Adverse Selection, Slow Moving Capital and Misallocation

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Adverse Selection, Slow Moving Capital and
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Abstract

Adverse selection is commonly used to explain trade inefficiencies in many markets. In this paper, we embed an informational asymmetry into a decentralized economy with heterogeneous capital and study its implications for aggregate dynamics in a general equilibrium environment. We show that the information friction leads to slow moving capital and persistent misallocation of resources. The model can help explain why economies recover slowly, even from shocks that do not affect potential output. It also provides a micro-foundation for convex adjustment costs, and a link between the underlying economic environment and the magnitude of adjustment costs.

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An important factor in determining aggregate productivity in an economy is the allocative efficiency of its resources. For example, there is growing consensus among economists that misallocation is large enough to explain a significant part of the TFP gap across rich and poor countries.\(^1\) Part of this persistence in misallocation has been linked to the failure of markets.\(^2\) Indeed, reallocating productive resources often requires a transaction between two parties. These resources are also naturally heterogenous in their quality and therefore, it is natural to suspect an information asymmetry between transacting parties as a potential culprit for misallocation. We model this consideration in a dynamic model of capital reallocation and show that adverse selection can lead to slow moving capital, persistent misallocation of resources and provides a micro-foundation for convex adjustment cost models.

The model consists of a decentralized two-sector dynamic economy in which sectoral productivity shifts arrive randomly and create a reason for reallocating capital from one sector to the other. Capital reallocation takes place in a competitive market; the ‘sellers’ are firms in the less productive sector who own capital and the ‘buyers’ are firms in the more productive sector who demand capital. In the absence of any frictions, capital is immediately reallocated to the more productive sector following a productivity shift.

We introduce an information asymmetry by allowing capital to vary in quality and firms to privately observe the quality of the capital they own and operate. Firms looking to purchase capital are at an information disadvantage and face an adverse selection problem (Akerlof, 1970). In a static environment, this friction can lead to a complete breakdown in the market for capital. Within our (dynamic) economy, the adverse selection problem translates into a slow moving reallocation process; capital moves gradually from the less productive sector to the more productive one. In particular, there is a unique separating equilibrium in which the capital quality is distinguished by the price at which it trades and the length of inefficient delay incurred. Because lower quality capital is less productive, firms with low quality capital are more anxious to sell their capital than firms with high quality capital. As a result, lower quality capital is reallocated more quickly, but at a lower price. Delays in reallocation generate real economic costs because capital, especially higher quality capital, continues to operate in the less productive sector following a productivity shift.

In order to isolate the implications of this mechanism, our analysis begins by assuming households are risk neutral households (hence the interest rate is fixed). We first examine the process of capital reallocation from the less productive to the more productive sector following a *permanent* shift. We derive a link between the production technology and the

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\(^1\)For instance, Hsieh and Klenow (2009) compare the dispersion in productivity between the US, India and China and find substantially lower dispersion in the US. Using a fairly general model, they argue that if the dispersion in TFP in India and China were equal to US levels, TFP would be 30-60% higher.

\(^2\)See, for example, Banerjee and Duflo (2005).
rate at which capital reallocation takes place. With this link, the model yields qualitative predictions about the reallocation dynamics. For example, the rate of reallocation increases over time when the factor inputs (i.e., quality and productivity) are complimentary, and decreases over time when inputs are substitutes. We relate these findings to commonly used convex adjustment cost specifications.

When shocks are transitory, the possibility of future productivity shifts affects the market value of capital today. Specifically, capital prices in the market depend not only on productivity, but also on the (in)efficiencies associated with capital reallocation. A firm that invests capital today accounts for the delays associated with reallocating the capital in the future. The anticipated possibility of misallocation leads to an endogenous discount in capital prices, which captures the social costs of misallocation. The discount varies with the severity of the adverse selection problem; as the dispersion of capital quality increases, the discount increases. Since higher quality capital takes longer to be reallocated, higher quality capital is associated with a larger discount. In equilibrium, the discount and the rate of reallocation are jointly determined. Perhaps surprisingly, this discount serves to speed up the reallocation process. Specifically, the presence of the discount makes capital values less sensitive to quality. This decreased sensitivity of price to quality serves to ameliorate the adverse selection problem and increases the rate of reallocation.

To understand the general equilibrium implications, we introduce households with CRRA utility. We illustrate how the results can be extended to this case. The primary additional consideration is that the stochastic discount factor (SDF) depends on the equilibrium strategies of firms, which in turns depend on the SDF. We then highlight several novel general equilibrium effects. First, the desire to smooth consumption increases the firms’ cost of delay and translates into faster reallocation. Second, the model predicts that large downturns are followed by fast recoveries whereas smaller negative shocks are followed by slower recoveries. It may be worth noting that both of these predictions are in contrast to convex adjustment cost models. Third, with risk averse households and transitory shocks, the rate of reallocation may reach zero prior to all capital being reallocated. Thus, the mechanism not only generates delays, it can halt the entire process.

We further extend the model to study new investment. We introduce entrepreneurs who have the ability to create new units of capital – projects or firms – upon the arrival of an investment opportunity. Entrepreneurs are heterogenous in ability: highly skilled entrepreneurs create projects of higher quality. Entrepreneurs have limited capacity, so in order to start a new firm they must first sell their existing projects, about which they are privately informed. This model generates delay in the response of the economy to investment opportunities (e.g., technological innovations) and gradual increases in the measured productivity of the new
sector. When entrepreneurs’ ability is sufficiently persistent across investment opportunities, aggregate measured productivity drops in response to innovations. This obtains because the first adopters of the new technology are the lower-ability entrepreneurs.

The dynamics implied by our model are qualitatively similar to those implied by models with convex adjustment costs, a standard feature in workhorse macroeconomic models. We consider several commonly used convex adjustment cost formulations and identify the conditions under which our model delivers dynamics consistent with each formulation. For example, adjustment cost specifications that penalize change in the fraction of capital stock – implying a declining rate of reallocation – are consistent with the model with adverse selection in which the economic gain from reallocation is decreasing in capital quality. Specifications that penalize the change in the rate of reallocation – implying an increasing rate of reallocation – are consistent with a model in which the economic gain from reallocation is increasing in capital quality. In summary, our model is flexible enough to generate dynamics that are consistent with several adjustment cost specifications. But, perhaps more importantly, the model provides a link between the underlying primitives of the economy and the corresponding adjustment cost specification.

Recent papers argue that time-variation in these adjustment costs may be important in explaining features of the data. For instance, Eisfeldt and Rampini (2006) argue that counter-cyclical reallocation costs are needed to reconcile the fact that reallocation activity is pro-cyclical, while the gains from reallocation – measured as the dispersion in productivity or Tobin’s Q – is counter-cyclical. Further, Justiniano, Primiceri, and Tambalotti (2011) estimate a medium-scale DSGE model and find that shocks to marginal adjustment costs account for a substantial fraction of business cycle fluctuations. By endogenizing the costs of reallocating capital to the economic environment, our paper allows us to interpret these shocks in the context of parameter shifts in our model. To illustrate this connection, we conduct impulse responses to shifts in three key parameters in our model: the dispersion in capital quality, the frequency of sectoral shocks and the level of the interest rate. When the dispersion of capital quality increases, the degree of adverse selection increases which reduces the allocative efficiency and therefore aggregate productivity and output. Perhaps surprisingly, a similar result obtains in response to reduction in the interest rate. Increasing the frequency of sectoral shocks can increase or decrease the rate of reallocation but leads to an unambiguous drop in aggregate output.

Clearly, our mechanism is not the only one that can generate misallocation. An alternative mechanism that delivers persistent misallocation is financial constraints. Financial constraints – typically based on moral hazard – prevent poor but potentially highly productive entrepreneurs from efficiently deploying capital (for a workhorse model, see Banerjee and Newman, 1993).
However, several pieces of evidence suggest it is unlikely that financial constraints are the only reason for persistent misallocation. The reason is that if the sole force preventing reallocation is a moral-hazard style borrowing constraint, high-type entrepreneurs quickly save out of their borrowing constraint.\footnote{First, \cite{Gilchrist2013} find that, for misallocation due to financial constraints to account for a significant fraction of measured TFP differentials across countries, the dispersion in borrowing costs has to be an order of magnitude higher than that observed in the data. Second, the quantitative performance of these models in generating sizable losses from reallocation is mixed \cite{Buera2011, Midrigan2010}. Along the same lines, \cite{Banerjee2010} argue that the persistence of misallocation – especially on the intensive margin – is puzzling.} Our theory provides an explanation for the persistence in misallocation based on adverse selection rather than financial constraints. Adverse selection may play a key role in preserving the large differences in productivity that have been documented even in countries where financial constraints may be less important, such as the US.\footnote{For instance, \cite{Syverson2004} reports that, within narrowly defined industries in the U.S., the ratio between the 90-th and the 10-th percentiles of the firm-level productivity distributions is approximately equal to two.} Other frictions, such as physical (convex) costs, search, learning, time-to-build and are also surely important components in the allocation of new and existing capital. We have abstracted away from these considerations in order to highlight the key ideas of the paper.

In financial economics, models with adverse selection are commonly used to study the sale of claims on firms’ capital (see, for instance \cite{Leland1977, Myers1984, Brennan1987, Lucas1990, Korajczyk1991}). Our paper contributes to a small but growing literature that introduces adverse selection into dynamic macro-finance models. The most closely related papers are \cite{Eisfeldt2004}, \cite{House2004} and \cite{Kurlat2013}. \cite{Eisfeldt2004} and \cite{House2004} study the problem of equity issuance and consumer’s choice of a durable good in an environment with adverse selection. Both papers find that increasing the variance of the underlying shock increases non-informational motives for trade and thus ameliorates the adverse selection problem. \cite{Kurlat2013} studies a setting in which entrepreneurs have private information that lasts for one period. He shows that this is mathematically equivalent to a tax on capital, which leads to an amplification mechanism in response to aggregate shocks. By contrast, the duration of the information asymmetry is endogenously determined in our model and our focus is on how the information friction affects both the duration and the magnitude of the effect on aggregate dynamics in response to a shock.

That adverse selection can generate delays in trade between buyers and sellers is, by now, well understood within the dynamic adverse selection literature. \cite{Janssen2002} derive a competitive equilibrium in which the price mechanism sorts sellers of different qualities into different (discrete) time periods. \cite{Horner2009} use a game-theoretic
approach to investigate the implications of public versus private offers in a discrete-time model. Daley and Green (2012) study trade dynamics in a continuous-time model in which information about the seller’s quality is revealed gradually. Formally, our modeling approach is most similar to Fuchs and Skrzypacz (2013), who study the costs and benefits of temporarily closing the market. Our contribution to this literature is to embed dynamic adverse selection into a macroeconomic model and the study general equilibrium effects and implications for aggregate dynamics.

The remainder of the paper is organized as follows. In Section 1, we illustrate how adverse selection generates slow movements in capital across sectors and describe the relation to various convex adjustment cost models. In Section 2, we embed the mechanism into a stationary model with transitory shifts. Section 3 analyzes equilibrium of the model with risk-neutral households. In Section 4, we study the general equilibrium effects on the aggregate dynamics. In Section 5, we explore the impulse response of output and productivity to a variety of shocks. Section 6 extends the model to study new investment. Section 7 discusses empirical implications and some preliminary evidence. Section 8 concludes. Proofs are located in the Appendix A.

1 A Motivating Example

To illustrate the main ideas in the paper, we start with a motivating example. Consider an economy with two productive sectors, \( i \in \{A, B\} \). This is a small open economy, so the interest rate is fixed at \( r \). There is a mass \( M > 1 \) firms in each sector. Firm cannot migrate across sectors. They are risk neutral, have an infinite horizon, and maximize total discounted profits, which includes the purchase or sale of any capital.

There is a unit mass of capital. Capital is heterogenous in its quality, also referred to as type and denoted by \( \theta \), which is distributed according to a uniform distribution with support \( \Theta = [\theta, \bar{\theta}] \subset \mathbb{R}_{++} \). Output of the capital stock depends on sector productivity \( z_i \) and capital quality. Quality is observable only to the firm who owns and operates the capital. If the firm does not have any capital, it remains idle and generates a constant output normalized to zero. For simplicity, we assume here that capital does not depreciate and there is no inflow of investment (the model in Section 2 incorporates such features).

A unit of capital of quality \( \theta \), henceforth a “\( \theta \)-unit,” operated by a firm in sector \( i \), generates a flow of output per unit time of

\[
\pi_i(\theta) = (\beta \theta^\alpha + (1 - \beta) z_i^\alpha)^{\frac{1}{\alpha}},
\]

where \( \beta \) captures the importance of capital quality in production, and \((1 - \alpha)^{-1}\) represents
the elasticity of substitution between capital quality and productivity.

We are interested in the process by which capital is reallocated from sector A to sector B. Therefore, assume that at $t = 0$, all capital is allocated to firms in sector A and that productivity is higher in sector B, $z_B > z_A$.\(^5\) Prior to analyzing the role of adverse selection, it is useful to establish two benchmarks: (1) the frictionless benchmark, and (2) a model with exogenously specified adjustment costs.

1.1 Benchmark 1: Frictionless Environment

In the absence of adjustment costs, a social planner would immediately reallocate all capital from sector A to sector B. In a decentralized economy, the same outcome obtains without the information friction. To see this, suppose that $\theta$ is perfectly observable and therefore prices can be conditioned on capital quality $\theta$.

At any point in time, a sector B firm is willing to pay up to $\pi_B(\theta)/r$ to buy a $\theta$-unit of capital. Since capital is scarce and sector B firms are identical and competitive, the price for a $\theta$-unit will get bid up to exactly this amount. Each sector A firm will sell at $t = 0$ at a price equal to the present value of the output the capital generates in sector B. Since there is no informational friction and there are gains from reallocation, all capital is immediately and efficiently reallocated.

1.2 Benchmark 2: Exogenous Adjustment costs

A second useful benchmark is the case in which capital is homogeneous (i.e., $\theta = \bar{\theta}$), but there are exogenous costs to reallocating capital. We consider three formulations for these adjustment costs that correspond to the cases most commonly used in the literature. In line with the literature on convex adjustment costs, we specify these costs as a function of the aggregate mass of capital being reallocated at a point in time and focus on the central planner’s problem. We denote by $k$ the capital stock in sector B.

The first formulation corresponds to the case where adjustment costs are convex in the rate of reallocation $\dot{k}$:

$$c(\dot{k}) = \frac{1}{2} c \left( \dot{k} \right)^2. \quad (2)$$

These costs are in line with the formulation in Abel (1983). We refer to this as the ‘kdot’ model. The second formulation is closely related to (2), except that it specifies the adjustment cost

\(^5\)Perhaps, due to a demand shock or recent technological innovation in sector B.
in terms of the growth rate of capital being reallocated

\[ c(k, \dot{k}) = \frac{1}{2} c \left( \frac{\dot{k}}{1 - k} \right)^2 (1 - k). \]  

(3)

This type of adjustment costs is commonly used in the literature studying investment and reallocation dynamics (Abel and Eberly, 1994; Eisfeldt and Rampini, 2006; Eberly and Wang, 2009). We refer to these costs as the ‘ik’ model.

The last adjustment cost formulation penalizes changes in the flow rate of reallocation \( \dot{k} \)

\[ c(\ddot{k}) = \frac{1}{2} c (\ddot{k})^2, \]

(4)

and is based on the adjustment costs proposed by Christiano, Eichenbaum, and Evans (2005). We refer to these costs as the ‘idot’ model.

We compare the reallocation dynamics across the three adjustment cost models in Figure 1. In terms of capital stock, the ‘kdot’ model implies a linear response of capital. By contrast, the ‘ik’ model generates strictly concave dynamics for the capital stock, whereas the ‘idot’ model generates S-shaped path for the capital stock. Relative to the first two formulations, the ‘idot’ model generates more delayed responses of capital flow to a sectoral productivity shift. The rate of capital reallocation in the ‘ik’ model spikes on impact and decays smoothly over time. By contrast, in the ‘idot’ model, the rate of capital reallocation increases slowly over time. This slow increase occurs because the formulation in (4) severely penalizes large adjustments to the rate. Christiano, Eichenbaum, and Evans (2005) argue that this feature is crucial in explaining the response of aggregate investment to shocks. In what follows, we show that adverse selection can generate dynamics similar to each of the convex adjustment costs models and derive the economic conditions under which each obtains.

1.3 Heterogeneous Capital and Adverse Selection

Having established two important benchmarks, here and from now on, capital is heterogeneous in quality \( (\theta < \bar{\theta}) \) and is privately observed by the firm who owns it. Further, suppose that \( \pi_A(\bar{\theta}) > \int \pi_B(\theta) dF(\theta) \), so that a firm with the highest quality capital in sector \( A \) would prefer to retain her capital rather than trade at the average value to firms in sector \( B \). To capture the effects of the informational friction, we study the competitive equilibrium of the decentralized economy in which reallocation decisions are made by firms.

In order for a unit of capital to be reallocated, a transaction must take place: a firm in sector \( B \) must purchase the capital from a firm in sector \( A \). This occurs in a dynamic market;
at every $t \geq 0$ a firm in sector $A$ who wishes to sell its unit of capital can trade with firms in sector $B$ who wish to purchase capital. There are no institutional frictions in the market (e.g., transactions costs or search). The only friction is an informational one. That is, buyers cannot observe the quality of capital in the market prior to purchasing it (or, alternatively, it is too costly to do so). Therefore, sector $B$ firms face a potential adverse selection problem in the market for capital.

A competitive equilibrium of this environment can be characterized by (1) a path of prices $P_t$, and (2) a time for each $\theta, \tau(\theta)$, at which a $\theta$-unit of capital is reallocated.\footnote{We allow for the possibility that certain types of capital are never reallocated, in which case $\tau(\theta) = \infty$.} We formalize our notion of equilibrium in Section 2 (see Definition 1). Roughly, it requires that (i) given the path of prices, sector $A$ firms with capital choose the optimal time to trade, (ii) firms in sector $B$ make zero expected profits and (iii) that the market for capital clears.

Since quality is unobservable, prices cannot be conditioned on $\theta$ and the first-best reallocation cannot be part of an equilibrium. To see why, suppose that all sector $A$ firms sell their capital at $t = 0$. For sector $B$ firms to break even requires that $P_0 = \frac{1}{r} \int \pi_B(\theta) dF(\theta)$. But given this price, a firm with capital of quality $\bar{\theta}$ in sector $A$ would prefer to retain her capital than trade it away to sector $B$ at the current market price. An alternative conjecture is that all firms with capital quality below some threshold $\theta^c \in (\underline{\theta}, \bar{\theta})$ trade at $t = 0$. In this case, the remaining capital in sector $A$ is of discretely higher quality and the equilibrium price would jump upward. Clearly then firms that sold capital at $t = 0$ would prefer to wait.

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**Figure 1:** Comparison across the ‘kdot’ (red dotted), ‘ik’ (black solid), and ‘idot’ (blue dashed) adjustment cost models. The left panel illustrates the capital quality that switches at time $t$, the right panel illustrates the rate at which capital is reallocated. See Appendix for details.
Having ruled out a mass of reallocation at date zero, we will show that the unique separating equilibrium involves smooth and gradual reallocation of capital; firms trade off the immediate gains from reallocation versus preserving the option to sell it in the future. Firms with lower quality capital are effectively more anxious to sell – since their capital is less productive – and do so sooner than firms with high quality capital. Since higher quality capital gets reallocated later, the market price of capital will gradually increase over time.

To construct this equilibrium, let $\chi_t$ denote the quality of capital that is reallocated at date $t$. In order for sector $B$ firms to break even, it must be that

$$P_t = \frac{\pi_B(\chi_t)}{r}.$$  \hspace{1cm} (5)

For this to be an optimal strategy, the firm who owns a $\chi_t$-unit of capital must be locally indifferent between trading immediately or waiting an instant for a higher price:

$$rP_t - \pi_A(\chi_t) = \frac{d}{dt}P(t).$$ \hspace{1cm} (6)

The left hand side of (6) corresponds to the cost that a firm with a $\chi_t$-unit in sector $A$ gives up by delaying trade. Using (5), the right hand side can be rewritten as:

$$\frac{d}{dt}P_t = \frac{\pi'_B(\chi_t)}{r} \dot{\chi}_t,$$ \hspace{1cm} (7)

where $\dot{\chi}_t = \frac{d\chi_t}{dt}$ represents the rate at which capital is reallocated to the more productive sector. Combining (6) and (7), we have that

$$\dot{\chi}_t = \frac{r}{\pi'_B(\chi_t)} \frac{[\pi_B(\chi_t) - \pi_A(\chi_t)]}{\pi_B(\chi_t)}$$ \hspace{1cm} (8)

This simple differential equation characterizes the equilibrium rate at which capital transitions to sector $B$. It is based on the first two equilibrium requirements, (i) that sector $A$ firms optimize their selling decisions and (ii) that sector $B$ firms break even. One immediate observation from (8) is that the rate of reallocation is proportional to the gains from doing so (i.e., $\pi_B - \pi_A$).

The boundary condition is pinned down by the market clearing condition, which requires the price at time zero to be at least $\pi_B(\theta)/r$. This implies that the lowest quality capital trades immediately

$$\chi_0 = \theta.$$ \hspace{1cm} (9)
For any set of production technologies, (8) and (9) pin down the equilibrium reallocation dynamics. The main takeaway is that adverse selection inhibits the reallocation of capital, resulting in a slow transition of resources to the more productive sector.

The equilibrium reallocation dynamics depends, in part, on the production technology and specifically, on the elasticity of substitution between capital quality and productivity. Using our CES formulation, we focus on three values for the elasticity, \( \alpha \in \{0, 1, 2\} \), the first two of which permit analytic solutions.

First, in the case \( \alpha = 1 \), the production technology is linear, and as a result, there are constant gains from reallocation. Equation (8) becomes

\[
\dot{\chi}_t = \left( \frac{1 - \beta}{\beta} \right) (z_B - z_A) r.
\]

Since the right hand side is a constant, the equilibrium reallocation rate is constant over time. Combining with (9), the solution is given by

\[
\chi_t = \theta + \left( \frac{1 - \beta}{\beta} \right) (z_B - z_A) rt.
\]

where the above holds for \( t \leq \tau(\bar{\theta}) \), where \( \tau \equiv \chi^{-1} \). At this point all capital has been reallocated to sector \( B \) and the transition dynamics terminate.

Second, as \( \alpha \to 0 \), the production technology tends to a Cobb-Douglas. In this case, the gains from reallocation are increasing with quality. Equation (8) becomes

\[
\dot{\chi}_t = \kappa \chi_t,
\]

where \( \kappa = \left( 1 - \left( \frac{z_A}{z_B} \right)^\beta \right) (1 - \beta)^{-1} r. \) Combining with (9), for \( t \leq \tau(\bar{\theta}) \), the solution is given by

\[
\chi_t = \theta e^{\kappa t},
\]

In this case, the equilibrium reallocation rate is increasing exponentially over time.\(^7\)

We plot the implied reallocation dynamics for the three cases in Figure 2. As we see in panel (a), the quality of capital that is reallocated increases over time in all three cases. This property is true regardless of the production technology; lower quality capital will reallocate sooner than higher quality capital for all specifications of the model. More importantly, panel (b) shows that the qualitative features of the equilibrium reallocation rate depend

\(^7\) Figure 2 also illustrates the case in which \( \alpha = 2 \). In this case, the differential equation does not admit an analytic solution, however, it is straightforward to compute it numerically.
Figure 2: Equilibrium reallocation with CES production technology for $\alpha = 0$ (blue dashed line), $\alpha = 1$ (black solid line) and $\alpha = 2$ (red dotted line). The left panel illustrates the capital quality that switches at time $t$, the right panel illustrates the rate at which capital is reallocated.

on the elasticity of substitution between factors. A quite striking similarity emerges when comparing Figure 2 to Figure 1. Specifically, that increasing gains from trade ($\alpha = 0$) generate an increasing rate of reallocation in line with the ‘idot’ models of adjustment costs, while decreasing gains from trade ($\alpha = 2$) generate a decreasing rate of reallocation in line with the ‘ik’ models of adjustment costs. When the gains from reallocation are constant ($\alpha = 0$), the dynamics match those of the ‘kdot’ model. The following result formalizes the findings illustrated in Figure 2:

Proposition 1.1. Until all capital has been reallocated to the efficient sector:

- If $\alpha < 1$, the equilibrium rate of reallocation is strictly increasing over time.
- If $\alpha > 1$, the equilibrium rate of reallocation is strictly decreasing over time.
- If $\alpha = 1$, the equilibrium rate of reallocation is constant over time.

In the next section, we generalize this dependance as well as the impact on prices (and subsequently $\dot{\chi}$) when firms anticipate costly reallocation in the future. We then analyze the implications for the aggregate dynamics of the economy in response to a variety of shocks in Section 5.

2 Stationary Model of Capital Reallocation

Our motivating example in the previous section considers a single transitionary period since reallocation occurs only once. Here, we allow productivity to vary across sectors and over
time in a stochastic manner. In this case, firms will internalize the possibility of costly future reallocation in their decisions. Further, the frequency of these shocks affect the equilibrium price of capital and, in turn, the reallocation dynamics.

Technology. Consumption goods are produced using capital. Capital can be located in one of two sectors (A and B). Capital is heterogenous in its quality, where quality is indexed by $\theta$. Quality is observable only to the owner of the capital unit; capital quality is distributed according to $F(\theta)$, which is continuous with strictly positive density over the support $\Theta = [\theta, \bar{\theta}]$. The flow output of a unit of capital depends on its quality $\theta$, the sector in which it is currently allocated $i$ and the aggregate state $x$ according to

$$y_i^t(\theta) = \pi_i(\theta, x)dt,$$

where $\pi_i$ is strictly positive, increasing and twice differentiable in $\theta$, with uniformly bounded first and second derivatives.\(^8\) We incorporate shocks to the model by allowing the production technology to vary stochastically over time. Specifically, we introduce a Markov switching process $X(\omega) = \{X_t(\omega), 0 \leq t \leq \infty\}$ defined on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $X_t(\omega) \in \{x_A, x_B\}$ represents the state of the economy at date $t$. Henceforth, we omit the argument $\omega$, and use a $t$ subscript as a place holder for the argument $(t, \omega)$. Existing capital depreciates at rate $\delta$.

Markets, Information and Prices. Reallocation of capital occurs in a competitive market; this market is open continuously at all $t \geq 0$. All firms observe the path of the exogenous state variable $X = \{X_s, 0 \leq s \leq \infty\}$. We let $\{\mathcal{F}_t\}_{t \geq 0}$ denote the filtration encoding the information observed by all firms prior to date $t$. In addition, a firm who currently owns a unit of capital privately observes its quality. The quality of each unit of capital is unobservable to all other firms. However, firms can observe to which sector the capital is currently allocated. For this reason, at each point in time $t$, there will be two prices in the market; one for capital currently located in sector A, denoted by $P_t^A$, and one for capital currently located in sector B, denoted by $P_t^B$.

Financial markets are complete with respect to the underlying probability space. In equilibrium, a complete financial market can be implemented with a risk-free asset and a market index. The state price density (the price of Arrow-Debreu securities per unit of probability) is given by

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \lambda_t d\tilde{X}_t,$$

\(^8\)Formally, there exists $a, A$ such that $0 < a < A < \infty$ and $\frac{\partial}{\partial q} \pi_i, \frac{\partial^2}{\partial q^2} \pi_i \in (a, A)$ for all $(i, q, x)$. Without imposing any structure on the distribution of capital quality, it is without loss to normalize either $\bar{\pi}$ or $\pi$. We have not done so here because at various points we will put additional structure on the production technology.
where \( r_t \) is the risk-free rate of return and \( \lambda_t \) is the price of risk associated with unexpected changes in the aggregate state \( (d\tilde{X}_t) \). Both are endogenously determined in equilibrium. We will require that the state price density satisfy the transversality condition that \( \lim_{t \to \infty} \xi_t = 0 \).

**Firms.** There exists a mass \( M > 1 \) of competitive firms located in each sector. Firms maximize their market value by undertaking a capital allocation decision. Consider a sector \( i \) firm who purchases a unit of capital at date \( t \). Upon doing so, the firm will observe the capital quality, \( \theta \), and operate the capital until it is no longer optimal to do so. The decision facing the firm is when to reallocate (i.e., sell) their existing capital. Let \( V^i_t(\theta) \) denote the firm’s value for the unit of capital. Given an \( (\mathcal{F}_t\text{-adapted}) \) price process, \( P^i_t \), the firm’s problem can be written as

\[
V^i_t(\theta) = \sup_{\tau \geq t} E_t \left[ \frac{1}{\xi_t} \int_t^\tau e^{-\delta(s-t)} \xi_s \pi_i(\theta, X_s) ds + e^{-\delta(\tau-t)} \xi_\tau P^i_\tau \right].
\] (10)

Last, there is a mass \( \delta \, dt \) of new firms created each period. New firms optimally choose in which sector to operate.

**Households.** There exist a continuum of identical households, indexed by \( h \in [0, 1] \). The households problem is to choose a consumption process, \( c^h_t = \{c^h_t : 0 \leq t \leq \infty\} \), that maximizes their lifetime utility,

\[
\sup_c E_0 \left[ \int_0^\infty e^{-\beta t} u(c_t) dt \right],
\] (11)

subject to the budget constraint,

\[
w_0 \geq E_0 \left[ \int_0^\infty \xi_t c_t dt \right].
\] (12)

Here, \( \beta > 0 \) is their rate of time preference and \( W_0 \) is the value of their initial endowment. We assume that \( u \) is a “smooth” (weakly) concave function. We will focus on the case of risk-neutral households in Section 3. In Section 4, we incorporate risk aversion.

**Equilibrium Concept.** To rigorously define an equilibrium of the economy, we will need the following notation and definitions. Aggregate consumption is denoted by \( C_t = \int c^h_t dh \). \( T^i_t(\theta) \) denotes the policy of a firm in sector \( i \) who acquires a unit of capital of quality \( \theta \) at time \( t \). The policy is admissible if it is both adapted to the filtration \( \{\mathcal{F}_s\}_{s \geq 0} \) and weakly larger than \( t \). \( \Theta^i_t \equiv \{\theta : T^i_s(\theta) = t, s \leq t\} \) denotes the set of capital qualities sold at date \( t \) from sector \( i \). Finally, \( F^i_t \) denotes the distribution of capital quality and \( \theta^i_t \equiv \inf\{\theta : T^i_s(\theta) \geq t, s \leq t\} \) denotes the lowest quality of capital allocated to sector \( i \) at date \( t \).
Definition 1. A competitive equilibrium of the decentralized economy consists of admissible policies, \( T_i^t(\theta) : \Omega \to \mathbb{R}_+ \) and \( \mathcal{F} \)-adapted consumption, price and state density processes \( c^h, P^i, \xi : [0, \infty] \times \Omega \to \mathbb{R} \) such that for each \( i \in \{A, B\}, t \geq 0, \theta \in \Theta, j \neq i, h \in [0, 1] \):

1. Firm’s capital allocation decision is optimal: \( T_i^t(\theta) \) solves (10).

2. Household’s consumption process is optimal: \( c^h \) solves (11) subject to (12).

3. The market for the consumption good clears: \( C_t = Y_t \equiv \sum_i \int \pi_i(\theta, x) dF_i^t(\theta) \).

4. The market for capital clears: if \( \Theta_i^t = \emptyset \), \( P_i^t \geq \inf \{ V_j^t(\theta) : \theta \geq \theta_i^t \} \).

5. New firms make zero profit: if \( \Theta_i^t \neq \emptyset \) then \( P_i^t = \mathbb{E} [ V_j^t(\theta) | \theta \in \Theta_i^t, \mathcal{F}_t ] \).

Conditions 1-3 are straightforward. Condition 4 requires that the price for a unit of capital in sector \( i \) cannot be less than the lowest possible value for that unit of capital in sector \( j \). If the price was strictly less, then all firms in sector \( j \) would demand capital at that price and demand would exceed supply.\(^9\) Condition 5 is motivated by free entry and says that the price of capital at time \( t \) must be equal to the expected value of the reallocated capital at time \( t \), which implies a firm who purchases a unit of capital cannot make positive (or negative) expected profits.

Preliminary analysis of the model leads to several standard, but useful, properties.

Lemma 2.1. In any competitive equilibrium, the state-price density is given by \( \xi_t \propto e^{-\beta t} u'(C_t) \).

In addition, the skimming property must hold. That is, lower quality capital is reallocated sooner than higher quality capital.

Lemma 2.2 (Skimming). In any competitive equilibrium, \( T_i^t(\theta) \) is weakly increasing in \( \theta \).

The intuition is the same as in the motivating example; firms with lower quality capital are more anxious to sell their capital, because their outside option to wait is less valuable due to lower output in the interim.

For both tractability and ease of exposition, we conduct our analysis within the class of symmetric economies. In a symmetric economy, the output of a firm depends only on the quality of its capital and whether that capital is allocated efficiently (i.e., to the more productive sector given the current state).

\(^9\)Besides having a natural economic interpretation, this condition rules out trivial candidate equilibria, such as one in which prices are always very low and trade never takes place.
Definition 2 (Symmetric economies). The economy is symmetric if there exists a pair of functions \( \{\bar{\pi}, \bar{\pi}\} \) and scalar \( \lambda \) such that \( \pi_i(\theta, x_i) = \bar{\pi}(\theta) \) for \( i \in \{A, B\} \), \( \pi_i(\theta, x_j) = \bar{\pi}(\theta) \), and \( \lambda_{ij} = \lambda \) for \( i \neq j \).

For the remainder of the paper, we will restrict attention to symmetric economies. It is straightforward, though more notationally cumbersome, to extend results to a setting in which the economy is not symmetric. A symmetric economy is fully described by \( \Gamma \equiv \{\bar{\pi}, \bar{\pi}, u, \beta, \delta, \lambda, F\} \). We refer to the production technology as a pair of functions \( \{\pi, \bar{\pi}\} : \Theta \rightarrow \mathbb{R} \).

Assumption 2.3 (Gains from trade). The production technology satisfies \( \bar{\pi}(\theta) > \pi(\theta) \) for all \( \theta < \theta \).

We refer to the efficient sector at any given time \( t \) as the sector in which output is given by \( \bar{\pi} \) at date \( t \) (i.e., \( i \) such that \( X_t = x_i \)).

Remark 2.4 (Human Capital). The model can alternatively be interpreted as being about the (re)allocation of human capital. Relabel ‘capital’ as ‘workers’, ‘quality’ as ‘ability’ and ‘prices’ as ‘wages’. Workers are privately informed of their ability \( \theta \). Rather than the firms decision of when to sell its capital, it becomes the worker’s decision of when to migrate to sector \( B \). Firms from sector \( B \) do not observe workers ability, but compete for workers from sector \( A \) through the timing and the wage they offer.\(^\text{10}\)

3 Equilibrium with Risk-Neutral Households

We begin by focusing on the setting with risk neutral households, \( u(c) = c \). In this case, \( \xi_t = e^{-\beta t} \) (Lemma 2.1) and the short-term interest rate is simply equal to household’s impatience, \( \tau_t = \beta \). With the state-price density pinned down, the natural extension of the equilibrium from Section 1 can be characterized by two functions. The first is \( \tau(\theta) \), which represents how long it takes a \( \theta \)-unit of capital to be reallocated following a productivity shock (and provided that no other shocks arrive in the interim). The second is \( \bar{V}(\theta) \), which is the (endogenous) value of efficiently allocated capital as it depends on \( \theta \). As in Section 1, we will construct a fully separating equilibrium, which requires that \( \tau \) is strictly increasing in \( \theta \). Here again, it will sometimes be easier to use the inverse of \( \tau \), which we denote by \( \chi_t \equiv \tau^{-1}(t) \).

To formalize the connection to the equilibrium objects in Definition 1, let \( m_t \equiv t - \sup\{s \leq t : x_{s+} \neq x_{s-}\} \) denote the amount of time that has elapsed since the last shock arrived.

\(^{10}\)In the context of technological progress, the elasticity of substitution between worker quality and productivity \( (1 - \alpha)^{-1} \) can be interpreted as the technology being skill biased (\( \alpha < 1 \)) or ‘unskill’-biased (\( \alpha > 1 \)). We discuss the model’s implications for this alternative interpretation in Section 7.
Definition 3. The firm strategies and capital prices that are consistent with \((\tau, \bar{V})\) are given by:

\[
T^i_t(\theta) = \inf \{ s \geq t : m_s = \tau(\theta), x_s \neq x_i \} \tag{13}
\]

\[
P^i_t = \begin{cases} 
\bar{V}(\chi(m_t)) & \text{if } x_t \neq x_i \text{ and } m_t < \tau(\theta) \\
\bar{V}(\theta) & \text{otherwise}
\end{cases} \tag{14}
\]

The main result of this section is the following.

Theorem 3.1. In a symmetric economy with risk-neutral households and strict gains, there exists a unique \((\tau^*, V^*)\) such that the firm strategies and capital prices consistent with \((\tau^*, V^*)\) are part of a fully-separating competitive equilibrium.

To sketch the argument, we proceed with a heuristic construction of the equilibrium based on necessary conditions, which can be reduced to a single initial value problem. This initial value problem can be shown to have a unique solution, which proves that a unique candidate exists. We then verify that these necessary conditions are also sufficient.

According to the candidate equilibrium, the value a firm derives from capital depends only on its quality if it is efficiently allocated. If it is inefficiently allocated, the value derived also depends the lowest quality of capital remaining in the inefficient sector (or equivalently, \(m_t\)). Let \(V(\theta, \chi)\) denote the value of an inefficiently allocated \(\theta\)-unit when the lowest remaining quality of capital in the inefficient sector is \(\chi \leq \theta\). According to \((\tau, \bar{V})\), the firm waits until \(\chi = \theta\) to trade. Therefore, the evolution of \(V\) for \(\chi < \theta\) is given by

\[
\beta V(\theta, \chi) = \bar{V}(\theta) - \delta V(\theta, \chi) + \lambda (\bar{V}(\theta) - V(\theta, \chi)) + \frac{\partial}{\partial \chi} V(\theta, \chi) \dot{\chi}_t \tag{15}
\]

When \(\theta = \chi\), a firm with a misallocated \(\theta\)-unit sells at a price equal to \(V(\chi)\). We abuse notation by letting \(P(\theta)\) denote the price at which a firm in the inefficient sector sells a \(\theta\)-unit to a firm in the efficient sector. Hence, a necessary boundary condition for \(V\) is given by

\[
V(\theta, \theta) = P(\theta). \tag{16}
\]

The (local) optimality condition—required to ensure that firm optimality holds—is that when \(\theta = \chi\), the firm with a \(\theta\)-unit is just indifferent between selling immediately and waiting an “instant”. In other words, the firm’s value function must smoothly paste to the path of prices.

\[
P'(\chi) = \frac{\partial}{\partial \chi} V(\theta, \chi) \bigg|_{\theta=\chi} \tag{17}
\]
In order for the zero profit condition to hold, the price at which capital transacts must be equal to its value in the efficient sector. This requires that

\[ P(\theta) = \bar{V}(\theta). \]  

(18)

Evaluating (15) at \( \theta = \chi_t \) using (16)-(18), we arrive at

\[ \dot{\chi}_t = \frac{\rho \bar{V}(\chi_t) - \bar{\pi}(\chi_t)}{\bar{V}'(\chi_t)}, \]  

(19)

where \( \rho = \beta + \delta \), represents the firm’s effective discount rate. Note that (19) is analogous to (8), where \( \pi_B/r \) is replaced with \( \bar{V} \). It is also worth noting that the rate at which productivity shocks arrive, \( \lambda \), does not enter directly into (19). This is because the price the firm gets upon selling capital is equal to the value of that capital if another shock were to arrive (in which case the firm would retain possession). Nevertheless, \( \lambda \) does play an important role in determining the equilibrium capital values and prices.

**Equilibrium Value of Capital**

Consider an arbitrary \( (\tau, V) \) and note that the value of a unit of inefficiently allocated capital when \( \chi = \theta \) can be written as

\[ V(\theta, \theta) = f(\tau(\theta)) \frac{\bar{\pi}(\theta)}{\rho} + (1 - f(\tau(\theta))) \bar{V}(\theta), \]  

(20)

where

\[ f(\tau) \equiv \int_0^\tau (1 - e^{-\rho t}) \lambda e^{-\lambda t} dt + e^{-\lambda \tau}(1 - e^{-\rho \tau}), \]  

(21)

denotes the expected discount factor until either (i) the state switches back, or (ii) the capital gets reallocated to the other sector. Similarly, for an arbitrary \( \bar{V}(\theta, \theta) \), the value of an efficiently allocated \( \theta \)-unit is given by

\[ \bar{V}(\theta) = \frac{\rho}{\rho + \lambda} \bar{\pi}(\theta) + \frac{\lambda}{\rho + \lambda} \bar{V}(\theta, \theta). \]  

(22)

Solving (20) and (22) jointly, we arrive at

\[ \bar{V}(\theta) = g(\tau(\theta)) \frac{\bar{\pi}(\theta)}{\rho} + (1 - g(\tau(\theta))) \frac{\bar{\pi}(\theta)}{\rho}, \]  

(23)

where \( g(\tau) \equiv \frac{\lambda}{\rho + \lambda f(\tau)} f(\tau) \). The expression in (23) has an intuitive form. Capital spends some fraction of the time allocated efficiently and some fraction of the time misallocated.
Therefore, its value is simply a weighted average of the value were it to be permanently efficiently allocated (i.e., $\bar{\pi}$) and permanently misallocated (i.e., $\bar{\pi}$). The amount the time it takes to get reallocated is determined by (19), which in turn depends on $V$; this illuminates the nature of the fixed point. The solution turns out to be quite tractable. By substituting $\chi_t$ for $\theta$ into (23) and substituting back into (19), we arrive at

$$\dot{\chi}_t = \frac{r \left(1 - g(t) + \frac{\dot{g}(t)}{r}\right)(\bar{\pi}(\chi) - \pi(\chi))}{g(t)\pi'(\chi) + (1 - g(t))\bar{\pi}'(\chi)}.$$

As before, the boundary condition is pinned down by the fact that in any separating equilibrium, the lowest type must reallocate immediately after the productivity shock and therefore

$$\chi_0 = \hat{\theta}.$$  

The regularity conditions imposed on $\bar{\pi}$ and $\pi$ ensure a unique solution exists and that this solution is monotonically increasing (see Lemma A.1). The last step in the proof of Theorem 3.1 is to verify that the candidate satisfies the remaining equilibrium conditions. The zero profit condition follows from the fact that the equilibrium is fully separating and capital of quality $\theta$ trades at a price of $V(\theta)$. Capital market clearing follows immediately from (14) and that $V(\theta)$ is equal to the value derived from a $\theta$-unit. Finally, in the appendix, we demonstrate that a firm who owns capital does not have a profitable deviation by showing that the Spence-Mirlees condition holds for firms’ objective function, which verifies firm optimality.

**Remark 3.2 (Market breakdown).** Equation (24) illustrates the importance of having strict gains from reallocating capital (Assumption 2.3) as it ensures that the numerator is strictly positive and thus $\chi$ and (hence $\tau$) are strictly increasing. On the other hand, if $\pi(\bar{\theta}) \geq \pi(\underline{\theta})$ over some interval of $\Theta$, then in equilibrium, the market for used capital would breakdown completely and the reallocation process would get “stuck”; capital with quality in and above the interval would never be reallocated. Whether the reallocation process from one sector to another is completed in finite time also depends the gains from trade at the upper end of the distribution; if $\pi(\bar{\theta}) > \pi(\underline{\theta})$, then all capital gets reallocated in finite time, whereas if $\pi(\bar{\theta}) = \pi(\underline{\theta})$ then $\tau(\bar{\theta}) = \infty$.

### 3.1 Reallocation following a Permanent Shift

A special case of the model is when the productivity shift is *permanent*. To study the transition dynamics for this case, let $\lambda = 0$, assume that all capital is originally allocated to sector $A$, and the productivity shift occurs at $t = 0$ so that $B$ is the more productive sector.
for all $t \geq 0$.

This situation is effectively the same as that in Section 1: because sector $B$ is more productive, capital will transition from $A$ to $B$; due to adverse selection, the reallocation process occurs slowly over time. Since there are no further technological shocks, firms in sector $B$ retain the capital until it fully depreciates. Hence, firms have a value $\bar{\pi}(\theta)/\rho$ for a $\theta$-unit of capital. Thus, in any separating equilibrium, the rate at which at $\theta$-unit of capital is reallocated does not impact the price at which it trades.

**Proposition 3.3.** Suppose that the productivity shift is permanent. Then, $g(t) = 0$ for all $t$ and (24) reduces to

$$\dot{x}_t = \rho \frac{\bar{\pi}(\chi_t) - \pi(\chi_t)}{\pi'(\chi_t)}. \quad (26)$$

As expected, the expression for $\dot{x}$ in (26) is effectively the same as (8) in the example. Therefore, the unique separating equilibrium in the case of permanent productivity shifts is precisely the one characterized in Section 1. We revisit it here because it is useful for highlighting the economic environments under which various patterns in the rate of reallocation obtain. The numerator in (26) measures the magnitude of the productivity gains from reallocation as they depend on the quality of the capital; the larger the benefit of reallocation, the faster it takes place. The denominator measures the marginal productivity of capital quality in the efficient sector. It is perhaps surprising that higher marginal productivity of quality leads to slower reallocation. The intuition for this comes from the indifference condition of the cutoff type. Recall that the total change in prices with respect to time is given by

$$dP_t = \frac{\bar{\pi}'(\chi)}{\rho} \cdot \dot{x}_t dt.$$

Fixing $\dot{x}_t$, increasing the marginal productivity of capital quality increases the rate at which prices increase over time. In order for the cutoff type to remain indifferent, the reallocation rate must decrease. Using (26) and noting that $\dot{x}_t$ is strictly positive, we have the following result.

**Proposition 3.4.** Suppose that the productivity shift is permanent. Then, the equilibrium rate of reallocation will increase (decrease) over time until all capital has been reallocated (i.e., $t \in (0, \tau(\bar{\theta}))$) if and only if $(\bar{\pi} - \pi)/\bar{\pi}'$ is increasing (decreasing) over $\theta \in \Theta$.

### 3.2 Reallocation with Transitory Shifts

When the sectoral productivity shift is transitory, firms investing in capital today will become sellers of capital at some point in the future. Therefore, in considering their willingness to pay for a $\theta$-unit of capital, firms must account for the potential costs associated with reallocation,
which are then incorporated into equilibrium prices. Recall that when the shift is permanent, the price at which a $\theta$-unit trades is equal $\bar{\pi}(\theta)/\rho$, which is the present value of the future output that it generates for a firm in the efficient sector. With transitory shifts, this is no longer the case.

**Proposition 3.5.** If the productivity shift is transitory ($\lambda > 0$), the price at which a $\theta$-unit of capital trades is strictly less than $\bar{\pi}(\theta)/\rho$ for all $\theta > \bar{\theta}$.

Capital trades at a discount relative to the value that it could generate were it always efficiently allocated. The discount is endogenously determined by the degree of adverse selection. One way to see this is through a comparative static on the dispersion of capital quality, as measured by an increase in the support of capital quality. Increasing the support of capital quality increases buyer’s uncertainty about capital quality and hence the severity of the adverse selection problem. Reallocation will then be slower and more costly leading to a larger discount. To measure the discount, we compute the difference between the full information price and the price at which capital sells when firms are privately informed,

$$\text{Discount}(\theta) = \frac{\bar{\pi}(\theta)}{\rho} - \bar{V}(\theta).$$

Figure 3 illustrates the discount as it depends on $\theta$ and the dispersion of capital quality.

**Figure 3:** The effect of capital quality dispersion on the discount. The solid line represents an economy with higher dispersion ($\Theta = [0.3, 1]$) whereas the dashed line involves lower dispersion ($\Theta = [0.5, 1]$). The figure uses CES production technology with $\alpha = 1$.

Naturally, the discount affects the price at which a firm can sell its capital and affects its eagerness to do so. Therefore, we now explore how changes in the frequency of sectoral shifts, $\lambda$, impacts the rate of reallocation. It may be intuitive to think that firms will have less
incentive to reallocate because the shock is only temporary and hence the rate of reallocation should decrease with $\lambda$. As the next result demonstrates, this intuition is not entirely correct.

**Proposition 3.6.** Consider any two symmetric economies $\Gamma_x$ and $\Gamma_y$, which are identical except that $\lambda_x < \lambda_y$. There exists a $\bar{t} > 0$ such that the rate of reallocation is strictly higher in $\Gamma_y$ than in $\Gamma_x$ prior to $\bar{t}$, i.e., $\chi_y'(t) > \chi_x'(t)$ for all $t \in [0, \bar{t}]$.

We should note that this behavior is in stark contrast to what one would expect in a standard model with exogenous reallocation costs, in which increasing the volatility of sectorial shocks typically leads to more delay as the option value of waiting increases. In our model, increasing volatility leads to an increase in the rate of reallocation for at least some fraction (and perhaps all) of firms and can increase the rate of reallocation for all firms.

To see the intuition behind this result, recall that the lowest-quality capital is always efficiently allocated and therefore $\rho V(\theta) = \pi_1(\theta)$ in both $\Gamma_x$ and $\Gamma_y$. Fixing the equilibrium strategies from $\Gamma_x$, consider the effect of an increase from $\lambda_x$ to $\lambda_y$. Since the state is now switching more frequently and the rate of reallocation remains unchanged, firms with capital of quality $\theta > \bar{\theta}$ endure more misallocation, which gives them more incentive to imitate the lowest type. Now recall that by construction, types arbitrarily close to $\bar{\theta}$ were indifferent in $\Gamma_x$ between accepting $P(\theta)$ or waiting an instant. Hence, the increase in $\lambda$ will cause these types to strictly prefer to imitate $\bar{\theta}$. To restore the equilibrium in $\Gamma_y$, types near $\bar{\theta}$ must trade faster and the reallocation of capital increases, as we see in Figure 5.

![Figure 4:](image)

**Figure 4:** The effect of transitory shocks on the price of capital. The dashed blue corresponds to $\lambda = 0.1$ and the dotted red lines corresponds $\lambda = 1$. The black line represents the case when the shock is permanent ($\lambda = 0$), which also corresponds to the fully efficient value of capital. The fainter blue (red) dotted lines represent the hypothetical value of a unit of capital if it is never reallocated for $\lambda = 0.1$ ($\lambda = 1$), which approaches for $\lambda = 1$ and $\theta$ large. The figure uses CES production technology with $\alpha = 1$.  

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This behavior implies that equilibrium prices become less sensitive to capital quality for $\theta$ near $\bar{\theta}$. As a result, there is endogenously less adverse selection at the bottom of the distribution. Despite this decrease in the level of adverse selection, since the frequency with which costly reallocation occurs increases, the overall efficiency decreases leading to lower equilibrium prices as illustrated in Figure 4.

Finally, using the CES technology, we study the interaction of the complementary of quality and productivity of the sector with the persistence of the shocks. Figure 5 illustrates how the transitory nature of shocks affects the reallocation dynamics. In particular, the transitory nature of shocks tends to make $\dot{\chi}_t$, decreasing offsetting the effects of complementarity between quality and productivity ($\alpha = 0$) which generates an increasing rate of reallocation when $\lambda = 0$. Another important consideration which will shape the aggregate dynamics is the distribution of capital quality, which we explore next.

![Figure 5: Equilibrium reallocation with transitory shocks and CES production technology for $\alpha = 1$ (left) and $\alpha = 0$ (right). The other parameters used are $\beta = 0.45$, $r = 0.15$, $z_A = \frac{1}{2}$, $z_B = 1$, $\Theta = [0.5, 1]$.](image)

### 3.3 Reallocation and the distribution of capital quality

Maintaining the CES production technology, we now consider how the distribution of capital affects the predictions.$^{11}$ In general, the total measure of capital allocated in sector $i$ equals

$$k_t^i = \int dF_t^i(\theta), \tag{27}$$

where $F_t^i(\theta)$ is the cumulative distribution of capital quality in sector $i$ at time $t$. Hence, given the equilibrium rate of skimming through types, $\chi_t$, the rate of capital reallocation

$^{11}$Note that it would be without loss to normalize the distribution for arbitrary production technologies.
from sector $i$ to sector $j$ equals
\[ \frac{dk^i(t)}{dt} = \dot{\chi}_t dF^i(\chi_t). \] (28)

Holding constant the rate at which types transition across sectors, varying the distribution of capital quality leads to different reallocation dynamics. For instance, in the case where the rate of type transition $\dot{\chi}_t$ is a constant, the dynamics of capital reallocation are driven entirely by the shape of $F$. In the special case of uniform quality, $dF^i(\theta) = (\bar{\theta} - \theta)^{-1}$, the capital reallocated to sector $B$ is proportional to $\chi_t$, and the rate of capital reallocation is equal to $\dot{\chi}_t$.

We compare the dynamics implied by the uniform distribution for $F(\theta)$ in $[\theta, \bar{\theta}]$ to those of a beta distribution with the same support. We consider three parameterizations of the beta distribution: a right-skewed $(2,1)$, a symmetric $(2,2)$, and a left-skewed $(1,2)$ version. To illustrate the effect of the distribution of quality on the dynamics of capital reallocation, we focus on the case of constant gains from trade, $\alpha = 1$.

\begin{figure}[h]
  \centering
  \begin{subfigure}{0.45\textwidth}
    \centering
    \includegraphics[width=\textwidth]{fraction.png}
    \caption{Fraction of capital reallocated}
  \end{subfigure}\hspace{0.5cm}
  \begin{subfigure}{0.45\textwidth}
    \centering
    \includegraphics[width=\textwidth]{rate.png}
    \caption{Rate of capital reallocation}
  \end{subfigure}
  \caption{Comparison of capital reallocation dynamics across different distributions of capital quality, $F$. The solid black line refers to the case of uniform distribution of quality. The dashed blue line corresponds to a beta $(2,2)$; the dotted red line refers to a beta $(2,1)$; the thick gray line corresponds to a beta $(1,2)$. The figure uses constant gains from trade $\alpha = 1$ and transitory shocks $\lambda = 1/10$.}
\end{figure}

The dynamics of capital reallocation inherit the dynamics of $\chi_t$ along with the shape of the distribution as shown in Figure 6. Comparing Figure 6 to Figure 1, we see a striking qualitative similarity between the ‘ik’ and ‘idot’ adjustment cost models and the beta $(1,2)$ and beta $(2,1)$ cases respectively. When the distribution of quality is hump shaped (as is the case quality follows a beta $(2,2)$, the rate of reallocation is hump-shaped and the dynamics of capital have an $S$-shape.
4 Risk Averse Households

To this point, we have ignored general equilibrium effects on the interest rate by focusing on a setting with risk-neutral households. Let us now suppose that households exhibit CRRA utility: \( u(c) = \frac{c^{1-\gamma}}{1-\gamma} \). From Lemma 2.1, this implies that the state price density is given by

\[ \xi_t \propto \exp(-\beta t)C_t^{-\gamma}. \]

The crucial difference here is that the SDF (and hence the equilibrium reallocation dynamics) will depend on total output and therefore the distribution of capital. In contrast, with risk-neutral households, \( \dot{\chi}_t \) is independent of the distribution of capital.

We will start by studying how the desire to smooth consumption over time affects reallocation dynamics in response to a permanent productivity shift. We illustrate how our results from the previous sections can be extended and highlight two novel general equilibrium effects. First, the desire to smooth consumption increases the cost of delay and translates into faster reallocation. Second, the model predicts that large downturns are followed by fast recoveries whereas smaller negative shocks are followed by slower recoveries. Both of these predictions are in contrast to convex adjustment cost models in which the opposite prediction obtains.

We then re-incorporate aggregate risk into the economy with multiple transitory shifts. With sufficiently risk averse households, some capital remains misallocated despite the fact that output would increase by reallocating it. That is, the rate of reallocation reaches zero prior to all capital being reallocated. The intuition is that misallocated capital can serve as a hedge against a subsequent productivity shift. Thus, informational frictions not only generates delays in reallocation but can halt the reallocation process entirely.

4.1 Permanent Shift

Suppose that at \( t < 0 \), both sectors are equally productive. At \( t = 0 \), a productivity shift arrives that makes sector \( B \) relatively more productive (\( \pi_B > \pi_A \)). Capital will then gradually flow from sector \( A \) to sector \( B \). Our interest will be in characterizing the equilibrium rate of reallocation and how it depends on \( \gamma \) as well as the initial distribution of capital across sectors, which we allow to be arbitrarily distributed according to smooth, strictly positive density functions \( f_A, f_B \) over \( \Theta \). For simplicity ignore both depreciation and new investment by setting setting \( \delta = 0 \). The primary additional consideration here is that \( \xi_t \) depends on output dynamics and hence the rate of reallocation, \( \dot{\chi}_t \). Therefore, the equilibrium value of capital will be determined endogenously. In this way, the effect is similar to the case with risk-neutral households and transitory shifts. However, here the mechanism works through
the discount rate whereas with transitory shifts, the endogeneity worked through the cash flow channel.

To see this, recall that $\chi_t$ denotes the lowest quality capital allocated to sector $A$ at time $t$ and therefore aggregate output and consumption can be written as

$$C_t = Y_t = \int_{\chi_t}^{\bar{\chi}} \pi_A(\theta) f^A(\theta) d\theta + \int_{\bar{\chi}}^{\chi_t} \pi_B(\theta) f^B(\theta) d\theta.$$  

Hence consumption grows according to

$$dC_t = (\pi_B(\chi_t) - \pi_A(\chi_t)) f^A(\chi) \dot{\chi}_t dt,$$

and thus $\dot{\chi}_t$ enters into the evolution of $\xi_t$

$$\frac{d\xi_t}{\xi_t} = - (\beta + \gamma C_t^{-1}(\pi_B(\chi_t) - \pi_A(\chi_t)) f^A(\chi) \dot{\chi}_t dt,$$

and leads to a short-term interest rate that is given by

$$r(\chi_t) = \beta + \gamma C_t^{-1}(\pi_B(\chi_t) - \pi_A(\chi_t)) f^A(\chi) \dot{\chi}_t.$$  (29)

The value of an efficiently allocated $\theta$-unit of capital at time $t$ (i.e., in state $\chi_t$) can be written as

$$V(\theta, \chi_t) = \nu(\chi_t) \pi_B(\theta),$$  (30)

where $\nu(\chi_t)$ is simply the price of an perpetuity at time $t$. Using standard arguments, $\nu$ satisfies

$$r(\chi_t) \nu(\chi_t) = 1 + \dot{\chi}_t \nu'(\chi_t), \quad \nu(\bar{\theta}) = \beta^{-1}$$  (31)

and the zero-profit condition requires that

$$P_t = \nu(\chi_t) \pi_B(\chi_t).$$  (32)

For a fixed $\dot{\chi}_t$, we have now fully characterized the equilibrium price. The next phase of the analysis follows closely that in Section 3.1. That is, we take the price as given and derive necessary conditions on $\dot{\chi}_t$. Analogous to (17), the optimality conditions for firms requires their value function smoothly pastes to prices. Letting $V(\theta, \chi_t)$ denote the value of
an inefficiently allocated unit of capital, the (local) optimality condition requires that

$$\left. \frac{d}{dt} V(\theta, \chi_t) \right|_{\theta=\chi_t} = \frac{d}{dt} P_t$$

(33)

and value matching requires that

$$V(\chi_t, \chi_t) = P_t.$$  

(34)

Using the law of motion for $\bar{V}$ and $\bar{V}$ along with (32)-(34), one arrives at

$$\dot{\chi}_t = \frac{1}{\nu(\chi_t)} \frac{\pi_B(\chi_t) - \pi_A(\chi_t)}{\pi'_B(\chi_t)}, \quad \chi_0 = \theta.$$  

(35)

We are left with a pair of initial boundary problems (i.e., (31) and (35)) to which the (unique) fixed point characterizes the equilibrium.

**Theorem 4.1.** In any economy in which households have CRRA utility and the productivity shift is permanent, there exists a unique $(\tau^{**}, V^{**})$ such that the firm strategies and capital prices consistent with $(\tau^{**}, V^{**})$ are part of a fully-separating competitive equilibrium.

The effect of consumption-smoothing motives on equilibrium reallocation is illustrated in Figure 7. The higher is $\gamma$, the stronger is the desire to smooth consumption. This increases the short-term interest rate, which makes it more costly for firms to delay reallocation and, in turn, speeds up the reallocation process. Note that increasing $\gamma$ will have the same qualitative implications for reallocation dynamics as a reduction the marginal adjustment cost.

![Figure 7](image)

**Figure 7:** This figure illustrates how the reallocation dynamics depend on $\gamma$ for the case of a permanent productivity shift.

Another novel feature of the general equilibrium environment is that the rate at which the economy recovers from a productivity shift depends on the allocation of capital upon
its arrival. To fix ideas, consider the case in which sector $A$ experiences a negative shock to productivity at $t = 0$. If all capital is initially allocated in Sector $A$ when the shock arrives, the economy will suffer a severe drop in output but the rate of reallocation will be high and the recovery process will be relatively quick. On the other hand, if capital is more evenly split across the two sectors when the shock arrives, then the drop in output will be smaller but the recovery process will be slower. The intuition is that when there is more capital to reallocate, the growth rate of consumption will be higher, which in return requires higher interest rates and lower $\nu$. This in turn raises the cost to firms in sector $A$ from delaying the sale of their capital and increases $\dot{\chi}_t$.\footnote{A similar comparative static prediction obtains with respect to either (i) fraction that sectors $A$ and $B$ constitute of the larger (unmodeled) economy and (ii) the magnitude of the productivity shift.} These dynamics are illustrated in Figure 8 for the case of a negative productivity shift to sector $A$.

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{output_change.png}
\caption{Change in Output}
\end{subfigure}
\hfill
\begin{subfigure}{0.45\textwidth}
\includegraphics[width=\textwidth]{reallocation_rate.png}
\caption{Rate of reallocation}
\end{subfigure}
\caption{Recovery from a negative productivity shift to sector $A$ as it depends on the initial distribution of capital. The solid black line corresponds to the case where all of the capital is initially allocated in Sector $A$. The dotted red line (dashed blue line) correspond to the case where 50% (10%) of the capital is initially allocated to Sector $A$.}
\end{figure}

### 4.2 Aggregate Risk and Transitory Shifts

In the previous subsection, aggregate risk does not play a role in the reallocation decision of firms. The economic implications were driven by households’ desire to smooth consumption over time. In this section, we explore how aggregate risk and households’ desire to smooth consumption across aggregate states affects reallocation dynamics. We do so by extending the analysis from the previous subsection to a situation in which there are multiple transitory shifts. We focus attention on the reallocation dynamics from sector $A$ to sector $B$ when
sector $B$ is currently more productive ($\pi_B > \pi_A$), but sector $A$ will become more productive at some (random) point in the future.

By using backward induction on the number of shocks yet to arrive, we derive the system of differential equations characterizing the reallocation dynamics in Appendix B. We solve this system numerically by applying standard techniques. Figure 9 illustrates an important finding from this exercise. Namely, that the rate of reallocation reaches zero prior to the all of the capital being reallocated to sector $B$. This implies that some capital remains *persistently* misallocated. The intuition is that misallocated capital can serve as a hedge against a subsequent productivity shift. Thus, informational frictions not only generates delays in reallocation but can halt the reallocation process entirely. In an adjustment cost model, this would correspond to an arbitrarily large adjustment cost beyond a certain threshold.

![Figure 9: Reallocation dynamics in the presence of aggregate risk. This figure plots the dynamics prior to the arrival of the last shock.](image)

One might be tempted to consider a model in which shocks continue to arrive ad infinitum. We expect qualitatively similar results to obtain in such a model. However, analogous to macroeconomic models with heterogeneous households, solving for the equilibrium of such a model requires keeping track the entire distribution of capital across sectors and thus an arbitrarily large state space. In order to overcome this problem, one would need to develop an approximate solution method (Krusell and Smith, 1998). This exercise is outside the scope of this paper, but may be a useful one to undertake in future work.
5 Response of output and productivity to shocks

5.1 Reallocation with Permanent Shifts

Here, we examine the behavior of aggregate and sectoral output and productivity, in response to several types of shocks. For simplicity, we focus on the case with risk-neutral households. The output of sector \( i \) at time \( t \) depends on the current distribution of project quality in that sector

\[
Y_t^i = \int y_t^i(\theta) dF_t^i(\theta),
\]

where \( y_t^i(\theta) \) denotes the output of a unit of capital of quality \( q \) in sector \( i \) at time \( t \). Aggregate output is then equal to \( Y_t = Y_t^A + Y_t^B \).

We also compute the average productivity of capital in each sector as

\[
X_t^i = \frac{Y_t^i}{k_t^i}.
\]

Since aggregate capital is constant, aggregate productivity is equal to total output, \( X_t = Y_t \).

5.2 Response to a sectoral productivity shift

First, we examine the response of the economy to a sectoral productivity shift. We focus on the case where the gains from trade are constant, \( a = 1 \), and the overall distribution of quality is distributed as a truncated normal on \( \Theta \). We show the results in Figure 10.

Recall that a productivity shift causes the sectoral productivity of \( A \) to fall and of \( B \) to rise. Since all capital is initially allocated in sector \( A \), aggregate output falls on impact, as we see in Panel (c). As the economy reallocates capital, output in sector \( A \) continues to fall while output in sector \( B \) rises. Once all capital is reallocated from sector \( A \) to sector \( B \), total output is restored to the pre-shock level. In this model, the response of output to a sectoral shift is qualitatively similar to that of a model with adjustment costs. However, the behavior of total factor productivity exhibits dynamics that are markedly different to a model with adjustment costs. In particular, Panels (d) and (e) show that productivity rises over time in both the sector from which capital exits (A) and in the sector to which it is being reallocated (B). In contrast, in the standard adjustment cost models, productivity would either be flat or display opposite patterns in each sector.\(^{13}\)

The behavior of average productivity in response to a shock is a distinguishing feature

\(^{13}\)Specifically, with constant returns average productivity of capital would be flat. With decreasing returns, productivity in sector \( A \) would increase while average productivity in sector \( B \) would decrease. Increasing returns to scale would generate the opposite pattern.
Figure 10: Response to a sectoral productivity shift, where at \( t = 0 \), sector B becomes the more productive sector. The distribution of quality \( F(q) \) is beta in \( \theta \) and \( \bar{\theta} \) with shape parameters \( a = b = 2 \). The figure uses constant gains from trade \( \alpha = 1 \) and transitory shocks \( \lambda = 1/10 \).

of our model. Productivity improves in both sectors because the distribution of capital quality \( \theta \) across sectors varies over time. As time passes, the quality of the marginal unit being transferred from sector \( A \) to \( B \) is higher, implying the average quality for both sectors is increasing. This prediction is in line with the data. Collard-Wexler and Loecker (2013) study the reallocation of resources in the steel industry following the introduction of the minimill technology. They find that productivity increased over time among plants that used minimills, but also among vertically integrated plants (the old technology). The improvement in productivity among vertically integrated plants took place primarily through the exit of low productivity plants. Further, this mechanism may help to explain a puzzling fact from the international literature, which is that developing countries experience measured increases in productivity along side capital outflows (Gourinchas and Jeanne, 2006; Prasad, Rajan,
5.3 Response of the economy to structural changes

Here, we consider the effect of an unanticipated change in the model’s structural parameters on aggregate output, the level of misallocation, and the rate of capital reallocation. Several papers have recently argued that shocks to adjustment costs can be useful for explaining features of the data (Eisfeldt and Rampini, 2006; Justiniano et al., 2011). Yet, one may find it difficult to interpret the underlying economic environment in which adjustment costs are exposed to such shocks. One benefit of conducting this exercise is to link a change in adjustment costs to a shift in the parameters of our model.

We consider three types of shocks. First, we examine an increase in the dispersion of capital quality. Second, we consider unanticipated changes to the frequency of sectoral productivity shifts, modeled as an increase in $\lambda$. We can interpret this change as an increase in the volatility of sectoral shocks. Finally, we consider the effect of a change to the interest rate.

To understand the dynamic response of these shocks, we compute the level of misallocation at time $t$ as the percent of total potential output lost due to misallocation of capital:

$$M_t = \frac{\int \bar{\pi}(\theta)dF(\theta) - Y_t}{\int \bar{\pi}(\theta)dF(\theta)}.$$

(38)

We construct impulse responses with respect to these structural changes as follows. We first simulate a sequence of shocks assuming no structural shifts. Holding the sequence of shocks fixed, we then permute the model by introducing an unanticipated parameter change at time 0 and compute the deviation across the two paths. We repeat this procedure 1,000,000 times and report mean deviations over all simulations in the rate of reallocation ($\Delta R_t = R_t - R_t^{SS}$), the percentage of misallocation ($\Delta M_t$), and total output ($\Delta \log(Y_t)$).

5.3.1 Increase in quality dispersion

First, we consider an unanticipated increase in the dispersion of capital quality. As before, we model this as an expansion in the support of the quality distribution of new capital inflows holding the mean quality constant. The results are plotted in Figure 11.

Increasing the dispersion of quality for new capital has no effect on impact, since new capital flows in slowly and is initially efficiently allocated. However, upon the arrival of the next productivity shift, the distribution of quality in the divesting sector is now greater. This increase in the degree of adverse selection implies that the rate of reallocation is slower. As buyers become more uncertain about capital quality, sellers need to wait longer to sell
in order to signal their type. This decrease in the speed of transaction leads to a higher likelihood of capital misallocation, and therefore to lower aggregate output and productivity. Consistent with Eisfeldt and Rampini (2006), the model predicts that the dispersion in capital productivity is counter-cyclical.

**Figure 11:** Response to an increase in the dispersion of capital quality. Figures plot mean difference from steady state across simulations.

### 5.3.2 Increase in volatility

Next, we consider the effect of an unanticipated increase in the frequency of sectoral shifts $\lambda$ on equilibrium outcomes. We show the results in Figure 12. An increase in the volatility of sectoral shifts $\lambda$ increases the rate of reallocation.

This increase in the rate of reallocation is not purely the result of the increased frequency of shocks. It also results from the fact that lower quality types reallocate faster as $\lambda$ increases, as we saw in Proposition 3.6. However, despite the increase in the rate that capital is reallocated, the likelihood that it is misallocated is still increasing with $\lambda$. As a result, average output is lower following an increase in $\lambda$. 

![Figure showing response to an increase in the dispersion of capital quality](image-url)
In summary, our model generates a negative causal link from volatility of sectoral productivity to the degree of misallocation, and consequently to the level of aggregate productivity. Consistent with this prediction, Collard-Wexler, Asker, and Loecker (2013) document a negative relation between the time-series volatility of sectoral TFP shocks and the degree of resource misallocation – measured by the dispersion in TFP. Further, the negative relation between volatility – or adverse selection – and output is an alternative mechanism that is consistent with the findings of Bloom (2009), who documents a negative relation between measures of uncertainty and aggregate productivity.

5.3.3 Expansionary monetary policy

Last, we analyze the impact of an unanticipated expansion in monetary policy, modelled as a reduction in the interest rate \( r \). In standard models, lowering the rate at which agents discount the future increases the present value of the benefits from reallocating capital. This increase in valuations leads to faster reallocation of capital and an increase in efficiency. By contrast, in our setting, lowering the discount rate lowers the opportunity cost of delay for firms in the less productive sector (i.e., the left-hand side of (6)). To separate themselves, firms with higher quality capital must wait even longer. As we see in Figure 13, lowering the discount rate leads to a lower rate of capital reallocation, more misallocation and thus lower productivity. This comparative static not only illustrates the benefit of endogenizing the costs of reallocation, but also has implications on the impact of monetary policy on the efficient allocation of capital. Indeed, despite the fact that the Federal Reserve dropped the interest rate to zero following the recent housing and financial crisis, turnover in many asset markets remained extremely low.
6 Extension: New Investment and Technology Adoption

To this point, we have focused on how adverse selection affects the reallocation of existing capital. Here, we incorporate the same mechanism into a model of new investment. What changes is the type of capital that gets reallocated. Specifically, rather than physical capital or labor, the factor in limited supply is entrepreneurial talent, namely the ability to take advantage of investment opportunities to create productive projects. Below, we flesh this out in a simple model and discuss the several novel findings.

The economy has a mass of entrepreneurs and investors. When an investment opportunity or innovation arrives, entrepreneurs can take advantage of it to create projects. Entrepreneurs can also manage the projects they create. Investors can manage projects, but cannot create them. Projects are heterogeneous in their profitability, they have a quality $\theta$ distributed according to $F(\theta)$ with a continuous and strictly positive density on the support $\Theta = [\underline{\theta}, \overline{\theta}]$. A project of quality $\theta$ using innovation $i$ produces a flow of cash flows $\pi_i(\theta)$, where $\pi_{i+1}(\theta) \geq \pi_i(\theta) \ \forall i, \theta$. Entrepreneurs exhibit persistence in their ability to create projects. Specifically, with probability $\kappa \in [0, 1]$ the next project the entrepreneur creates is of the same quality, $\theta$, as her current project and with probability $(1 - \kappa)$ the quality of the new project is drawn from $F(\theta)$.

For simplicity, assume all entrepreneurs are initially managing projects from innovation 0 and an investment opportunity (using innovation 1) arrives at $t = 0$. Since entrepreneurs and investors generate the same cash flows from managing projects, there are no gains from trade prior to $t = 0$. Entrepreneurs have limited capacity; they cannot create a new project while they are currently managing an older one. Hence, in order to be able to take advantage of the new investment opportunity, the must sell their current project to an investor. However, as before, entrepreneurs are privately informed about the quality of their current project.
As long as an entrepreneur with the highest quality project is not willing to trade at the price for the average quality firm, not all entrepreneurs are willing to sell their firms immediately. As before, the equilibrium will have a gradual sale of gradual sale of firms in sector 0 to investors, and consequently, gradual investment in the new technology.

6.1 Equilibrium

The equilibrium construction follows steps similar to those in previous sections so we will omit formal details here. The equilibrium will be separating, hence there is only one type trading each instant. Investors are competitive, hence their break-even condition implies that, the price of a project of quality \( \theta \) in sector \( i = 0 \) is \( P(\theta) = \frac{\pi_0(\theta)}{r} \).

The next step involves determining the time, \( \tau(\theta) \), that an entrepreneur owning a project of type \( \theta \) will sell to investors. Once the entrepreneur has started his new firm of quality \( \tilde{\theta} \), the firm will generate a profit flow of \( \pi_1(\tilde{\theta}) \) forever. Hence, his valuation of the new firm once it is created equals

\[
\bar{V}(\tilde{\theta}) = \frac{\pi_1(\tilde{\theta})}{r}. \tag{39}
\]

After the arrival of the innovation, but prior to the creation of a new project, the entrepreneur’s expected payoff is equal to his conditional expectation of (39),

\[
E_{\tilde{\theta}}[\bar{V}(\tilde{\theta})] = \kappa \bar{V}(\theta) + (1 - \kappa) \int \bar{V}(\theta) dF(\theta), \tag{40}
\]

discounted for the fact that the entrepreneur needs to wait until \( \tau(\theta) \) to sell. Consequently, an entrepreneur of type \( \theta \) has a continuation value that is a function of the lowest remaining entrepreneur in sector \( i = 0 \), denoted by \( \chi \)

\[
V_0(\theta, \chi) = \frac{\pi_0(\theta)}{r} + e^{-r(\tau(\theta) - \tau(\chi))} \left( \kappa \bar{V}(\theta) + (1 - \kappa) \int \bar{V}(\theta) dF(\theta) - I \right). \tag{41}
\]

In equilibrium, the entrepreneur of type \( \chi \) must be indifferent between locally speeding up or delaying the sale of his project. This indifference condition can be written as

\[
P'(\chi) \dot{\chi} = r \left( \kappa V_1(\theta) + (1 - \kappa) \int \bar{V}(\theta) dF(\theta) - I \right). \tag{42}
\]

Combining the two equations above yields a differential equation in \( \chi \)

\[
\dot{\chi} = r \frac{\kappa \pi_1(\chi) + (1 - \kappa) \int \pi_1(\theta) dF(\theta) - I}{\pi_0'(\chi)} \tag{43}
\]
Equation (43), along with the boundary condition that \( \chi(0) = \theta \), pins down the unique equilibrium. From this, one can begin to see the role played by the persistence parameter, \( \kappa \). If there is no persistence, the entrepreneurs’ expected return from investing in the new technology (i.e., the numerator in (43)) is constant in \( \theta \). For \( \kappa > 0 \) the numerator is increasing in \( \theta \) and increasingly so as \( \kappa \) becomes higher, or equivalently, as entrepreneurial talent and the new technology become more complementary. Recalling our discussion of the case where \( \alpha < 1 \) in the CES example in Section 5.1, when gains from trade are increasing in quality, trade is slower with the low types and then speeds up with higher types, or equivalently the rate of investment in the new technology \( \dot{\chi}_t \) is increasing over time. Comparing (43) to (8) illustrates the fact that the investment rate is increasing with time even when the gains from moving to the new technology, \( \pi_1(\theta) - \pi_0(\theta) \), are independent of quality \( \theta \), a feature which is in stark contrast with the motivating example.

### 6.2 Output and productivity

Next, we analyze the model’s implications for the dynamics of output and TFP of each technology and in the economy as a whole, defined as in (36) and (37) respectively.

Once the innovation becomes available and entrepreneurs start investing in the new technology, total factor productivity in the new sector is slowly increasing over time. This gradual increase in productivity occurs as progressively more talented entrepreneurs sell their old firms and create projects using the new technology. However, once the new technology becomes available, aggregate TFP can actually decrease. This productivity drop can occur because, even though the new sector is on average more productive, the first projects created using the new technology sector are of below average quality – since they are created by below-average entrepreneurs. The higher the persistence, the greater the drop in measured TFP. The following proposition states the conditions under which this drop in productivity occurs.

**Proposition 6.1.** Upon the arrival of an innovation, economy wide TFP is initially decreasing (and eventually increasing) over time if and only if \( \kappa \pi_1(\bar{\theta}) + (1 - \kappa) E_{\tilde{\theta}}[\pi_1(\tilde{\theta})] < E[\pi_0(\theta)] \).

Furthermore, the total magnitude of the TFP drop will be higher the greater the persistence in quality \( \kappa \). As we see in Figure 14, when entrepreneurial talent is more transferable to the new technology (high \( \kappa \)), the process of technology adoption is further delayed; investment responds with a lag, and aggregate productivity dips on impact.

The possibility that measured total factor productivity might drop at the onset of the arrival of a new technology is consistent with several empirical studies (David, 1990; Jovanovic and Rousseau, 2005). In their investigation of two major technological innovations,
Figure 14: Output, productivity and rate of technology adoption. Black is aggregate, red is new sector, blue is old sector. A solid line represents the high persistence case ($\kappa = 0.75$) and the dotted line represents lower persistence ($\kappa = 0.25$).

electrification and information technology (IT), Jovanovic and Rousseau (2005) point out that both technologies were accompanied by continued use of the old technology, slow adoption of the new technology, continued productivity improvements, and an initial decline in aggregate productivity. These patterns are consistent with panels (c) through (f) of Figure 14, respectively. Further evidence is provided by Collard-Wexler and Loecker (2013) who study the effect of the introduction of the minimill, a superior technology, to the US steel industry. They find a significant, but slow, increase in the productivity of the industry due to, first, reallocation of resources from the old to the new technology, and second, by improvement in productivity in plants employing the old technology due to the exit of inefficient producers. Further, despite the widespread belief in the superiority of the new technology, plants employing the old technology were 20% more productive in terms of
output per worker compared to plants that adopted the new technology, but this difference evaporated over time.

7 Discussion

Some of the model’s predictions may appear to run counter to what intuition would suggest. Specifically, one may naturally expect that, in contrast to our model, ‘higher types’ should reallocate faster than ‘lower types’. However, this intuition refers to observable characteristics. If higher types can receive a higher price regardless of the timing of their reallocation decision, then they will naturally reallocate more quickly than lower types. We should emphasize that the reallocation decision of firms in our model operates based on unobservable characteristics. That is, conditional on observable characteristics, types with worse unobservable characteristics will reallocate faster than types with better ones. That the model’s predictions pertain to unobservable characteristics makes developing direct tests of the mechanism inherently challenging. However, an empirical design can exploit the possibility that certain ex-ante unobservables may be correlated with ex-post observable measures; for instance, the unobservable quality of capital in our model is correlated with ex-post measures of profitability. A rigorous empirical analysis of this mechanism is outside the scope of this paper. However, we have some preliminary evidence that lends supports to the model’s main mechanism.

We focus on the length of time elapsed between a firm’s incorporation and its initial public offering (IPO). IPOs are a natural setting to test the predictions of our model. First, IPOs represent a change in ownership that is motivated by non-informational reasons; entrepreneurial risk aversion can be viewed as a higher flow operating cost while the firm is private. Second, the amount of public information available about the firm is scarce prior to its IPO. Third, IPOs are a setting in which we have available ex-post measures of operating performance for the asset being traded.

Our preliminary findings indicate that the length of time from a firm’s incorporation to its IPO is predictive of its future profitability at horizons of up to 5 years. This predictive relation is conditional on observable characteristics, including a firm’s size and its profitability at the time of the IPO. This finding supports our model’s prediction that entrepreneurs with higher quality capital delay the sale of their capital for longer as a signal of quality to the market. Importantly, even though a firm’s age predicts future profitability, it does not predict its stock returns following the IPO decision, suggesting that the firm’s price at the time of the IPO is fully revealing of its quality. Put differently, the higher ex-post profitability associated with older firms does not represent news to the market.

Without an empirical design of the nature described above, it is difficult to distinguish our theory from others. Nevertheless, our model’s predictions are consistent with existing
empirical evidence. For instance, Ramey and Shapiro (2001) document the following stylized facts that are consistent with our model: i) capital sells at a substantial discount relative to its replacement cost; ii) this discount is smaller if capital sells to other aerospace firms, which presumably have better ability to evaluate its quality; iii) the process of selling used equipment is lengthy.

Our model also provides an economic explanation for why disinvestment should be more costly than investment, which plays a prominent role in structural models of investment (see e.g., Abel and Eberly, 1994, 1996). This is because disinvestment involves the sale of used capital, where one naturally expects the adverse selection problem to be more severe, whereas investment often involves purchasing capital directly from its producers where the information friction is generally thought to be less severe (e.g., due to the reputational concerns of producers). Along these lines, Cooper and Haltiwanger (2006) estimate a structural model of convex and non-convex adjustment costs using plant-level data. Their estimates imply a substantial spread between the purchase and sale price of capital.

8 Conclusion

In this paper, we have incorporated adverse selection into a competitive decentralized economy to study the dynamics of capital allocation and new investment. The information friction leads to slow movements in capital reallocation, lagged investment following technological innovations, and provides a micro-foundation for convex adjustment cost models. The model generates a rich set of dynamics for reallocation, investment, output and productivity.

Clearly, our mechanism is not the only one that can generate some of these patterns. Physical (convex) costs, search, financial frictions, learning, time-to-build and other factors are surely important components in the allocation of new and existing capital. We have abstracted away from these considerations in order to highlight the key ideas of the paper. Further developing this framework into models that are suitable for calibration seems a promising direction for future work.
References


A Appendix

Proof of Lemma 2.1. Follows immediately from the households first order condition and the goods market clearing condition.

Proof of Lemma 2.2. If the skimming property did not hold, then there exists \((i, t)\) such that \(t_1 \equiv T_i^i(\theta) > t_0 \equiv T_i^i(\theta')\) for some \(\theta' > \theta\). Since \(q\) prefers to wait until \(T_i^i(\theta)\) then \(V_{t_0}^i(\theta) \geq P_{t_0}^i\). Since \(\theta'\) accepts at \(t_0\), \(V_{t_0}^i(\theta) = P_{t_0}^i\). But since \(\pi_i\) is increasing, \(\theta'\) could do strictly better by mimicking the type \(\theta\), which violates (10).

The proof of Theorem 3.1, relies on Lemma A.1, which we prove below.

Lemma A.1. There exists a unique \(\chi^*\) that satisfies (24) and (25). Furthermore, \(\chi^*\) is strictly increasing.

Proof. Note first that (24)-(25) is an initial value problem of the form

\[
\chi'(t) = f(t, \chi(t)), \quad \chi(0) = q
\]  

(44)

To verify existence and uniqueness of a solution, we will apply the Picard-Lindelof Theorem (see Zeidler (1998), Theorem 3.A.) To do so, it is sufficient to verify several properties of \(f\) : (i) \(f(t, x)\) is continuous on \([0, T] \times [q, \bar{q}]\); (ii) \(f\) is bounded (iii) that \(f(t, x)\) is Lipschitz. Property (i) is by inspection (since both \(g\) and \(\pi_i\) are continuously differentiable). Property (ii) follows immediately from the expression for \(g\) and the conditions placed on \(\pi_i\). To demonstrate (iii), it suffices to show that \(\frac{d}{dx} f(t, x)\) is bounded, which follows from the restriction that \(\pi_i\) have bounded first and second derivatives.

Proof of Theorem 3.1. From Lemma A.1, there is a unique candidate separating equilibrium. Thus, in order to prove the theorem, it suffices to check that the candidate satisfies the equilibrium conditions. The zero profit and capital market clearing conditions are satisfied by construction. To verify that firms optimize, note that no firm in the efficient sector strictly prefers to sell their capital since the price is \(V(\theta)\), which is the least a firm can expect to earn by continuing to operate their capital. It remains to verify that there are no profitable deviations for firms in the inefficient sector. Since \(\tau(\chi)\) and \(P(\chi)\) are monotonic functions and the sellers’ payoff satisfies the single crossing property. To see this, note that the sellers objective can be written as

\[
u_\theta(t, P) = (1 - f(t)) \frac{\pi(\theta)}{\rho} + f(t) P
\]
and therefore
\[ \frac{\partial}{\partial \theta} \left( \frac{\partial u_{\theta}}{\partial P} \frac{\partial u_{\theta}}{\partial t} \right) = \frac{f'(\tau)}{f'(t)(P - \pi(\theta)/\rho)} > 0 \]

Since this condition holds, the local IC constraint is sufficient to guarantee that there are no profitable global deviations. Note that \( \dot{\chi}_t \) was constructed as to ensure that the local IC constraint is satisfied. Therefore, there are no profitable deviations for firms in the inefficient sector.

**Proof of Proposition 3.3.** That \( g(t) = 0 \) for \( \lambda = 0 \) is by inspection. That (24) then reduces to (26) follows immediately.

**Proof of Proposition 3.4.** Taking the total derivative of the RHS of (26) with respect to time we get that
\[ \chi''(t) = \rho \cdot \frac{d}{d\chi} \left( \frac{\bar{\pi} - \pi}{\bar{\pi}'(\chi)} \right) \cdot \dot{\chi}_t \]

Since \( \dot{\chi}_t(t) > 0 \) for all \( t \in [0, \tau(\theta)] \). The derivative of \( \dot{\chi}_t \) with respect to time has the same sign as the derivative of \( \frac{\bar{\pi} - \pi}{\bar{\pi}'(\theta)} \) with respect to \( \theta \).

**Proof of Corollary 1.1.** Follows immediately from Proposition 3.4 and the fact that for CES production technology, \( \frac{d}{d\theta} \left( \frac{\bar{\pi} - \pi}{\bar{\pi}'} \right) \) is strictly positive for \( \alpha < 1 \), strictly negative for \( \alpha > 1 \), and equal to zero for \( \alpha = 1 \).

**Proof of Proposition 3.5.** This follows from (23) and the fact that (i) \( \tau(\theta) > 0 \) for all \( \theta > \bar{\theta} \), and (ii) \( g(t) > 0 \) for all \( t > 0 \).

**Proof of Proposition 3.6.** Using a subscript to represent elements of the relevant economy, we have that
\[ \chi'_2(0) - \chi'_1(0) = (1 - g(0; \lambda_2) + g(0; \lambda_2)) - (1 - g(0; \lambda_1) + g'(0; \lambda_1)) \]
\[ = g'(0; \lambda_2) - g'_1(0; \lambda_1) > 0 \]

Where the inequality follows from the fact that \( \frac{d}{d\lambda} g'(0; \lambda) > 0 \). Therefore, \( \chi'_2(0) - \chi'_1(0) > 0 \). By the continuity and boundedness of \( \chi'_1 \) and \( \chi'_2 \), there must exist \( \tilde{t} > 0 \) such that the inequality holds for \( t \in [0, \tilde{t}] \).

**Proof of Theorem 4.1.** The proof involves showing that there exists a unique candidate solution satisfying the joint system of differential equations and then verify that the strategies and prices consistent with the candidate satisfy the equilibrium requirements.
Fix an economy, which can be represented by \( \{ f^A, f^B, \pi_A, \pi_B, \gamma, \beta \} \). Define
\[
c(\chi) \equiv \int^\theta_\chi \pi_A(\theta) f^A(\theta) d\theta + \int^{\infty}_\theta \pi_B(\theta) f^B(\theta) d\theta.
\]

Let \( \tau, V \) denote an arbitrary candidate separating equilibrium and note that the zero profit condition requires that \( V(\theta, \chi) = \pi_B(\theta) \nu(\chi) \), therefore it is sufficient to characterize \( \tau, \nu \).
Assuming \( \tau \) is strictly increasing and therefore invertible, define \( \chi_t \equiv \tau^{-1} \) and \( \phi(\theta) = \frac{1}{\tau'(\theta)} \).
From (29), (31) and (35), we know that any candidate separating equilibrium must satisfy
\[
\phi(\theta) = \frac{\pi_B(\theta) - \pi_A(\theta)}{\pi_B'(\theta) \nu'(\theta)}, \quad \tau(\theta) = 0 \quad (45)
\]
\[
\left( \beta + \frac{\gamma}{c(\theta)} (\pi_B(\theta) - \pi_A(\theta)) f^A(\theta) \phi(\theta) \right) \nu'(\theta) = 1 + \phi(\theta) \nu'(\theta), \quad \nu(\theta) = \beta^{-1} \quad (46)
\]

Substituting the ODE from (45) into (46), and rearranging, we arrive at an initial value problem of the form
\[
\nu'(\theta) = f(\theta, \nu(\theta)), \quad \nu(\theta) = \beta^{-1} \quad (47)
\]

The proof of existence and uniqueness of a solution to (47) follows closely the proof of Lemma 3.1 and is therefore omitted. Letting \( \nu^{**} \) denote this solution, substitute it into (45), and apply the same argument to get existence and uniqueness of \( \tau^{**} \). Therefore, \( \tau^{**} \) is invertible. Let \( \chi_t^{**} \) denote its inverse. Thus, we have shown there exists a unique candidate separating equilibrium.

To verify the candidate is indeed part of an equilibrium, specify that \( \xi_t = \exp(-\beta t) c(\chi_t^{**})^\gamma \) and \( c_t^h = c(\chi_t^{**}) \). Household optimality and market clearing of the consumption good is immediate. That the capital market clears and new firms make zero profit in the candidate follows immediately from the fact that only \( \chi_t^{**} \) trades at time \( t \) and the solution satisfies (32).
Locally, firm optimality is by construction (i.e., (33)). That the firm’s strategy is optimal globally follows from the same arguments as used in the proof of Theorem 3.1.

**Proof of Proposition 6.1.** The expected productivity of a new firm is \( \rho \bar{\pi}(\bar{\theta}) + (1 - \rho) E_{\bar{\theta}} \left[ \bar{\pi}(\bar{\theta}) \right] \) and the average productivity when all firms are in the original sector is just the average productivity of the sector, \( E \left[ \bar{\pi}(\theta) \right] \). Hence, when the former is smaller than the latter the firms created upon arrival of new vintage are of below average TFP and thus lower the average measured TFP of the economy.
B Appendix

Multiple Transitory Shocks with Risk Averse Households

Suppose now that there are multiple transitory shocks and households are risk averse. Our analysis in Section 4.1 applies once the last shock arrives. Let us now consider the case in which there are two shocks. To fix ideas, suppose that at \( t < 0 \) all capital is allocated to sector \( A \). At \( t = 0 \), a shock arrives that makes sector \( B \) more productive. But this shift is not permanent: at some random time \( \tau > 0 \), another shock will arrive that will make sector \( A \) the more productive sector. As before use \( \bar{\pi} (\pi) \) to denote the productivity of capital allocated efficiently (inefficiently).

One can think of the model as having two regimes. In the first regime \(( t < \tau )\), capital transitions from sector \( A \) to sector \( B \). In the second regime \(( t > \tau )\), capital transitions back to sector \( A \). We use subscripts to denote to which regime the object refers. For example, \( T_1 (\theta) \) denotes the time at which a sector \( A \) firm sells capital of quality \( \theta \) to sector \( B \) in the first regime. Let \( \theta_1 \) denote the lowest type remaining in sector \( A \) at the end of the first regime, i.e., \( \theta_1 = \inf \{ \theta : T_1 (\theta) > \tau \} \).

B.1 Second Regime

We proceed by backward induction. Note that all \( \theta > \theta_1 \) are efficiently allocated at the beginning of the second regime. Hence, for all \( t \geq \tau \):

\[
Y_t = \int_0^{\bar{\pi}} \bar{\pi}(\theta) \, dF(\theta) - \int_{\chi_2(t)}^{\theta_1} (\bar{\pi}(\theta) - \bar{\pi}(\chi)) \, dF(\theta)
\]

\[= c_2(\chi_2(t), \theta_1), \tag{48}\]

where, \( \chi_2(t) \) denotes the lowest remaining type in the inefficient sector (sector \( B \)) during the second regime. Using the same argument as in Section 4.1, the solution consists of the rate at which types change

\[
\phi_2(\chi, \theta_1) = \frac{\bar{\pi}(\chi) - \bar{\pi}(\chi_B)}{\bar{\pi}'(\chi)} \frac{1}{\nu_2(\chi; \theta_1)} \tag{49}\]

where \( \nu_2(\chi; \theta_1) \) is the price of an annuity in the current state and solves the ODE

\[
\left( \rho + \gamma c_2(\chi, \theta_1)^{-1} \left( \bar{\pi}(\chi) - \bar{\pi}(\chi_B) \right) f(\chi) \phi_2(\chi, \theta_1) \right) \nu_2(\chi; \theta_1) = 1 + \phi_2(\chi, \theta_1) \nu_2'(\chi; \theta_1). \tag{50}\]
The boundary condition now becomes

$$\nu_2(\theta_1; \theta_1) = \rho^{-1}. \quad (51)$$

The value of an efficiently allocated unit of capital of quality $\theta$ in the second regime is therefore equal to

$$\bar{V}_2(\theta, \chi_2, \theta_1) = \nu_2(\chi_2, \theta_1) \hat{\pi}(\theta). \quad (52)$$

Next, we derive the value of an inefficiently allocated unit of capital during regime 2. It will be sufficient to compute this value evaluated at $\chi_2 = \theta$ for all $\theta_1$, which is given by

$$\xi_t \bar{V}_2(\theta, \theta, \theta_1) = \int_t^{T_2(\theta, \theta_1)} \xi_s \bar{\pi}(\theta) \, ds + \int_{T_2(\theta, \theta_1)}^{\infty} \xi_s \bar{\pi}(\theta) \, ds$$

$$= \int_t^{T_2(\theta, \theta_1)} \xi_s (\bar{\pi}(\theta) - \bar{\pi}(\theta)) \, ds + \int_t^{\infty} \xi_s \bar{\pi}(\theta) \, ds$$

$$= \xi_t \bar{V}_2(\theta, \theta, \theta_1) - \int_t^{T_2(\theta, \theta_1)} \xi_s (\bar{\pi}(\theta) - \bar{\pi}(\theta)) \, ds \quad (53)$$

where $T_2(\theta, \theta_1) = \int_\theta^\theta \frac{1}{\phi_2(y, \theta_1)} \, dy$ is the stopping rule used by a type $\theta$ seller in the second regime. Using a change a variables, equation (53) can be written as

$$V_2(\theta, \theta, \theta_1) = \bar{V}_2(\theta, \theta, \theta_1) - (\bar{\pi}(\theta) - \bar{\pi}(\theta)) \int_\theta^\theta \exp \left( - \rho T_2(y, \theta_1) \right) \left( \frac{c_2(y, \theta_1)}{c_2(\theta, \theta_1)} \right)^{-\gamma} \frac{1}{\phi_2(y, \theta_1)} \, dy \quad (54)$$

where instead of integrating over time, we integrate over types that switch before a type $\theta$ switches. Substituting in the expression for $T_2$, one can calculate $V_2(\theta, \theta, \theta_1)$ in terms of $V_2$ and $\phi_2$.

**B.2 First Regime**

By the zero-profit condition, the price must equal the value of an efficiently allocated unit of capital in the first regime—denoted by $\bar{V}_1$—satisfies

$$\xi_t \bar{V}_1(\theta, \chi_1(t)) = E \left[ \int_t^\tau \xi_s \bar{\pi}(\theta) \, ds + \xi_\tau V_2(\theta, \theta, \chi_1(\tau)) \right].$$

Here, $\chi_1(t)$ denotes the lowest quality of capital remaining in the inefficient sector during the first regime (sector $A$). Because $\chi_1(t)$ must be monotonic in a separating equilibrium, we
often omit $t$ arguments and write things in terms of the state variable $\chi_1$, using $\phi_1(\chi_1)$ to denote the rate of reallocation in the first regime. Aggregate consumption and output in the first regime is given by

$$c_1(\chi_1) \equiv \int_{\theta}^{\chi_1} \pi(\theta) d\theta + \int_{\chi_1}^{\theta} \pi(\theta) d\theta.$$ 

From Lemma 2.1, the stochastic discount factor is $\xi_t = e^{-\rho t} c_1(\chi_1)^{-\gamma}$, which satisfies

$$\frac{d\xi_t}{\xi_t} = -\rho dt - \gamma c_1(\chi_1)^{-1} \frac{\partial}{\partial \chi_1} c_1(\chi_1)\phi_1(\chi_1) dt + \left( \frac{c_2(\theta, \chi_1)}{c_1(\chi_1)} \right)^{-\gamma} - 1 dN_t,$$

$$E_t \left[ \frac{d\xi_t}{\xi_t} \right] = -\rho dt - \gamma c_1(\chi_1)^{-1} \frac{\partial}{\partial \chi_1} c_1(\chi_1)\phi_1(\chi_1) dt + \lambda \left( \frac{c_2(\theta, \chi_1)}{c_1(\chi_1(t))} \right)^{-\gamma} - 1 dt.$$

Define the discounted price process $\tilde{V}$, as

$$\tilde{V}(\theta, \chi_1) \equiv c_1(\chi_1)^{-\gamma} \tilde{V}_1(\theta, \chi_1)$$

$$= E \left[ \int_{\tau}^{\theta} e^{-\rho(s-t)} c_1(\chi_1(s))^{-\gamma} \pi(\theta) ds + e^{-\rho(\tau-t)} (c_2(\theta, \chi_1(\tau)))^{-\gamma} V_2(\theta, \theta, \chi_1(\tau)) \right].$$

Also note that

$$\tilde{V}_\chi \equiv \frac{d}{d\chi_1} (c_1(\chi)^{-\gamma} \tilde{V}_1(\theta, \chi_1))$$

$$= c_1(\chi)^{-\gamma} \tilde{V}_\chi(\theta, \chi) - \gamma c_1(\chi)^{-1} \pi(\chi) f(\chi) \tilde{V}_1(\theta, \chi)$$

By the martingale property, $\tilde{V}(\theta, \chi_1)$ satisfies the ODE

$$\rho \tilde{V} = c_1(\chi_1)^{-\gamma} \pi(\theta) + \tilde{V}_\chi \phi_1(\chi_1) + \lambda \left( \frac{c_2(\theta, \chi_1)}{c_1(\chi_1)} \right)^{-\gamma} V_2(\theta, \theta, \chi_1) - \tilde{V}$$

or after substituting for $\tilde{V}$

$$r_1(\chi) \tilde{V}_1(\theta, \chi) = \pi(\theta) + \frac{\partial}{\partial \chi_1} \tilde{V}_1(\theta, \chi) \phi(\chi) + \lambda \left( \frac{c_2(\theta, \chi_1)}{c_1(\chi_1)} \right)^{-\gamma} (V_2(\theta, \theta, \chi) - \tilde{V}_1(\theta, \chi))$$
Since this is the value of an efficiently allocated unit of capital, the above equation holds only for \( \theta \leq \chi \). For the boundary condition, consider what happens to an efficiently allocated unit of capital when all the capital has moved, but before the second shock hits, \( \chi_1 = \overline{\theta} \) and \( t < \tau \).

The value of capital at the boundary must solve

\[
\rho \hat{V}_1(\theta, \overline{\theta}) = \bar{\pi}(\theta) + \lambda \left( \frac{c_2(\theta, \overline{\theta})}{c_1(\overline{\theta})} \right)^{-\gamma} \hat{V}_2(\theta, \theta, \overline{\theta}) - \hat{V}_1(\theta, \overline{\theta})
\]

\[
\hat{V}_1(\theta, \overline{\theta}) = \frac{1}{\rho + \lambda} \bar{\pi}(\theta) + \frac{\lambda}{\rho + \lambda} \left( \frac{c_2(\theta, \overline{\theta})}{c_1(\overline{\theta})} \right)^{-\gamma} \hat{V}_2(\theta, \theta, \overline{\theta})
\]

Next, we solve for the value of an inefficient unit of capital. Following the same steps as before, the value of an inefficiently allocated unit \( \overline{V}_1 \) satisfies

\[
\rho \hat{V}_1(\theta, \overline{\theta}) = \bar{\pi}(\theta) + \phi_1(\overline{\theta}) \frac{\partial}{\partial \chi} V_1(\theta, \chi) + \lambda \left( \frac{c_2(\theta, \overline{\theta})}{c_1(\overline{\theta})} \right)^{-\gamma} \left( \hat{V}_2(\theta, \theta, \chi) - V_1(\theta, \chi) \right)
\]

Zero profit requires that

\[
P_1(\chi) = \overline{V}_1(\chi, \chi).
\]

At the instant where type \( \theta \) trades he has to be locally indifferent between waiting or not. So at the boundary, we have that

\[
P_1(\chi) = \hat{V}_1(\chi, \chi) = V_1(\chi, \chi)
\]

and

\[
\phi_1(\chi) \frac{\partial}{\partial \chi} V_1(\theta, \chi) \bigg|_{\theta = \chi} = \phi_1(\chi) \frac{d}{d\chi} P(\chi)
\]

\[
= \left( \frac{\partial}{\partial \theta} \hat{V}_1(\theta, \chi) |_{\theta = \chi} + \frac{\partial}{\partial \chi} \hat{V}_1(\theta, \chi) |_{\theta = \chi} \right) \phi_1(\chi)
\]

Replacing the partials with respect to \( \chi \) in the LHS and the RHS using the two ODEs for \( V_1 \) and \( \hat{V}_1 \), we arrive at

\[
\bar{\pi}(\chi) - \bar{\pi}(\chi) - \lambda \left( \frac{c_2(\theta, \chi)}{c_1(\chi)} \right)^{-\gamma} \left( \hat{V}_2(\theta, \theta, \chi) - V_2(\theta, \theta, \chi) \right) = \left( \frac{\partial}{\partial \theta} \hat{V}_1(\theta, \chi) |_{\theta = \chi} \right) \phi_1(\chi).
\]
Therefore, in equilibrium, the rate of reallocation is given by

$$\phi_1(\chi) = \max \left\{ \frac{\bar{\pi}(\chi) - \pi(\chi) - \lambda \left( \frac{c_2(\theta, \chi)}{c_1(\chi)} \right)^{-\gamma} (V_2(\theta, \bar{\theta}, \chi) - V_2(\theta, \bar{\theta}, \chi))}{\left. \frac{\partial}{\partial \theta} V_1(\theta, \chi) \right|_{\theta = \chi}}, 0 \right\}.$$ 

### B.3 Numerical Solution Method

The numerical solution also works by backward induction. Starting in period 2, we first solve for $\nu_2$ using equations (50) and (51). Using equation (49), we can then find the rate of reallocation in the second period. From this, we then solve for the equilibrium value functions in the second period ($\bar{V}_2$ and $\bar{V}_2$) using (52) and (54). After replacing equation (49) into (52) and (54), we obtain two non-linear ODEs in $\chi$, with an initial condition at $\chi = \theta$. We solve these ODEs numerically using an explicit Runge-Kutta (4,5) formula – implemented in Matab’s ode45 solver. Moving back to the first regime, we solve for the two value functions in the first period ($\bar{V}_1$ and $\bar{V}_1$) using the same methodology, while taking the solutions ($\bar{V}_2$ and $\bar{V}_2$) as given.