)^2 + E_2(t) \rho(t) + E_3(t)] dt, \]

where coefficients \( E_1(t), E_2(t) \) and \( E_3(t) \) are given by:

\[
\begin{align*}
    E_1(t) &= \frac{q_{11}q_{12} + q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} - m_{21}\sqrt{\Sigma_{11}(t)} - m_{12}\sqrt{\Sigma_{22}(t)}, \\
    E_2(t) &= -\frac{\omega_{11}}{2\Sigma_{11}} - \frac{\omega_{22}}{2\Sigma_{22}} - \frac{1}{2} \left( \frac{q_{11}^2 + q_{21}^2}{\Sigma_{11}(t)} + \frac{q_{12}^2 + q_{22}^2}{\Sigma_{22}(t)} \right), \\
    E_3(t) &= \frac{\omega_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} - \frac{2q_{11}q_{12} + q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} + m_{21}\sqrt{\Sigma_{11}(t)} + m_{12}\sqrt{\Sigma_{22}(t)}
\end{align*}
\]

The instantaneous conditional variance of the correlation process is easily obtained from equation (A2) and it is a third order polynomial with state dependent coefficients:

\[ \mathbb{E} [d\rho(t)^2| \mathcal{F}_t] = \left( 1 - \rho^2(t) \right) \left( \frac{q_{11}^2 + q_{21}^2}{\Sigma_{11}(t)} + \frac{q_{12}^2 + q_{22}^2}{\Sigma_{22}(t)} - 2\rho(t) \frac{q_{11}q_{12} + q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} \right) dt. \]

This concludes the proof. \( \Box \)

Proof of Proposition 2

Since markets are incomplete, we follow He and Pearson (1991) and represent any market price of risk as the sum of two orthogonal components, one of which is spanned by the asset returns. Since Brownian motion \( W \) can be rewritten as \( W = B\rho + Z\sqrt{1-\rho^2} \), for a standard bivariate Brownian motion \( Z \) independent of \( B \), we rewrite the innovation component of the opportunity set dynamics as \( \Sigma^{1/2}[Z,B|L] \), with \( L = \)
\[ t \] for the following Radon-Nykodim derivative with respect to the physical measure 

\[ \nu \]

where the Lagrange multiplier for the static budget constraint is 

\[ \text{tr} \]

Using (A9) and (A13), it is noticeable that the solution requires the computation of the expected value of 

\[ (A13) \]

It then follows:

\[ (A12) \]

Using (A9) and (A13), it is noticeable that the solution requires the computation of the expected value of the exponential of a stochastic integral. A simple change of measure reduces the problem to the calculation of the exponential of a deterministic integral. Let \( P^\gamma \) be the probability measure defined by the following Radon-Nykodim derivative with respect to the physical measure \( P \):

\[ (A14) \]

We denote expectations under \( P^\gamma \) by \( \mathbb{E}^\gamma [\cdot] \). Then, the minimizer of (A13) is the solution of the following problem:

\[ J(0, \Sigma_0) = \inf_{\nu} \mathbb{E}^\gamma \left[ \xi_\nu(T)^{-\gamma} \right] 
\]

\[ = \inf_{\nu} \mathbb{E}^\gamma \left[ e^{-\frac{1}{\gamma} t \text{tr} \left( f^T \Theta_\nu(s) \Theta_\nu(s) ds \right) + \frac{1}{2} \left( \frac{\gamma-1}{\gamma} \right) f^T \Theta_\nu(s) \Theta_\nu(s) ds} \right] 
\]

\[ = \inf_{\nu} \mathbb{E}^\gamma \left[ e^{-\frac{1}{\gamma} t \text{tr} \left( \Sigma \lambda \nu^2 + \nu^2 \Sigma \lambda \nu^2 \right) ds} \right] 
\]

\[ (A15) \]

\[ 25 \text{Remember that } W = BL \]

\[ 26 \text{Strictly speaking, this holds for } \gamma \in (0,1). \text{ For } \gamma > 1, \text{ minimizations are replaced by maximizations and all formulas follow with the same type of arguments.} \]
Notice that the expression in the exponential of the expectation in (A15) is affine in $\Sigma$. By Girsanov Theorem, under the measure $P^\gamma$, the stochastic process $B^\gamma$, defined as

$$B^\gamma(t) = B(t) + \frac{\gamma - 1}{\gamma} \int_0^t \Theta_\nu(s) ds$$

is a $2 \times 2$ matrix of standard Brownian motions. Therefore, the process $\Sigma$ is an affine process also under the new probability measure $P^\gamma$:

$$d\Sigma(t) = \left[ \left( \nu^{1/2} \left( M - \frac{\gamma - 1}{\gamma} Q'(\bar{p}\lambda' + \bar{\nu}) \right) \right)' \right] dt + \Sigma^{1/2}(t)dB^\gamma(t)Q + Q'dB(t)\Sigma^{1/2}(t).$$

(A16)

Using Feynman Kac formula, it is known that if the optimal $\nu$ and $\hat{J}$ solve the probabilistic problem (A15), then they must also be a solution of the following Hamilton Jacobi Bellman (HJB) equation:

$$0 = \frac{\partial \hat{J}}{\partial t} + \inf_{\nu} \left\{ A\hat{J} + \hat{J} \left[ -\frac{\gamma - 1}{\gamma} r + \frac{1 - \gamma}{\gamma^2} tr \left( \Sigma \left( \lambda' + \nu\lambda' \right) I_2 + \frac{\nu\nu'}{1 - \nu\nu} \right) \right] \right\},$$

(A17)

subject to the terminal condition $\hat{J}(T, \Sigma) = 1$, where $A$ is the infinitesimal generator of the matrix-valued diffusion (A16), which is given by:

$$A = tr \left( \left( \nu^{1/2} \left( M - \frac{\gamma - 1}{\gamma} Q'(\bar{p}\lambda' + \bar{\nu}) \right) \right)' \right) + tr(2\Sigma DQ'QD),$$

(A18)

and

$$D := \left( \frac{\partial^2}{\partial \Sigma_{11}} \quad \frac{\partial^2}{\partial \Sigma_{12}} \right).$$

(A19)

The generator is affine in $\Sigma$. The optimality condition for the optimal control $\nu$, implied by HJB equation (A17), is:

$$-\frac{1}{\gamma} \nu \Sigma \left( I_2 + \frac{\nu\nu'}{1 - \nu\nu} \right) = \frac{\partial}{\partial \nu} tr \left( \left( \Sigma \nu' + \Sigma \nu \right) \frac{\partial \hat{J}}{\partial \nu} \right) = \frac{\partial}{\partial \nu} tr \left( \left( \Sigma \nu' + \Sigma \nu \right) \frac{\partial \hat{J}}{\partial \nu} \right).$$

Applying rules for the derivative of trace operators, the right hand side can be written as $\Sigma \left( \frac{\partial \hat{J}}{\partial \nu} + \frac{\partial \hat{Q}}{\partial \nu} \right) Q'$. It follows that

$$\bar{\nu} = -\gamma \left( \frac{\partial \hat{J}}{\partial \nu} + \frac{\partial \hat{Q}}{\partial \nu} \right) Q' \left( I_2 + \frac{\nu\nu'}{1 - \nu\nu} \right)^{-1}. \tag{A20}$$

Note that $\left( I_2 + \frac{\nu\nu'}{1 - \nu\nu} \right)^{-1} = I_2 - \nu\nu'$. Substituting the expression for $\nu$ in equation (A18), we obtain the generator

$$A = tr \left( \left( \nu^{1/2} \left( M - \frac{\gamma - 1}{\gamma} Q'(\bar{p}\lambda' + \bar{\nu}) \right) \right)' \right) + \left( \frac{\partial \hat{J}}{\partial \nu} + \frac{\partial \hat{Q}}{\partial \nu} \right) tr \left( \left( \Sigma \nu' + \Sigma \nu \right) \frac{\partial \hat{J}}{\partial \nu} \right)$$

$$+ tr \left( \left( \nu^{1/2} \left( M - \frac{\gamma - 1}{\gamma} Q'(\bar{p}\lambda' + \bar{\nu}) \right) \right)' \right) + \left( \frac{\partial \hat{J}}{\partial \nu} + \frac{\partial \hat{Q}}{\partial \nu} \right) tr \left( \left( \Sigma \nu' + \Sigma \nu \right) \frac{\partial \hat{J}}{\partial \nu} \right)$$

$$- \frac{\gamma}{\gamma - 1} tr \left( \left( I_2 + \frac{\nu\nu'}{1 - \nu\nu} \right) \Sigma \left( \frac{\partial \hat{J}}{\partial \nu} + \frac{\partial \hat{Q}}{\partial \nu} \right) Q' \left( \frac{\partial \hat{J}}{\partial \nu} + \frac{\partial \hat{Q}}{\partial \nu} \right) \right).$$
Substitution of the last expression for $A$ into the HJB equation (A17) yields the following partial differential equation for $\hat{J}$:

$$
-\frac{\partial \hat{J}}{\partial t} = tr \left( \left( \Omega \Omega' + \left( M - \frac{\gamma - 1}{\gamma} Q \rho \lambda' \right) \Sigma + \Sigma \left( M - \frac{\gamma - 1}{\gamma} Q \rho \lambda' \right)' \right) D + 2 \Sigma Q' Q D \right) \hat{J}
+ \frac{\gamma - 1}{\gamma} \hat{J} \left( -r - \frac{tr(\Sigma \lambda \lambda')}{2\gamma} \right) + \frac{1}{2} \gamma \hat{J} tr \left( (I_2 - \rho \rho') \Sigma \left( \frac{D \hat{J}}{J} + \frac{D \hat{P}}{J} \right)' Q Q \left( \frac{D \hat{J}}{J} + \frac{D \hat{P}}{J} \right)' \right),
$$

subject to the boundary condition $\hat{J}(\Sigma, T) = 1$. The affine structure of this problem suggests an exponentially affine functional form for its solution:

$$
\hat{J}(t, \Sigma) = \exp(B(t, T) + tr(A(t, T) \Sigma),
$$

for some state independent coefficients $B(t, T)$ and $A(t, T)$. After inserting this functional form into the differential equation for $\hat{J}$, the guess can be easily verified. The coefficients $B$ and $A$ are the solutions of the following system of Riccati equations:

$$
-\frac{dB}{dt} = tr(\Omega \Omega') - \frac{\gamma - 1}{\gamma} r,
-\frac{dA}{dt} tr(\Sigma) = tr \left( \Gamma' A \Sigma + A \Gamma \Sigma + 2 A Q' Q A \Sigma + \frac{1-\gamma}{2} (A' + A) Q' (I_2 - \rho \rho') Q (A' + A) \Sigma + C \Sigma \right),
$$

with terminal conditions $B(T, T) = 0_{2 \times 2}$ and $A(T, T) = 0_{2 \times 2}$, where

$$
\Gamma = M - \frac{\gamma - 1}{\gamma} Q \rho \lambda',
C = \frac{1-\gamma}{2\gamma} \lambda \lambda'.
$$

For a symmetric matrix function $A$, the second differential equation implies the following matrix Riccati equation:

$$
0_{2 \times 2} = \frac{dA}{dt} + \Gamma' A + A \Gamma + 2 A \Lambda A + C.
$$

where

$$
\Lambda = Q' (I_2 \gamma + (1 - \gamma) \rho \rho') Q
$$

This is the system of matrix Riccati equations in the statement of Proposition 2. These differential equations are completely integrable, so that closed-form expressions for $\hat{J}$ (and hence for $J$) can be computed. For convenience, we consider coefficients $A$ and $B$ as parameterized by $\tau = T - t$. This change of variable implies the following simple modification of the above system of equations:

$$
\frac{dB}{d\tau} = tr(\Omega \Omega') - \frac{\gamma - 1}{\gamma} r,
\frac{dA}{d\tau} = \Gamma' A + A \Gamma + 2 A \Lambda A + C,
$$

subject to initial conditions $A(0) = 0_{2 \times 2}$ and $B(0) = 0_{2 \times 2}$. Given a solution for $A$, function $B$ is obtained by simple integration:

$$
B(\tau) = tr \left( \int_0^\tau A(s) \Omega \Omega' ds \right) - \frac{\gamma - 1}{\gamma} r \tau.
$$

To solve equation (A26), we use the linearization method applied in Da Fonseca, Grasselli, and Tebaldi (2005). Let us represent the function $A(\tau)$ as:

$$
A(\tau) = H(\tau)^{-1} K(\tau),
$$

where $H(\tau)$ and $K(\tau)$ are square matrices, with $H(\tau)$ invertible. Premultiplying (A26) by $H(\tau)$ we obtain:

$$
H \frac{dA}{d\tau} = H \Gamma' A + H A \Gamma + 2 H A \Lambda A + HC.
$$
Where no confusion may arise, we suppress the argument $\tau$ for brevity. On the other hand, in light of (A27), differentiation of
\[ HA = K \] results in:
\[ H \frac{dA}{d\tau} = \frac{d}{d\tau}(HA) - \frac{dH}{d\tau} A, \] (A30)
and:
\[ \frac{d}{d\tau}(HA) = \frac{dK}{d\tau}. \] (A31)
Substituting (A29), (A30), and (A31) into (A28) we get
\[ \frac{dK}{d\tau} - \frac{dH}{d\tau} A = H \Gamma'A + K \Gamma + 2 K \Lambda A + HC. \]
After collecting coefficients of $A$, we conclude that the last equation is equivalent to the following matrix system of ODEs:
\[ \frac{dK}{d\tau} = K \Gamma + HC, \] (A32)
\[ \frac{dH}{d\tau} = -2 K \Lambda - H \Gamma', \] (A33)
or:
\[ \frac{d}{d\tau} (K \ H) = (K \ H) \begin{pmatrix} \Gamma & -2 \Lambda \\ C & -\Gamma' \end{pmatrix}. \]
The above ODE can be solved by exponentiation:
\[
\begin{pmatrix}
K(\tau) & H(\tau)
\end{pmatrix}
= \begin{pmatrix}
K(0) & H(0)
\end{pmatrix}
\exp\left[\tau \begin{pmatrix}
\Gamma & -2 \Lambda \\ C & -\Gamma'
\end{pmatrix}\right]
= \begin{pmatrix}
A(0) & I_2
\end{pmatrix}
\exp\left[\tau \begin{pmatrix}
\Gamma & -2 \Lambda \\ C & -\Gamma'
\end{pmatrix}\right]
= \begin{pmatrix}
A(0) F_{11}(\tau) + F_{21}(\tau) & A(0) F_{12}(\tau) + F_{22}(\tau)
\end{pmatrix},
\]
We conclude from equation (A27) that the solution to (A26) is given by:
\[ A(\tau) = F_{22}(\tau)^{-1} F_{21}(\tau). \] (A34)
This concludes the proof. □

Proof of Proposition 3
In order to recover the optimal portfolio policy we have, from the proof of Proposition 2
\[ X^*(t) := \frac{1}{\xi(t)} 1_{\xi(t)}(T) X^*(T) | F_t = \psi^{-\frac{1}{2}} \xi(t)^{-\frac{1}{4}} \tilde{J}(t, \Sigma(t)). \] (A35)
For the Wishart dynamics (11), Itô’s lemma applied to both sides of (A35) gives, for every state $\Sigma$:
\[ X^*(t) \text{tr} \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \Sigma^{1/2} d\mathcal{B} = X^*(t) \text{tr} \left( \frac{1}{\gamma} \Theta(t) d\mathcal{B} + \frac{D \tilde{J}}{J} \left( \Sigma^{1/2} dUQ + Q'U'd\mathcal{B}' \Sigma'^{1/2} \right) \right), \] (A36)
where matrix $U$ is given by:
\[ U = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \]
This implies
\[ L \begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \Sigma^{1/2} = \frac{1}{\gamma} (L \xi + \nu') \Sigma^{1/2} + 2 UQA\Sigma^{1/2}. \]
For volatilities, the same argument gives, for a neighborhood of \( \tau \), then
\[
A
\]
Lemma 2

We first prove a useful technical result on the form of the inverse covariance matrix \( \Sigma \)
\[
\text{Proof of Proposition 6.} \quad \Box
\]

This concludes the proof of the proposition. \( \Box \)

Proof of Proposition 6

We apply the following lemma, similar to a result in Buraschi, Cieslak and Trojani (2007), to which we refer for a proof.

Lemma 1

Consider the solution \( A(\tau) \) of matrix Riccati equation \([12]\). If matrix \( C \) is negative semidefinite, then \( A(\tau) \) is negative-semidefinite and monotonically decreasing for any \( \tau \), i.e. \( A(\tau_2) - A(\tau_1) \) is a negative semidefinite matrix for any \( \tau_2 > \tau_1 \).

Since \( C = (1 - \gamma)/(2\gamma^2)\lambda Y \), if \( \gamma > 1 \) then \( C \) is negative semidefinite and from Lemma 1, \( A(\tau) \) is also negative semidefinite. It follows that \( A_{11}(\tau) \leq 0 \) and, taking the symmetry of \( A(\tau) \) into account, that \( A_{22} \leq 0 \). Inequality \( |A_{12}| \leq |A_{11} + A_{22}|/2 \) follows from the properties of negative semidefinite matrices. Now consider a neighborhood of \( \tau = 0 \) of arbitrary small length \( \epsilon \). By the fundamental theorem of calculus, we have:
\[
A(\epsilon) = A(0) + \left. \frac{dA(\tau)}{d\tau} \right|_{\tau=0} \epsilon
\]

But \( A(0) = 0 \) and \( \left. \frac{dA(\tau)}{d\tau} \right|_{\tau=0} = C \). If \( \lambda_1 \) and \( \lambda_2 \) agree in sign and \( \gamma < 1 \) then \( C_{12} > 0 \) and \( A_{12}(\epsilon) < 0 \). If, in addition, \( \lambda_1 > \lambda_2 \), we have \( \lambda_1^2 > \lambda_1 \lambda_2 > \lambda_2^2 \), that is \( |C_{11}| > |C_{12}| > |C_{22}| \). We conclude from \([A38]\) that \( |A_{11}| > |A_{12}| > |A_{22}| \). This concludes the proof of the proposition. \( \Box \)

Proof of Proposition 6

To obtain the optimal hedging demand in terms of the state variables \( \Sigma_{11}, \Sigma_{22}, \) and \( \rho \), write \( \Sigma_{12} = \rho \sqrt{\Sigma_{11} \Sigma_{22}} \) and note that:
\[
-\frac{\partial^2 J}{\partial X \partial \rho} = -\frac{\partial^2 J}{\partial \Sigma_{12}} \sqrt{\frac{\partial \Sigma_{12}}{\partial \rho}} = A_{12} \sqrt{\frac{\Sigma_{11}}{\Sigma_{12}}},
\]

This is the closed form expression for the wealth-scaled ratio of marginal utilities with respect to \( \rho \) and \( X \). For volatilities, the same argument gives, for \( i, j = 1, 2 \), where \( i \neq j \):
\[
-\frac{\partial^2 J}{\partial \Sigma_{ij} \partial X} = A_{ii} + \frac{\partial^2 J}{\partial \Sigma_{12} \partial X} \sqrt{\frac{\Sigma_{11}}{\Sigma_{12}}} = A_{ii} + 2 A_{12} \rho \frac{\Sigma_{ij}}{\Sigma_{ii}},
\]

This is the closed form expression for the wealth-scaled ratio of marginal utilities with respect to \( \Sigma_{ij} \) and \( X \), when \( \rho \) is treated as an explicit state variable, in addition to \( \Sigma_{11} \) and \( \Sigma_{22} \). The first term on the right hand side of equation \([A39]\) is the one that corresponds to the direct effect of \( \Sigma_{ij} \) on the value function. The second term is the one that corresponds to the indirect effect of \( \Sigma_{ij} \), via the feedback of \( \Sigma_{ij} \) on \( \Sigma_{12} \). To compute the corresponding hedging demand it is then enough to use Merton’s (1969) results and to calculate the projection coefficients of \( d\Sigma_{11}, d\Sigma_{22} \) and \( d\rho \) on the space spanned by \( dS_1/S_1 \) and \( dS_2/S_2 \), using the available dynamics. After collecting terms proportional to \( A_{12} \sqrt{\Sigma_{11}/\Sigma_{22}}, A_{11}, A_{22}, A_{12} \rho \sqrt{\Sigma_{11}/\Sigma_{22}} \) and \( A_{12} \rho \sqrt{\Sigma_{22}/\Sigma_{11}} \), respectively, the desired decomposition follows. This concludes the proof of the proposition. \( \Box \)

Proof of Proposition 6

We first prove a useful technical result on the form of the inverse covariance matrix \( \Sigma^{-1} = (SYS')^{-1} \) when \( SS' = id_{2 \times 2} \).

Lemma 2

Let \( SS' = id_{2 \times 2} \). It then follows:
\[
(SYS')^{-1} = SY^{-1}S'.
\]
Proof of Lemma 2
In that $SYS'$ is symmetric, we have:

$$SYS' = QΛQ', \quad (SYS')^{-1} = QΛ^{-1}Q', \quad (A40)$$

where $Q$ is a $2 \times 2$ matrix of eigenvectors of $SYS'$ and $Λ$ a diagonal $2 \times 2$ matrix of eigenvalues. Similarly,

$$Y = QΛQ', \quad Y^{-1} = QΛ^{-1}Q', \quad (A41)$$

where $Q$ is a $3 \times 3$ matrix of eigenvectors of $Y$ and $Λ$ a diagonal matrix of eigenvalues. We first show that the eigenvectors of $SYS'$ are all vectors $q_i$ such that $S'q_i$ is an eigenvector of $Y$. Indeed, let $q_i = S'q_i$ be an eigenvector of $Y$. It then follows,

$$SYS'q_i = SYq_i = λ_i Sq_i = λ_i q_i,$$

where $λ_i$ is an eigenvalue of both $SYS$ and $Y$. In particular, the non-zero elements of $Λ$ are a subset of the nonzero elements of $Λ$. We also have, for all eigenvectors $q_i$ of $SYS$:

$$S\bar{q}_i = SS'q_i = q_i.$$

Since $S$ has rank 2, one eigenvector $\bar{q}_i$ of $SYS'$ must be such that $S\bar{q}_i = 0$. Without loss of generality, let this eigenvector be $\bar{q}_3$. We then have:

$$S\bar{q} = \begin{bmatrix} Q & 0_{2 \times 1} \end{bmatrix}$$

and

$$SY^{-1}S' = S\bar{q}Λ^{-1}\bar{q}'S' = \begin{bmatrix} Q & 0_{2 \times 1} \end{bmatrix}Λ^{-1} \begin{pmatrix} Q' \\ 0_{1 \times 2} \end{pmatrix} = QΛ^{-1}Q',$$

because the non zero elements in $Λ$ are a subset of those in $Λ$. From (A40), we conclude:

$$(SYS')^{-1} = SY^{-1}S',$$

as desired. This concludes the proof of the Lemma. □

We now proceed with the proof of Proposition 6. By analogy with the proof of Proposition 2, we rewrite the innovation component of the opportunity set dynamics as $Σ^{1/2}[Z, B]L$, with $L = [\sqrt{1-\bar{p}} \tilde{p}_1, \tilde{p}_2, \tilde{p}_3]'$. By definition, market price of risk $Θ_ν$ satisfies:

$$Σ^{1/2}Θ_ν L = μ^e , \quad (A41)$$

from which

$$Θ_ν = Σ^{-1/2}μ^e L + Y^{1/2}ν , \quad (A42)$$

where $Σ^{-1/2} = SY^{1/2}$ and $ν$ is a $3 \times 4$ matrix-valued process such that $νL = 0_{3 \times 4}$, i.e. $ν = [-ν^{p} \sqrt{1-ν^{p}} ν]$. $ν^{p}$ is a $3 \times 3$ matrix pricing the shocks that drive the Wishart state variable $Y$.

It turns out, that the value function can be written in the form:

$$J(x, Y_0) = xγ \inf_ν \frac{1}{1-γ} \mathbb{E} \left[ ξ_ν(T) \right]^{\frac{1}{γ} - 1} = \frac{1}{1-γ} = \frac{1}{1-γ} \mathbb{E} \left[ J(0, Y_0)^{\gamma} - 1 \right],$$

where

$$\mathbb{E} \left[ ξ_ν(T) \right]^{\frac{1}{γ} - 1} \quad (A43)$$

for a probability measure $P^γ$ defined by the density:

$$\frac{dP^γ}{dP} = e^{-tr \left( \frac{1}{2γ} J^T_0 Θ_ν(x) dB + \frac{1}{2} \frac{1}{γ} J^T_0 Θ_ν(x) Θ_ν(x) ds \right)}.$$
The dynamics of $Y$ under the probability $P^\gamma$ are:

$$dY(t) = \left[ \Omega\Omega' + \left( M - \frac{\gamma - 1}{\gamma} Q'(\bar{p}\mu' S + \bar{\pi}') \right) Y(t) + Y(t) \left( M - \frac{\gamma - 1}{\gamma} Q'(\bar{p}\mu' S + \bar{\pi}') \right)' \right] dt$$

$$+ Y^{1/2}(t)dB'(t)Q + Q'dB(t)^{1/2}Y^{1/2}(t). \tag{A44}$$

These dynamics are affine in $Y$. It follows that the function $\hat{J}$ is a solution of the following Hamilton Jacobi Bellman (HJB) equation:

$$0 = \frac{\partial \hat{J}}{\partial t} + \inf_{\bar{\pi}} \left\{ A\hat{J} + \hat{J} \left[ -\frac{\gamma - 1}{\gamma} (r_0 + tr(YD)) + \frac{1-\gamma}{2\gamma^2} \frac{1}{Y} \left( Y \left( S'\mu'\mu' S + \bar{\pi}' \left( I_3 + \frac{\bar{\pi}'}{1-\bar{\pi}} \right) \right) \right) D \right]$$

$$+ tr(2YDQ'QD), \tag{A45}$$

subject to the terminal condition $\hat{J}(T, Y) = 1$, where $A$ is the infinitesimal generator of the matrix-valued diffusion $\left[ A44 \right]$, which is given by:

$$A = tr \left( \left( \Omega\Omega' + \left( M - \frac{\gamma - 1}{\gamma} Q'(\bar{p}\mu' S + \bar{\pi}') \right) Y + Y \left( M - \frac{\gamma - 1}{\gamma} Q'(\bar{p}\mu' S + \bar{\pi}') \right)' \right) D \right)$$

$$+ tr(2YDQ'QD), \tag{A46}$$

The generator is affine in $Y$. The optimality condition for the optimal control $\bar{\pi}$ yields, similarly to the proof of Proposition $[2]$ to:

$$\bar{\pi} = -\gamma \left( \frac{D\hat{J}}{J} + \frac{D\hat{J}}{J} \right) Q' \left( I_3 + \frac{\bar{\pi}'}{1-\bar{\pi}} \right)^{-1}. \tag{A47}$$

Note that $\left( I_3 + \frac{\bar{\pi}'}{1-\bar{\pi}} \right)^{-1} = I_3 - \bar{\pi}$. Substituting the expression for $\bar{\pi}$ in equation $[A45]$, we obtain the following partial differential equation for $\hat{J}$:

$$-\frac{\partial \hat{J}}{\partial t} = tr \left( \left( \Omega\Omega' + \left( M - \frac{\gamma - 1}{\gamma} Q'(\bar{p}\mu' S) \right) Y + Y \left( M - \frac{\gamma - 1}{\gamma} Q'(\bar{p}\mu' S) \right)' \right) D + 2YDQ'QD \right) \hat{J}$$

$$+ \frac{\gamma - 1}{\gamma} \left( -r_0 - tr(YD) - \frac{tr(Y S'\mu'\mu' S)}{2\gamma} \right) - \frac{1-\gamma}{2} \frac{1}{Y} \left( I_3 - \bar{\pi}' \right) tr \left( (I_3 - \bar{\pi}')Y \left( \frac{D\hat{J}}{J} + \frac{D\hat{J}}{J} \right) Q'Q \left( \frac{D\hat{J}}{J} + \frac{D\hat{J}}{J} \right)' \right),$$

subject to the boundary condition $\hat{J}(\Sigma, T) = 1$. The affine structure of this problem suggests an exponentially affine functional form for its solution:

$$\hat{J}(t, \Sigma) = \exp(B(t, T) + tr(A(t, T)Y),$$

for some state independent coefficients $B(t, T)$ and $A(t, T)$. After inserting this functional form into the differential equation for $\hat{J}$, the guess can be easily verified. The coefficients $B$ and $A$ are the solutions of the following system of Riccati equations:

$$-\frac{dB}{dt} = tr(A\Omega') - \frac{\gamma}{\gamma - 1} r_0,$$

$$-tr \left( \frac{dA}{dt} Y \right) = tr \left( \Gamma'AY + A'GY + 2AQ'QAY - \frac{1-\gamma}{2} (A' + A)Q'(I_3 - \bar{\pi}')Q(A' + A)Y + CY \right),$$

with terminal conditions $B(T, T) = 0_{3\times3}$ and $A(T, T) = 0_{3\times3}$, where

$$\Gamma = M - \frac{\gamma - 1}{\gamma} Q'\bar{p}\mu' S \tag{A48}$$

$$C = \frac{1-\gamma}{2\gamma^2} S'\mu'\mu' S - \frac{1-\gamma}{\gamma} D. \tag{A49}$$

Explicit solutions for $B(t, T)$ and $A(t, T)$ are computed as in the proof of Proposition $[2]$. 

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By the same argument applied in the Proof of Proposition 6, the following equality must hold:

$$X^*(t) \operatorname{tr} \left( [\pi_1 \quad \pi_2] \Sigma^{1/2} dB \right) = X^*(t) \operatorname{tr} \left( \frac{1}{\gamma} \Theta' \cdot dB + \frac{D^2 \hat{J}}{J} Y^{1/2} dBUQ + Q' U' dB'Y \right).$$

where matrix $U$ is given by:

$$U = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

This implies

$$L \left[ \pi_1 \quad \pi_2 \right] \Sigma^{1/2} = \frac{1}{\gamma} \left( L \mu' \Sigma^{-1/2} + \nu' Y^{1/2} \right) + 2UQAY^{1/2}.$$ 

Premultiplying both sides by $L'$, postmultiplying them by $\Sigma^{-1/2}$, recalling that $L' \nu' = 0_{1 \times 3}$ and that $\Sigma^{1/2} = SY^{-1/2}$, we conclude that portfolio weight $\pi = (\pi_1, \pi_2)'$ is

$$\pi = \frac{1}{\gamma} \Sigma^{-1} \mu' + 2 \Sigma^{-1} SAQ' \rho.$$ 

This concludes the proof of Proposition 6 $\square$

Proposition 7 and its proof.

**Proposition 7** Let $\pi^H$ denote agent’s intertemporal hedging demand, as reported in (16). Then, $\pi^H$ allows for the following decomposition:

$$\pi^H = \pi^H_{\text{vol/cor}} + \pi^H_{\text{pred}},$$

where $\pi^H_{\text{vol/cor}}$, hedging demand for volatility-correlation risk, is $\pi^H_{\text{vol/cor}} = -\pi^H$, and $\pi^H_{\text{pred}}$, hedging demand for predictability risk, is $\pi^H_{\text{pred}} = 2\pi^H$.

We first need to provide an alternative representation of optimal hedging demands, obtained by Malliavin calculus methods.

According to the Proof of Proposition 3 the optimal portfolio allocation can be obtained from the following relation

$$X^*(t) \operatorname{tr} \left( [\pi_1 \quad \pi_2] \Sigma^{1/2} dB \right) = X^*(t) \left[ \operatorname{tr} \left( \frac{1}{\gamma} \Theta' \cdot dB \right) + \operatorname{Diff} \frac{\hat{J}}{J} \right]$$

where $\operatorname{Diff} \frac{\hat{J}}{J}$ denotes the diffusion component in the Ito representation of the value function $\hat{J}$. The next Lemma provides this diffusion component.

**Lemma 3** Let $\hat{M}_t \cdot$ denote the following matrix Malliavin differential operator:

$$\hat{M}_t \cdot = \begin{bmatrix} \hat{M}_t^{B_{11}} \cdot & \hat{M}_t^{B_{12}} \cdot \\ \hat{M}_t^{B_{21}} \cdot & \hat{M}_t^{B_{22}} \cdot \end{bmatrix}$$

where $\hat{M}_t^{B_{ij}} \cdot$ is the Malliavin derivative with respect to the Brownian component $B_{ij}$. Let also $K(t,T)$ denote the matrix

$$K(t,T) = \lambda \lambda' + \overline{\rho} \left( I_2 + \frac{\overline{p} \overline{p}'}{1 - \overline{p} \overline{p}} \right),$$

with $\overline{\rho}$ as in (A20), and $K_{ij}(t,T)$ the $i - j$ entry of $K(t,T)$. Then $\operatorname{Diff} \frac{\hat{J}}{J}$ is given by:

$$\operatorname{Diff} \frac{\hat{J}}{J} (t, \Sigma(t)) = \hat{J}(t, \Sigma(t)) \frac{1 - \gamma^2}{2 \gamma^2} \operatorname{tr} \left( \mathbb{E} \left[ \int_t^T \sum_{i,j=1,2} K_{ij}(t,s) \hat{M}_t \Sigma_{ij}(s) ds \right] F_t \right) dB(t)$$

where $\mathbb{E} [ \cdot ]$ denotes expectation with respect to the probability measure characterized in (A50) below.
Proof of Lemma\textsuperscript{3}

From the proof of Proposition\textsuperscript{2}, the value function $\hat{J}$ reads:

$$
\hat{J}(t, \Sigma(t)) = \inf_{\nu} \mathbb{E}^{\gamma} \left[ e^{-\frac{1}{2\gamma} \nu (T-t) + \frac{1}{2\gamma} \nu \left( f_t^T \Sigma(s) \left( \lambda \lambda^T + \Sigma T \Sigma (T_2 + \Sigma \Sigma^T) \right) ds \right) } \right]
$$

(A53)

Indeed, letting

$$
f(t, T) = e^{-\frac{1}{2\gamma} \nu (T-t) + \frac{1}{2\gamma} \nu \left( f_t^T \Sigma(s) \left( \lambda \lambda^T + \Sigma T \Sigma (T_2 + \Sigma \Sigma^T) \right) ds \right) }
$$

Diff $\hat{J}$ can be equivalently characterized as

$$
\text{Diff} \hat{J} = \frac{\phi(t)}{f(0, t)} \text{vec}(dB)
$$

where $\phi(t)$ is the integrand in the stochastic integral representation of the martingale $\mathbb{E}^{\gamma} [f(0, T)|\mathcal{F}_t]$ with respect to the vector Brownian motion vec$(dB)$. But according to Clark-Ocone formula\textsuperscript{26} we have

$$
\phi(t) = \mathbb{E}^{\gamma} [\mathcal{M}_t f(0, T)|\mathcal{F}_t]
$$

Where $\mathcal{M}$ is the Malliavin derivative operator with respect to vec$(dB)$. By the chain rule of Malliavin calculus and the $\mathcal{F}_t$-measurability of vec$(dB(t))$ we obtain\textsuperscript{27}

$$
\frac{\mathbb{E}^{\gamma}[\mathcal{M}_t f(0, T)|\mathcal{F}_t]}{f(0, t)} \text{vec}(dB(t)) = \mathbb{E}^{\gamma} \left[ f(t, T) \frac{1-\gamma}{2\gamma^2} \int_t^T \text{vec}(K(t, s))' \mathcal{M}_t \text{vec}(\Sigma(s)) \text{vec}(dB(t)) ds \bigg| \mathcal{F}_t \right]
$$

(A54)

But

$$
\mathcal{M}_t \text{vec}(\Sigma(s)) \text{vec}(dB(t)) = \text{vec} \left[ \begin{bmatrix} \text{tr} \left( \mathcal{M}_t \Sigma_{11}(s) dB(t) \right) & \text{tr} \left( \mathcal{M}_t \Sigma_{12}(s) dB(t) \right) \\ \text{tr} \left( \mathcal{M}_t \Sigma_{21}(s) dB(t) \right) & \text{tr} \left( \mathcal{M}_t \Sigma_{22}(s) dB(t) \right) \end{bmatrix} \right]
$$

from which

$$
\text{vec}(K(t, s))' \mathcal{M}_t \text{vec}(\Sigma(s)) \text{vec}(dB(t)) = \text{tr} \left( K(t, s) \begin{bmatrix} \text{tr} \left( \mathcal{M}_t \Sigma_{11}(s) dB(t) \right) & \text{tr} \left( \mathcal{M}_t \Sigma_{12}(s) dB(t) \right) \\ \text{tr} \left( \mathcal{M}_t \Sigma_{21}(s) dB(t) \right) & \text{tr} \left( \mathcal{M}_t \Sigma_{22}(s) dB(t) \right) \end{bmatrix} \right)
$$

Computing explicitly the trace in the last expression we conclude that

$$
\text{Diff} \hat{J}(t, \Sigma(t)) = \frac{1-\gamma}{2\gamma^2} \text{tr} \left( \mathbb{E}^{\gamma} \left[ f(t, T) \int_t^T \left( \sum_{i,j=1,2} K_{ij}(t, s) \mathcal{M}_t \Sigma_{ij}(s) ds \right) \bigg| \mathcal{F}_t \right] dB(t) \right)
$$

(A55)

$$
= \frac{1-\gamma}{2\gamma^2} \mathbb{E}^{\gamma} [f(t, T)|\mathcal{F}_t] \text{tr} \left( \mathbb{E}^{\gamma} \left[ \int_t^T \left( \sum_{i,j=1,2} K_{ij}(t, s) \mathcal{M}_t \Sigma_{ij}(s) ds \right) \bigg| \mathcal{F}_t \right] dB(t) \right)
$$

(A56)

\textsuperscript{27}See the monograph of Nualart (1995) for the Clark-Ocone formula and other results on Malliavin calculus applied in this proof.

\textsuperscript{28}Note that $K(t, T)$ is deterministic, because

$$
\nabla = -\gamma \left( \frac{\partial \hat{J}}{\hat{J}} + \frac{\partial \hat{J}}{\hat{J}} \right) Q' \left( I_2 + \frac{\eta \eta'}{1 - \rho \rho} \right)^{-1} = -\gamma (A + A') Q' \left( I_2 + \frac{\eta \eta'}{1 - \rho \rho} \right)^{-1}
$$

Functions $A$ are given in (A34). It follows that $\mathcal{M}_t K(t, T) = 0$. 

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where $E^J[\cdot]$ denotes expectation with respect to the probability measure under which the Wishart variance-covariance matrix $\Sigma(t)$ evolves as:

$$d\Sigma(t) = \left[ \Omega\Omega' + \left( M - \frac{\gamma - 1}{\gamma} Q'\bar{\nu} + \bar{\nu} \right) + Q'QA \right] \Sigma(t) + \Sigma(t) \left( M - \frac{\gamma - 1}{\gamma} Q'\bar{\nu} + \bar{\nu} \right) dt$$

$$+ \Sigma(t) d\gamma(t) + Q'dB(t) \gamma(t)\Sigma(t),$$

where $A$ is given in (A34). Expression (A56) and dynamics (A57) follow from (A55) by a standard change of numeraire technique. Note that $E^\gamma[f(t,T)|\mathcal{F}_t]$ is the value function $\hat{J}(t,\Sigma(t))$.

This ends the proof of the Lemma. □

In light of this Lemma, noting that $B = BU$, we identify from (A51) the intertemporal hedging demand:

$$\pi^H = \mathbb{E}^\hat{J}\left[ \int_t^T \left( \sum_{i,j=1,2} K_{ij}(t,s)\hat{M}_t \Sigma_{ij}(s)ds \right) | \mathcal{F}_t \right] \Sigma^{-1/2}\bar{\nu}$$

REMARK. The Malliavin matrix derivatives $\hat{M}_t \Sigma_{ij}$ can be computed following the rules of Malliavin calculus:

$$\hat{M}_t \Sigma_{ij}(s) = \hat{M}_t \Sigma_{ij}(t) \exp \left( 2e_i'(t) MS^{ij}(s) e_j(s) - \frac{1}{4} \int_t^s e_i'(t) S^{ij} \Sigma^{-1}(s) S^{ij} Q e_j ds \right)$$

$$+ \frac{1}{2} \int_t^s \left( e_i'(s) \Sigma^{-1/2}(s) S^{ij} dB(s) Q e_j + e_i'(s) Q^2 dB(s) \Sigma^{-1/2}(s) S^{ij} e_j \right), \quad i,j = 1,2, \quad s > t.$$

where $S^{ij}$ is a $2 \times 2$ matrix of zeros, with the exception of the $ij$-th entry being equal to one, and $e_i$ is a bivariate column vector of zeros, with the exception of the $i$-th entry being equal to one. The initial condition of the Malliavin derivative, $\hat{M}_t \Sigma_{ij}(t)$, is the $2 \times 2$ matrix with $yz$-th entry $(y,z = 1,2)$ given by the element of the diffusion component of $\Sigma_j(t)$ proportional to the Brownian motion $B_{yz}$. The sample path of $\hat{M}_t \Sigma_{ij}(t)$ corresponding to a given sample path of $\Sigma_{ij}(t)$ can be easily simulated.

Remember that the market price of risk of our incomplete markets framework is written as:

$$\Theta_\nu = \Sigma^{1/2}(\Sigma^{1/2} \Sigma^{1/2})^{-1}(\mu_t - \gamma T_2)L' + \Sigma^{1/2} \nu,$$

$$= \Sigma^{1/2}(\Sigma \lambda)L' + \Sigma^{1/2} \nu,$$

where $\nu = [-\bar{\nu} \frac{\sigma}{\sqrt{1-\bar{\nu}^2}}, \bar{\nu}]$ and $\bar{\nu}$ is given in (A20). It follows, that the squared market price of risk appearing in the value function $\hat{J}$ can be written as

$$tr(\Theta_\nu \Theta_\nu) = tr \left( \frac{\lambda' \Sigma^{-1}}{\lambda' \Sigma^{-1} \Sigma \lambda \Sigma^{-1} \Sigma \lambda} + \frac{\Sigma \nu \nu'}{\Sigma \nu \nu' \Sigma} \right).$$

Term above represents the component of investment opportunities driven by time-varying equity premia, $C$ is a volatility-correlation risk component, whereas $D$ is the squared price of market incompleteness and can be traced back to both sources of time-variation. The idea behind the identification of volatility-correlation hedging in contrast to hedging for time-varying risk premia is to simply take separate Malliavin derivatives of the value function with respect to the components above. Mimicking the proof of Lemma \[\text{3}\] we restate

\[\text{We remind that the diffusion component of $\Sigma_{ij}$ is}

$$e_i' \Sigma^{1/2} dBQe_j + e_i' Q'dB' \Sigma^{1/2} e_j.$$
Term (A60) above is the innovation of the value function due to predictability risk, component (A61) is the innovation due to volatility-correlation risk, whereas (A62) is the innovation due to fluctuations in the price of market incompleteness, which needs in turn to be decomposed into a predictability and a volatility-correlation risk component. Consider components (A60) and (A61). After simple manipulations we have:

\[
E^\gamma \left[ f(t, T) \frac{1 - \gamma}{2\gamma^2} \int_t^T \left( \frac{\partial A'(CA)}{\partial A} \right)' \mathcal{M}_t A \right. \left. \text{vec}(dB(t)) \right] \bigg| \mathcal{F}_t \bigg] = E^\gamma \left[ f(t, T) \frac{1 - \gamma}{2\gamma^2} \int_t^T 2\lambda(I_2 \otimes \lambda') \mathcal{M}_t \text{vec}(\Sigma) \right. \left. \text{vec}(dB(t)) \right] \bigg| \mathcal{F}_t \bigg] = E^\gamma \left[ f(t, T) \frac{1 - \gamma}{2\gamma^2} \int_t^T 2\lambda \lambda' \mathcal{M}_t \text{vec}(\Sigma) \right. \left. \text{vec}(dB(t)) \right] \bigg| \mathcal{F}_t \bigg]
\]

and

\[
E^\gamma \left[ f(t, T) \frac{1 - \gamma}{2\gamma^2} \int_t^T \left( \frac{\partial A'(CA)}{\partial \text{vec}(C)} \right)' \mathcal{M}_t \text{vec}(C) \right. \left. \text{vec}(dB(t)) \right] \bigg| \mathcal{F}_t \bigg] = E^\gamma \left[ f(t, T) \frac{1 - \gamma}{2\gamma^2} \int_t^T -\text{vec}(\Sigma^{-1} \otimes \Sigma^{-1}) \mathcal{M}_t \text{vec}(\Sigma) \right. \left. \text{vec}(dB(t)) \right] \bigg| \mathcal{F}_t \bigg] = E^\gamma \left[ f(t, T) \frac{1 - \gamma}{2\gamma^2} \int_t^T -\text{vec}(\lambda \lambda') \mathcal{M}_t \text{vec}(\Sigma) \right. \left. \text{vec}(dB(t)) \right] \bigg| \mathcal{F}_t \bigg]
\]

As of component (A62), note that \( D = \Sigma \nu' \left( I_2 + \frac{\nu \nu'}{1 - \rho^2} \right) \), where, according to (A20), \( \nu \) is defined implicitly by the following equation

\[
\nu = -\gamma \left( \frac{\text{DIFF} \tilde{J}}{J} + \frac{\text{DIFF} \tilde{J}'}{J} \right) \left( I_2 + \frac{\nu \nu'}{1 - \rho^2} \right)^{-1}
\]

where \( \text{DIFF} \tilde{J} \) is the volatility of the value function \( \tilde{J} \) once we represent its diffusion component as

\[
\text{Diff} \tilde{J} = \text{tr} \left( \text{DIFF} \tilde{J} dB + dB' \text{DIFF} \tilde{J}' \right)
\]

But \( \text{DIFF} \tilde{J}' \) can be once again represented by Malliavin calculus, and expressions (A60) (A62) show that \( D = D_{\text{vol/cor}} + D_{\text{pred}} \) where \( D_{\text{vol/cor}} = 2D \) and \( D_{\text{pred}} = -D \). Summarizing, the innovation component of the value function \( \tilde{J} \) due to volatility-correlation risk can be written as

\[
E^\gamma \left[ f(t, T) \frac{1 - \gamma}{2\gamma^2} \int_t^T \left( \frac{\partial A'(CA)}{\partial A} \right)' \mathcal{M}_t A + \left( \frac{\partial \text{tr}(-D)}{\partial \text{vec}(D)} \right)' \mathcal{M}_t \text{vec}(D) \right. \left. \text{vec}(dB(t)) \right] \bigg| \mathcal{F}_t \bigg] = -E^\gamma \left[ f(t, T) \frac{1 - \gamma}{2\gamma^2} \int_t^T \text{vec}(K(t, s))' \mathcal{M}_t \text{vec}(\Sigma(s)) \right. \left. \text{vec}(dB(t)) \right] \bigg| \mathcal{F}_t \bigg] = -E^\gamma \left[ \frac{\mathcal{M}_t f(0, T)| \mathcal{F}_t}{f(0, t)} \right] \text{vec}(dB(t))
\]
where the last expression is the total innovation component of the value function changed in sign. In light of the proof of Lemma 3 this ends the proof of the Proposition. □
Appendix B: Moments restrictions in the GMM estimation

This Appendix provides detailed expressions for the moment conditions used in the GMM estimation of our model. The following computations make extensive use of the closed-form expressions for the moments of the Wishart process, which can be found, e.g., in the Appendix of Buraschi, Cieslak and Trojani (2007).

Let $\tau$ denote data sampling frequency. We have $\tau = 5/250$ for weekly data and $\tau = 22/250$ for monthly data.

1) **Unconditional risk premia of log-returns.**

The conditional risk premia of $i$-th asset’s logarithmic returns, at frequency $\tau$, $i=1,2$, are given by:

$$
\mathbb{E}_t \left[ \log S^i(t+\tau) - \log S^i(t) \right] - \int_t^{t+\tau} r ds = \mathbb{E}_t \left[ \int_t^{t+\tau} \epsilon_i^r(s)(\lambda - \frac{1}{2}\epsilon_i^2) ds \right] 
$$

Then the first two unconditional moments we use are:

$$
M_1 = \left( \mathbb{E}[\Sigma(t)]\lambda - \frac{1}{2} \left[ \epsilon_1^r \mathbb{E}[\Sigma(t)] \epsilon_1^2 \epsilon_2^r \mathbb{E}[\Sigma(t)] \epsilon_2^2 \right] \right) \tau 
$$

2) **Unconditional mean of the variance-covariance matrix of log-returns.**

$$
M_2 = \text{vech} (\mathbb{E}[\Sigma(t)]) \tau 
$$

where $\text{vech}(X)$ denotes lower triangular vectorization of a square matrix $X$.

3) **Unconditional second moment of the variance-covariance matrix of log-returns.**

$$
\mathbb{E} \left[ \int_t^{t+\tau} \text{vec}(\Sigma(s)) \text{vec}(\Sigma(s))' ds \right] = 2 \int_0^\tau dr_2 \int_0^{r_2} dr_1 \mathbb{E} \left[ \text{vec}(\Sigma(r_1)) \text{vec}(\Sigma(r_2))' \right] 
$$

Therefore

$$
M_3 = \text{vech} \left( \mathbb{E} \left[ \int_t^{t+\tau} \text{vec}(\Sigma(s)) \text{vec}(\Sigma(s))' ds \right] \right) 
$$

4) **Unconditional covariance between assets’ simple excess returns and the variance-covariance matrix of log-returns.** For asset $i$, $i=1,2$, and $s > t$, this quantity reads

$$
\lim_{t \to \infty} \mathbb{E}_t \left[ \exp \left( \int_s^{s+\tau} e_i \Sigma(u) dW(u) + \int_s^{s+\tau} e_i \Sigma(u) \lambda du - \frac{1}{2} \int_s^{s+\tau} e_i \Sigma(u) e_i' du \right) \otimes \int_s^{s+\tau} \Sigma(u) du \right] = 
$$

$$
\exp \left( A_t(\tau) + \tilde{A}_t(\infty) \right) \left( \int_0^\tau \exp(\tilde{M}(u)u) \mathbb{E} \left[ \Sigma(t) \right] \exp(\tilde{M}(u)'u) du + \int_0^\tau \int_0^u \exp(\tilde{M}(u)u) kQ'Q \exp(\tilde{M}(u)'u) du ds \right) 
$$

where

$$
\tilde{M}(\tau) = M + Q' \rho e_i' + Q'Q B_i(\tau) 
$$

$$
A_t(\tau) = k \int_0^\tau tr \left( B_i(s)Q'Q \right) ds 
$$

$$
A_t(\infty) = k \int_0^\infty tr \left( B_i(s)Q'Q \right) ds 
$$

$$
B_i(t) = B_{22}(t)^{-1}B_{21}(t) 
$$

$$
\tilde{B}_i(t) = (B_i(t)\tilde{B}_{12}(t) + \tilde{B}_{22}(t))^{-1}B_i(t)\tilde{B}_{11}(t) 
$$

55
and

\[
\begin{pmatrix}
B_{11}(t) & B_{12}(t) \\
B_{21}(t) & B_{22}(t)
\end{pmatrix} = \exp \left[ t \begin{pmatrix}
M + Q' \rho e_i' \\
\lambda e_i' - (M + Q' \rho e_i')'
\end{pmatrix} \right]
\]

\[
\begin{pmatrix}
\tilde{B}_{11}(t) & \tilde{B}_{12}(t) \\
\tilde{B}_{21}(t) & \tilde{B}_{22}(t)
\end{pmatrix} = \exp \left[ t \begin{pmatrix}
M & -2Q'Q \\
0 & -M'
\end{pmatrix} \right]
\]

The last set of moment conditions is therefore given by

\[
M_{3+i} = \text{vech} \left( \exp \left( A_l(\tau) + \tilde{A}_l(\infty) \right) \times \left( \int_0^\tau \exp(\tilde{M}(u) u) E[\Sigma(t)] \exp(\tilde{M}(u)' u) du + \int_0^\tau \int_0^s \exp(\tilde{M}(u) u) kQ'Q \exp(\tilde{M}(u)' u) du ds \right) \right)
\]

for \( i = 1, 2 \).

Summarizing, the vector-valued function \( \mu^\tau(M, Q, \lambda, \rho, k) \) of theoretical moment conditions, for sampling frequency \( \tau \), is given by:

\[
\mu^\tau(M, Q, \lambda, \rho, k) = \begin{bmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4 \\
M_5
\end{bmatrix}
\]

This is compared to its empirical counterpart \( \hat{\mu}^\tau \) based on historical returns, volatilities and covariances.
Figure 1. Joint Correlation Dynamics of Stock Index Returns. The figure plots the sample correlations between the S&P500 and FTSE100 Index ($x$-axis) versus the sample correlations of S&P500 and Nikkey225 Index ($y$-axis). Thus, each point in the graph represents couples of realized sample correlations between these stock indices. Sample correlations are computed using overlapping four-month windows of weekly returns during the time period April 2004 - April 2008.

Figure 2. Empirical correlations for different return sizes. Monthly returns on the S&P500 index (from January 87 to April 08) have been divided into 6 equal size bins. The $y$-axis of this figure shows the average empirical correlations between the S&P500 and the Nikkei225 (circle marks), and between the S&P500 and the FTSE100 (triangle marks), given S&P500 returns realizations. Returns bins are reported on the $x$-axis.
Figure 3. Drift and Volatility of the Correlation Process. The figure plots the instantaneous drift (Panel 1) and volatility (Panel 2) of the correlation process implied by the Wishart diffusion ([4]), as a function of the correlation level $\rho(t)$ and for different values of the volatility ratio $\sqrt{\Sigma_{11}(t)/\Sigma_{22}(t)}$. Solid lines are for $\sqrt{\Sigma_{11}(t)/\Sigma_{22}(t)} = 3$, which is approximately the ratio of the sample volatilities of S&P500 Index Futures and 30-year Treasury Futures returns in our data. Dotted lines are for $\sqrt{\Sigma_{11}(t)/\Sigma_{22}(t)} = 2$ and dashed-dotted lines are for $\sqrt{\Sigma_{11}(t)/\Sigma_{22}(t)} = 4$. The parameters used to plot the drift and volatility functions of the correlation are those estimated in Table II for monthly returns, realized volatilities and correlations.
Figure 4. Pull Function of the Volatility and Correlation Processes. Panels 1, 4 and 7 show the nonparametric pull functions for weekly and monthly data frequencies (dotted and solid lines, respectively) based on the estimation procedure in Conley, Hansen, Luttmer, and Scheinkman (1997). Panel 1 and 4 present the pull function estimates for the conditional volatility of S&P500 Index and 30-year Treasury Bond Futures returns, respectively. Panel 7 plots the pull function estimate for their conditional correlation. Panels 2, 5 and 8 present the corresponding pull function estimates for volatilities and correlations, but based on a long time series of weekly volatilities and correlations simulated from the Wishart variance-covariance process (4) using the weekly parameter estimates in Table II. Panels 3, 6 and 9 present corresponding pull function estimates for volatilities and correlations, but based on a long time series of monthly volatilities and correlations simulated from the Wishart variance-covariance process (4) using the monthly parameter estimates in Table II. In Panels 2, 3, 5, 6, 8, 9, the model implied pull functions are plotted together with a 95% confidence interval around the empirical pull functions of Panels 1, 4 and 7.
Figure 5. Volatility and Correlation Leverage. Using the monthly parameter estimates in Table II, we simulate time series of monthly returns, volatilities and correlations from the estimated model. Panel 1 presents a scatter plot of S&P500 Futures monthly returns and realized volatility changes. Panel 2 presents the scatter plot of 30-year Treasury Bond future monthly returns and realized volatility changes. Panel 3 presents the scatter plot of S&P500 Futures monthly returns and realized correlation changes. Panel 4 presents the scatter plots of 30-year Treasury Bond future monthly returns and realized correlation changes.
Figure 6. Comparative Statics of Optimal Hedging Demands with respect to \( m_{12} \), \( m_{22} \) and \( q_{11} \). Comparative statics for optimal correlation hedging demands (Panels 1, 3 and 5) and volatility hedging demands (Panels 2, 4 and 6), obtained by letting the values of parameters \( m_{12} \) (Panels 1 and 2), \( m_{22} \) (Panel 3 and 4) and \( q_{11} \) (Panels 5 and 6) vary in a confidence interval of one sample standard deviation around the monthly parameter estimates in Table II. \( m_{ij} \) and \( q_{ij} \) are the entries of parameter matrices \( M \) and \( Q \), respectively, appearing in the Wishart covariance matrix dynamics:

\[
d\Sigma(t) = (\Omega\Omega' + M\Sigma(t) + \Sigma(t)M')dt + \Sigma^{1/2}(t)dB(t)Q + Q'dB(t)\Sigma^{1/2}(t)
\]

Hedging demands for the S&P500 index (30-year Treasury bond) future are plotted with solid (dotted) lines.
Figure 7. Comparative Statics of Optimal Hedging Demands with respect to $\rho$. The figure plots the optimal correlation hedging demands (Panels 1 and 2) and volatility hedging demands (Panels 3 and 4) as a function of $\rho_1$ and $\rho_2$, when these parameters vary in a confidence interval of one sample standard deviation around the monthly point estimates in Table II. $\rho_1$ and $\rho_2$ are the entries of the vector $\rho$, which controls leverage by means of the following relation:

$$dW(t) = dB(t)\rho + \sqrt{1-\rho^2}dZ(t)$$

where $W(t)$ is the Brownian motion driving asset returns, $B(t)$ is the Brownian motion driving the Wishart covariance matrix dynamics and $Z(t)$ is a vector of independent Brownian motions. Hedging demands for the S&P 500 Index (30-year Treasury bond) Futures are given in Panels 1 and 3 (Panels 2 and 4).
Figure 8. The Effect of the Investment Horizon. Panel 1: Total hedging demands for the S&P500 Index Futures (solid line) and 30–year Treasury Futures (dotted line) in percentage of the Merton myopic portfolio are plotted as a function of the investment horizon (in years). These hedging demands are computed using the monthly parameter estimates in Table II for a relative risk aversion parameter $\gamma = 6$. Panel 2: Pure volatility hedging (dotted line) and correlation hedging (solid line) demands for the 30-year Treasury bond Futures (dotted and solid line, respectively) and the S&P500 Index Futures (dashed and dashed-dotted line, respectively) are plotted as functions of the investment horizon (in years). Both hedging demands are expressed in percentage of the Merton myopic portfolio. The same parameters as for Panel 1 are used to computed them.
Figure 9. The Effect of the Risk Aversion Parameter. Panel 1: Total hedging demands for the S&P500 Index Futures (solid line) and 30-year Treasury Futures (dotted line) in percentage of the Merton myopic portfolio are plotted as functions of the relative risk aversion coefficient for a fixed investment horizon of 5 years. To compute these policies, we use the monthly parameters estimates in Table II. Panel 2: Volatility hedging and correlation hedging for the 30-year Treasury bond Futures (dotted and solid line, respectively) and the S&P500 Index Futures (dashed and dashed-dotted line, respectively) in percentages of the Merton myopic portfolio are plotted as functions of the relative risk aversion coefficient. The same parameters as in Panel 1 are used to compute these policies. Panel 3: Same plots as in Panel 1, but with percentage hedging demands replaced by actual hedging portfolio weights. Panel 4: Total portfolio weights for correlation hedging (solid line) and for volatility hedging (dotted line), aggregated over risky assets, are plotted as functions of the Relative risk aversion parameter. The same parameters as in Panel 1 are used to compute these policies.
Figure 10. Joint Dynamics of CDO’s Implied Base Correlations. Scatter plot of implied base correlations for 7 year maturity mezzanine tranches of (U.S.) CDX and (Europe) iTraxx CDOs, sampled weekly in the period from September 2004 to April 2008.
Table I
Unconditional Moments of Asset Returns

*Panel A:* Sample mean, volatility and volatility of volatility implied by S&P500 Index and 30-year Treasury Bond Futures returns, sampled both weekly and monthly in the period from January 1990 to October 2003. Realized volatilities are computed from high-frequency data of Futures returns. *Panel B:* Sample mean and volatility of realized correlations implied by S&P500 Index and 30-year Treasury Bond Futures returns, sampled both weekly and monthly in the period from January 1990 to October 2003. Realized correlations are computed from high-frequency data of Futures returns.

<table>
<thead>
<tr>
<th>Panel A</th>
<th>Mean of Returns</th>
<th>Volatility of Returns</th>
<th>Volatility of volatility</th>
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<tr>
<td></td>
<td>Weekly</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S&amp;P500</td>
<td>0.00096</td>
<td>0.0226</td>
<td>0.0087</td>
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<tr>
<td>Treasury</td>
<td>0.00032</td>
<td>0.0147</td>
<td>0.0028</td>
</tr>
<tr>
<td></td>
<td>Monthly</td>
<td></td>
<td></td>
</tr>
<tr>
<td>S&amp;P500</td>
<td>0.0042</td>
<td>0.0474</td>
<td>0.0182</td>
</tr>
<tr>
<td>Treasury</td>
<td>0.0015</td>
<td>0.0312</td>
<td>0.0087</td>
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<table>
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<tr>
<th>Panel B</th>
<th>Mean of Correlation</th>
<th>Volatility of Correlation</th>
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<tr>
<td></td>
<td>Weekly</td>
<td>0.0433</td>
</tr>
<tr>
<td></td>
<td>Monthly</td>
<td>0.0412</td>
</tr>
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</table>
Table II
Estimation Results for the Model with 2 Risky Assets

Estimated matrices $M$ and $Q$ and vectors $\lambda$ and $\bar{\gamma}$ for the returns dynamics (1) under the Wishart variance covariance diffusion process:

$$d\Sigma(t) = (\Omega \Omega' + M \Sigma(t) + \Sigma(t)M')dt + \Sigma^{1/2}(t)dB(t)Q + Q'dB(t)'\Sigma^{1/2}(t)$$

where we have set $\Omega \Omega' = kQ'Q$ and $k = 10$. Parameters are estimated by GMM using time series of returns and realized variance-covariance matrices for S&P 500 Index and 30-year Treasury Bond Futures returns, computed both for a weekly and a monthly frequency. The detailed set of moment restrictions used for GMM estimation is given in Appendix B. We report parameters estimates and their standard errors (in parentheses), together with the $p$-values for Hansen’s J-test of over-identifying restrictions. An asterisk denotes estimated parameters that are not significant at the 5% significance level.

<table>
<thead>
<tr>
<th></th>
<th>$M$</th>
<th>$Q$</th>
<th>$\lambda$</th>
<th>$\bar{\gamma}$</th>
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<tr>
<td><strong>Weekly</strong></td>
<td></td>
<td></td>
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<tr>
<td>point estimates</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>(standard errors)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M$</td>
<td>$Q$</td>
<td>$\lambda$</td>
<td>$\bar{\gamma}$</td>
<td></td>
</tr>
<tr>
<td>weekly</td>
<td>$-1.210$</td>
<td>$0.491$</td>
<td>$0.167$</td>
<td>$0.033^*$</td>
</tr>
<tr>
<td></td>
<td>$(0.330)$</td>
<td>$(0.203)$</td>
<td>$(0.047)$</td>
<td>$(0.027)$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$-1.271$</td>
<td>$0.001^*$</td>
<td>$0.090$</td>
<td>$3.317$</td>
</tr>
<tr>
<td></td>
<td>$(0.127)$</td>
<td>$(0.363)$</td>
<td>$(0.020)$</td>
<td>$(0.044)$</td>
</tr>
<tr>
<td><strong>Monthly</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>point estimates</td>
<td></td>
<td></td>
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<tr>
<td>(standard errors)</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>$M$</td>
<td>$Q$</td>
<td>$\lambda$</td>
<td>$\bar{\gamma}$</td>
<td></td>
</tr>
<tr>
<td>monthly</td>
<td>$-1.122$</td>
<td>$0.747$</td>
<td>$0.160$</td>
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<tr>
<td></td>
<td>$(0.410)$</td>
<td>$(0.368)$</td>
<td>$(0.063)$</td>
<td>$(0.0413)$</td>
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<tr>
<td>$Q$</td>
<td>$-0.888$</td>
<td>$-0.021^*$</td>
<td>$0.009^*$</td>
<td>$2.891$</td>
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<tr>
<td></td>
<td>$(0.556)$</td>
<td>$(0.235)$</td>
<td>$(0.028)$</td>
<td>$(0.118)$</td>
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<tr>
<td><strong>p-values for Hansen’s J-test</strong></td>
<td><strong>Weekly</strong>: 0.14</td>
<td><strong>Monthly</strong>: 0.38</td>
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<td></td>
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</table>
Optimal correlation and volatility hedging demands, as a percentage of the myopic portfolio, for different investment horizons and relative risk aversion parameters. We compute these demands for both the weekly and monthly parameters estimates of Table III and assume a ratio $\sqrt{\Sigma_{11}(t)/\Sigma_{22}(t)}$ in equation (24) and (25), which is approximately the ratio of sample standard deviations of returns in our data set. Each entry in the Table is a vector with two components: The first component is the hedging demand for the S&P500 Index Futures and the second one is the hedging demand for the 30-year Treasury Futures.

### Table III

**Optimal Hedging Demands in the Model with Two Risky Assets**

#### Weekly

<table>
<thead>
<tr>
<th>RRA</th>
<th>$T$</th>
<th>3m</th>
<th>6m</th>
<th>1y</th>
<th>2y</th>
<th>5y</th>
<th>7y</th>
<th>10y</th>
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<tr>
<td>2</td>
<td>0.0151</td>
<td>0.0253</td>
<td>0.0359</td>
<td>0.0457</td>
<td>0.0553</td>
<td>0.0650</td>
<td>0.0746</td>
<td>0.0842</td>
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<tr>
<td>6</td>
<td>0.0254</td>
<td>0.0350</td>
<td>0.0449</td>
<td>0.0548</td>
<td>0.0647</td>
<td>0.0746</td>
<td>0.0845</td>
<td>0.0943</td>
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<td>8</td>
<td>0.0333</td>
<td>0.0555</td>
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<td>0.1196</td>
<td>0.1397</td>
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<tr>
<td>11</td>
<td>0.0277</td>
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<td>0.0649</td>
<td>0.0738</td>
<td>0.0827</td>
<td>0.0916</td>
<td>0.1005</td>
<td>0.1094</td>
</tr>
<tr>
<td>16</td>
<td>0.0357</td>
<td>0.0596</td>
<td>0.0836</td>
<td>0.0994</td>
<td>0.1153</td>
<td>0.1312</td>
<td>0.1471</td>
<td>0.1630</td>
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<tr>
<td>21</td>
<td>0.0363</td>
<td>0.0605</td>
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<td>0.1097</td>
<td>0.1217</td>
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<tr>
<td>41</td>
<td>0.0372</td>
<td>0.0620</td>
<td>0.0869</td>
<td>0.0985</td>
<td>0.1099</td>
<td>0.1214</td>
<td>0.1330</td>
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#### Volatility hedging (/ myopic)

<table>
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<tr>
<th>RRA</th>
<th>$T$</th>
<th>3m</th>
<th>6m</th>
<th>1y</th>
<th>2y</th>
<th>5y</th>
<th>7y</th>
<th>10y</th>
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<tbody>
<tr>
<td>2</td>
<td>0.0116</td>
<td>0.0188</td>
<td>0.0256</td>
<td>0.0289</td>
<td>0.0294</td>
<td>0.0294</td>
<td>0.0294</td>
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#### Volatility hedging (/ myopic)

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Table IV
Estimation Results and Hedging Demands for Univariate Stochastic Volatility Models

Panel A: We report point estimates and standard errors (in parentheses) for the parameters of the following univariate stochastic volatility model:

\[ dS_t = S_t(r + \lambda \sigma_t^2)dt + \sigma_t(\rho dW_t + \sqrt{1-\rho^2}dZ_t) \]
\[ d\sigma_t^2 = (kb^2 + 2m\sigma_t^2)dt + 2b\sigma_t dW_t \]  

(1T)

\( S_t \) is the Futures price of either the S&P500 Future, the 30-year Treasury Bond Futures or the Nikkei 225 Index Futures. \( \sigma_t \) is the stochastic volatility process of returns, modeled by a Heston (1993)-type model. \( W_t \) and \( Z_t \) are independent scalar Brownian motions and \((k, \lambda, \rho, b, m)\) is the vector of parameters of interest. We estimate model (1T) by GMM using monthly time series of returns and realized volatilities for the S&P500 Futures, 30-year Treasury Bond Futures and Nikkei 225 Index Futures returns. Panel B: We compute optimal (volatility) hedging demands for the univariate stochastic volatility model (1T), in percentage of the myopic portfolio, using the parameter estimates in Panel A and for different investment horizons and relative risk aversion coefficients. The notation S&P500, Trea and Nik225 corresponds to the hedging demands in the univariate models for the S&P500 Index Futures, the 30-year Treasury Bond Futures and the Nikkei225 Index Futures, respectively.

Panel A

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<th>( \rho )</th>
<th>( \lambda )</th>
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<td>(1.16)</td>
<td>(0.56)</td>
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Panel B

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We present parameter estimates, Hansen’s statistics and optimal hedging demands for model (4) with 3 risky assets. Panel A: We report parameter estimates for $M$, $Q$, $\lambda$ and $\overline{\lambda}$ (with standard errors in parentheses) in the returns dynamics (1)-(4), where $\Omega'Y = kQQ'$ for $k = 10$. The parameters are estimated using monthly returns, realized volatilities and correlations of S&P 500 index, 30-year US Treasury bond, and Nikkei 225 index future returns sampled at a monthly frequency. The GMM estimation procedure is similar to the one used to estimate the bivariate model and detailed moment restrictions are given in Appendix B. Parameters that are not significant at the 5% significance level are denoted by an asterisk. Panel B: We report optimal correlation and volatility hedging demands in percentage of the myopic portfolio. Each entry of the array in Panel B consists of three components, the first of which is the demand for the S&P500 Index Futures, the second one the demand for the 30-year Treasury bond Futures and the third one the demand for the Nikkei 225 Index Futures, respectively.

### Panel A

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
& M & Q & \overline{\lambda} & \lambda \\
\hline
\text{point estimates} & -0.762 & 0.390 & 0.005 & 0.064 & 0.069 & -0.210 & 2.482 \\
\text{standard errors} & (0.293) & (0.180) & (0.060) & (0.029) & (0.051) & (0.090) & (0.380) \\
\hline
\end{array}
$$

### Panel B

#### Volatility Hedging

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 RRA & T & 3m & 6m & 1y & 2y & 5y & 7y & 10y & 20y \\
\hline
 2 & 0.015 & 0.022 & 0.004 & 0.023 & 0.021 & 0.021 & 0.021 & \\
 6 & 0.033 & 0.097 & 0.127 & 0.135 & 0.137 & 0.137 & 0.137 & \\
 8 & 0.045 & 0.128 & 0.162 & 0.169 & 0.168 & 0.168 & 0.168 & \\
 11 & 0.059 & 0.150 & 0.189 & 0.194 & 0.194 & 0.194 & 0.194 & \\
 16 & 0.072 & 0.185 & 0.218 & 0.219 & 0.219 & 0.219 & 0.219 & \\
 21 & 0.088 & 0.215 & 0.247 & 0.276 & 0.276 & 0.276 & 0.276 & \\
 41 & 0.108 & 0.249 & 0.282 & 0.283 & 0.283 & 0.283 & 0.283 & \\
\hline
\end{array}
$$

#### Correlation Hedging

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
 RRA & T & 3m & 6m & 1y & 2y & 5y & 7y & 10y & 20y \\
\hline
 2 & 0.036 & 0.047 & 0.022 & 0.021 & 0.021 & 0.021 & 0.021 & \\
 6 & 0.055 & 0.095 & 0.158 & 0.176 & 0.177 & 0.177 & 0.177 & \\
 8 & 0.077 & 0.128 & 0.182 & 0.200 & 0.200 & 0.200 & 0.200 & \\
 11 & 0.098 & 0.169 & 0.217 & 0.235 & 0.235 & 0.235 & 0.235 & \\
 16 & 0.119 & 0.203 & 0.290 & 0.329 & 0.329 & 0.329 & 0.329 & \\
 21 & 0.140 & 0.230 & 0.330 & 0.390 & 0.390 & 0.390 & 0.390 & \\
 41 & 0.171 & 0.280 & 0.400 & 0.480 & 0.480 & 0.480 & 0.480 & \\
\hline
\end{array}
$$
Estimation Results and Hedging Demands for the Model with Constant Risk Premia

**Panel A:** We report parameter estimates, Hansen’s statistics and hedging demands for the following model specification:

\[
\begin{align*}
    dS(t) &= I_S \mu dt + I_S Y^{-1/2}(t)(dB(t)\overline{\rho} + \sqrt{1-\overline{\rho}^2}dZ(t)) \\
    dY(t) &= \left[\Omega \Omega' + MY(t) + Y(t)M'\right] dt + Y^{1/2}(t)dB(t)Q + Q'dB(t)'Y^{1/2}(t),
\end{align*}
\]

\(S(t)\) is the two dimensional vector of the prices of S&P500 Index and 30-year Treasury bond Futures. \(\mu\) is a bivariate vector of constants and the interest rate \(r\) is also constant. \(Y(t)\) models the information matrix \(\Sigma(t)^{-1}\) and follows a Wishart diffusion of the type (4). \(B(t)\) is a 2 × 2 matrix of standard Brownian motions and \(Z(t)\) is a 2 × 1 vector of Brownian motions independent of \(B(t)\). Vector \(\overline{\rho}\) and matrices \(M, Q\) are the remaining model parameters. Parameters have been estimated with the same GMM method as for the estimation of model (1)-(4), but now applied to the information matrix \(Y = \Sigma^{-1}\) sampled at a monthly frequency. An asterisk denotes parameter estimates that are not significant at the 5% significance level. **Panel B:** Optimal hedging demands in percentages of the myopic portfolio are given for different investment horizons and relative risk aversion parameters. Each entry of the array of Panel B is a two-dimensional vector, the first component of which is the hedging demand for the S&P500 Index Futures, and the second one the hedging demand for the 30-year Treasury Futures.

| \(M\) \(Q\) \(\overline{\rho}\) \(\mu\) |
|---|---|---|---|
| -0.149 | 0.114* | 0.706 | 0.494* | 0.381 | 0.0616 |
| (0.074) | (0.081) | (0.34) | (0.312) | (0.161) | (0.008) |
| 0.070 | -0.112 | 0.806 | 0.641* | 0.392 | 0.0114 |
| (0.036) | (0.055) | (0.371) | (0.591) | (0.189) | (0.009) |
| **p-value for Hansen’s \(J\)-test** | 0.254 |

**Table VI**

**Panel B**

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Table VII
Optimal Hedging Demands in the Discrete-Time Model

Using standard numerical dynamic programming methods, we compute optimal hedging demands in percentages of the myopic portfolio for the exact discretization of the continuous-time model (1)-(4), for different investment horizons and relative risk aversion parameters. The parameters used to compute the exact discrete time transition density of the model are the monthly estimates in Table II. We compute optimal discrete-time hedging demands for a daily (d), a weekly (w) and a monthly (m) rebalancing frequency, and denote by $\pi_1$ and $\pi_2$ the hedging demands for the S&P500 Index and the 30-year Treasury bond Futures, respectively.

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</tr>
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<td>0.1365</td>
</tr>
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</table>
Table VIII  
Optimal Hedging Demands in the Discrete-Time Model with Short-Selling Constraints

Using standard numerical dynamic programming methods, we compute optimal hedging demands in percentages of the myopic portfolio for the exact discretization of the continuous-time model (11-12) when short-selling constraints are applied, for different investment horizons and relative risk aversion parameters. The parameters used to compute the exact discrete time transition density of the model are the monthly estimates in Table III and the rebalancing frequency is monthly. We denote by $\pi_1$ and $\pi_2$ the hedging demands for the S&P500 Index and the 30-year Treasury bond Futures, respectively, and distinguish the cases $u$, $c_1$ and $c_2$ corresponding to the unconstrained solution, the solution for a position limit of the form $\pi \geq -1$, and the solution in the short-selling constrained case ($\pi \geq 0$), respectively. Total hedging demands are decomposed in a correlation and a volatility hedging component by means of a cross-sectional regression of simulated hedging demands on the wealth-scaled ratios of simulated indirect marginal utilities of correlation and variances.

### Correlation Hedging

<table>
<thead>
<tr>
<th>RRA</th>
<th>$T$</th>
<th>3m</th>
<th>6m</th>
<th>1y</th>
<th>2y</th>
<th>3y</th>
<th>5y</th>
<th>7y</th>
<th>10y</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td>$u$</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$u$</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$u$</td>
<td>$c_1$</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>$u$</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$u$</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$u$</td>
<td>$c_1$</td>
</tr>
<tr>
<td>21</td>
<td></td>
<td>$u$</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$u$</td>
<td>$c_1$</td>
<td>$c_2$</td>
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</table>

### Volatility Hedging

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<th>6m</th>
<th>1y</th>
<th>2y</th>
<th>3y</th>
<th>5y</th>
<th>7y</th>
<th>10y</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td>$u$</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$u$</td>
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<td>$c_2$</td>
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<td>$c_2$</td>
<td>$u$</td>
<td>$c_1$</td>
</tr>
<tr>
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<td>$u$</td>
<td>$c_1$</td>
<td>$c_2$</td>
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<td>$c_1$</td>
<td>$c_2$</td>
<td>$u$</td>
<td>$c_1$</td>
</tr>
</tbody>
</table>
Table IX
Optimal Hedging Demands in the Discrete-Time Model with VaR constraints

Optimal VaR-constrained volatility and correlation hedging demands in percentages of the myopic portfolio for the exact discretization of the continuous-time model (1)-(4), as a function of different investment horizons and relative risk aversion parameters. The parameters used to compute the exact discrete time transition density of the model are the monthly estimates in Table II and the rebalancing frequency is monthly. As in Cuoco, He and Issaenko (2001), the VaR constraint is updated at each trading date, by imposing a constant upper bound on the 99%-VaR of next trading date wealth. Total hedging demands are decomposed in a correlation and a volatility hedging component by means of a cross-sectional regression of simulated hedging demands on the wealth-scaled ratios of simulated indirect marginal utilities of correlation and variances. Each entry of the two arrays in the table is a two-dimensional vector, the first component of which is the hedging demand for the S&P500 Index Futures and the second one the hedging demand for the 30-year Treasury Bond Futures.

<table>
<thead>
<tr>
<th>Volatility Hedging</th>
<th>Correlation Hedging</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T$ 3m 6m 1y 2y</td>
</tr>
<tr>
<td><strong>RRA</strong></td>
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<tr>
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<tr>
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</tr>
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</tr>
<tr>
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<td>0.023 0.032 0.047 0.062</td>
</tr>
</tbody>
</table>