Sovereign Default, Exchange Rates, and International Asset Prices

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October 15, 2009

JOB MARKET PAPER / PRELIMINARY AND INCOMPLETE

Abstract
This article develops a framework that combines the economics behind the structural modeling of sovereign credit risk, corporate capital structure, and the equilibrium modeling of international asset pricing. The default decisions of the sovereign and the firm are derived optimally and embedded in a two-country, two-good consumption-based asset-pricing model with a representative risk-averse agent for each country. The foreign exchange market acts as the main channel through which shocks are transmitted internationally. This framework can explain (i) the first two moments of international equity returns; (ii) the co-movement across returns on equity, corporate debt, and sovereign debt, in addition to co-movement in international equity return volatilities; and (iii) the negative relationship between equity return volatility in developed economies and sovereign credit risk in emerging economies. A structural test using the general methods of moments provides strong support for the model. In particular, the risk of a sovereign default crisis is highly relevant in the explanation of the dynamics of international equity returns.

JEL Codes: F31, F34, G12, G13, G15

Keywords: Sovereign Debt, Corporate Debt, Credit Risk, Asset Pricing, International Financial Markets, Foreign Exchange

*Acknowledgements: I am deeply grateful to Bernard Dumas for insightful discussions and priceless comments. This paper has also greatly benefited from suggestions provided by Daniel Andrei, Harjoat Bhamra, Michael Brennan, Julien Cujean, Darrell Duffie, Christopher Hennessy, Julien Hugonnier, Jean Imbs, Erwan Morellec, Anna Pavlova, Norman Schuerhoff, and Eduardo Schwartz. Financial support by the National Centre of Competence in Research "Financial Valuation and Risk Management" (NCCR FINRISK) is gratefully acknowledged. The NCCR FINRISK is a research instrument of the Swiss National Science Foundation. All errors, conclusions and opinions contained herein are solely those of the author.

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1 Introduction

This paper proposes a model that explains the relationships between a country’s level of sovereign credit risk and international corporate asset prices. Thus far, little is known about (i) the theoretical diversification benefits of investing across different asset classes (i.e., equity, corporate debt, and sovereign debt); (ii) how the risk of a sovereign default crisis affects equity return and equity return volatility internationally; and (iii) why equity return volatility in the U.S. is strongly related to sovereign credit risk in emerging countries. The recent financial turmoil has clearly illustrated the importance of these issues, and thus, the aim of this paper is to fill this research gap.

An investigation of the relationships between international asset returns requires a precise understanding of the relationships between macroeconomic fundamentals and asset prices. This article develops a structural general equilibrium model that combines the economics behind the structural modeling of corporate and sovereign credit risk and the equilibrium modeling of international asset pricing. Within the model, the world consists of a developed and an emerging economy; each one includes a representative risk-averse household, a representative defaultable firm financed by equity and debt, and a government with debt. Economic shocks are transmitted internationally through the foreign exchange market. The structural general equilibrium model allows for an analysis of the effect of a country’s macroeconomic conditions on the exchange rate, on each firm’s as well as the emerging government’s default decisions, and thus, on each firm’s asset prices.

The main results of the paper are as follows: First, the model suggests strong co-movement among corporate debt, sovereign debt, and the equity markets. The co-movement is present internationally and domestically. In light of the recent credit crisis, the model’s prediction questions the benefits of investing across asset classes. Second, the presence of the risk a sovereign default crisis in an emerging country increases the volatility of international equity returns. Third, equity return volatility is predicted to be countercyclical, to display an asymmetric leverage effect, and to increase with corporate and sovereign indebtedness. Fourth, the model predicts substantial co-movement in equity return volatilities across countries. Finally, the model predicts that equity return volatility in the developed economy is strongly correlated with sovereign credit risk in the emerging economy, despite the absence of a causal relationship.1

1Longstaff et al. (2008), Pan and Singleton (2008), and Remolona et al. (2008) find that variation in sovereign credit risk in emerging markets can be largely explained by changes in equity return volatility in the U.S., as measured with the option-implied volatility index on the S&P500 (VIX). The model may partially change interpretations of their results: the positive correlation can be explained by the presence of a single exogenous economic shock that is transmitted internationally.
These predictions are consistent with the related empirical literature, along with new empirical evidence provided by data for Brazil and the U.S. covering the period from 1994 through 2008.

This paper also provides a structural test of the proposed model. The econometric approach consists of testing a set of overidentifying restrictions on a system of moment equations using the generalized method of moments (GMM) developed by Hansen (1982). The moments under consideration are the first two moments of equity returns in Brazil (MSCI Brazil in U.S. dollars) and in the U.S. (S&P500). I use information on monthly industrial production data for Brazil and the U.S. from June 1994 through December 2008 to compute the asset prices predicted by the model. The model is calibrated to match the dynamics of industrial production in both countries, the corporate leverage ratios, and the government debt-to-GDP ratio in Brazil. The estimation provides strong support for the model. In particular, the anticipated economic costs resulting from a sovereign default crisis are found to be statistically and economically significant. This finding implies, first, that a sovereign has large costs in defaulting on its debt and, second, that the presence of sovereign default risk can explain the international level of both equity premium and equity return volatility.

To understand the approach proposed in this paper, consider a two-country, two-good consumption-based asset-pricing model with a representative agent with logarithmic preferences for each country, as in Pavlova and Rigobon (2007, 2008). Embed in each economy a representative firm financed by equity and debt. Each economy then consists of a corporate debt market and an equity market. Furthermore, assume that there is a government in each country that collects tax revenues from corporate income and issues some debt. The government debt in the developed country is default risk-free, whereas the government debt in the emerging country is not. Defaulting on sovereign debt triggers a change in the macroeconomic regime, and the emerging economy enters recession. I thus embed sovereign credit risk into a general equilibrium model in the presence of defaultable firms with macroeconomic conditions. This framework thus connects a number of models belonging to separate strands of the international economics and finance literature.

Sovereign creditworthiness plays a crucial role in the model because it influences the likelihood of remaining in the current macroeconomic regime. The government in the emerging economy can decide the timing of the default. Building on the sovereign credit risk literature, this decision is based on a

\[\text{\footnotesize²For co-movement across returns of different assets, see, for example, Kwan (1996), Jeanneret (2009), and the references therein; Regarding co-movement in international equity return volatilities, see, for example, Forbes and Rigobon (2002) and Bae et al. (2003); For the effect of equity return volatility in the U.S. on sovereign credit risk in emerging markets, see, for example, Longstaff et al. (2008), Pan and Singleton (2008), Remolona et al. (2008), and Jeanneret (2009);}

\[\text{\footnotesize³See, for example, Gibson and Sundareshan (2001), François (2006), Yue (2006), Arellano (2008), and Jeanneret (2009).} \]
cost-benefit analysis. On the one hand, the sovereign repays only a fraction of its debt upon default, while still benefiting from the use of this debt to foster economic growth. On the other hand, it is assumed that defaulting causes the emerging economy to be in recession, which is characterized by a contraction in production. I build on the recent literature that brings macroeconomic conditions into the corporate capital structure model, assuming the regime is Markov.\(^4\) However, I model the change of regime as an endogenous decision of the government to default on its debt. Thus, avoidance of this default effect on economic performance provides the sovereign’s country motivation not to default. Not only are a firm’s earnings reduced in times of recession, but also default losses that occur during these times affect investors more through the higher price of risk. These factors make such a firm more likely to default in a recession. Following the capital structure literature,\(^5\) the shareholders determine the default decision by trading off the tax benefits of debt and bankruptcy costs in default. The probability of defaulting eventually affects the value of the emerging firm’s and the developed firm’s assets. This framework is thus useful in exploring the relationships between equity returns, corporate credit risk, and sovereign credit risk within a country. In addition, the international dimension of the model calls for an investigation of co-movement across asset returns traded in different countries.

The foreign exchange market acts as the main channel through which shocks are transmitted internationally,\(^6\) thus building on Cole and Obstfeld (1991). The exchange rate is obtained via equilibrium conditions from the endogenous pricing kernel for each economy and is equal to the terms of trade.\(^7\) Thus, the propagation of shocks from one country to another arises from a Ricardian response to economic shocks. Co-movement in firm earnings arises in response to a production shock in one of the countries despite the independence of output innovations among countries. As a result, the dynamics of the exchange rate obtained in the model play an important role in the explanation of co-movement in international asset prices. The consideration of an endogenous exchange rate is also crucial when one is investigating sovereign and corporate creditworthiness because of the so-called “goods mismatch” in both the sovereign and corporate balance sheets. This goods mismatch arises from balance sheets being heavily tilted towards debt denominated in a world numéraire and local-good denominated revenue. Exchange rate depreciation may then have a negative balance sheet effect because it decreases the worth of

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\(^4\) See, for example, Hackbarth, Miao, and Morellec (2006), Bhamra, Kuehn, and Strebulavaev (2009a,b), and Chen (2009).

\(^5\) See, for example, Mello and Parsons (1992), Leland (1994, 1998), and Morellec (2004).

\(^6\) The presence of a contagion effect of economic shocks through the foreign exchange market is in line with Bae, Karolyi, and Stulz (2003) and Ehrmann, Fratzscher, and Rigobon (2005). Both studies conclude that movements in exchange rates explain a large fraction of the contagion across international equity markets.

\(^7\) For reference, see Dumas (1992), Backus et al. (2001), Brandt et al. (2006), Pavlova and Rigobon (2007, 2008), and Bakshi et al. (2008).
government fiscal revenues and corporate earnings in terms of world numéraire, thereby reducing their capacity to service foreign debt. As such, changes in the exchange rate are intimately related to equity, corporate debt, and sovereign debt prices. This paper is the first attempt to link endogenous exchange rate movements with corporate and sovereign default risk.

The rest of the paper is organized as follows. Section 2 outlines a theoretical understanding of endogenous default policy for a firm and a sovereign country embedded in a two-country consumption-based asset-pricing model. A test of the model is described in Section 3. Section 4 offers new implications yielded by the model. I conclude my analysis in Section 5.

2 The Model

The world consists of two types of countries, namely, emerging and developed. A developed country is a large economy with a default-free government and a firm. An emerging country is a small market economy with a defaultable government and a firm. Each government levies taxes on corporate profit and is financed by external debt. The firms are financed by equity and debt. Households exist in both countries; they are the owners and the lenders of the firms and the lenders of the governments. Financial markets are complete before and after default. The tax environment consists of a constant tax rate $\tau$ for corporate income and a zero tax rate for individual income. All parameters in the model are assumed to be common knowledge.

2.1 Assumptions

2.1.1 Dynamics of the Economies and Macroeconomic Regimes

Each country consists of a representative firm that produces a country-specific perishable good. Let $Y_t$ denote the perpetual stream of output produced by the firm located in the developed economy at time $t$, which evolves according to the process

$$\frac{dY_t}{Y_t} = \theta_y dt + \sigma_y W^y_t$$

where $W^y_t$ is a Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The standard filtration of $W^y_t$ is $\mathcal{F}_t = \{\mathcal{F}_t : t \geq 0\}$. The conditional moments $\theta_y$ and $\sigma_y$ represent the expected growth rate and the volatility of the representative firm’s output in the developed economy.
Regarding the emerging economy, I build on Hackbarth, Miao, and Morellec (2006), Bhamra, Kuehn, and Streuballaev (2009a,b), and Chen (2009) to allow for the presence of different macroeconomic regimes. I presume that the emerging country is characterized by two different states, namely, a normal regime $H$ and a low, or recession, regime $L$. The dynamics of the perpetual stream of output generated by the emerging country’s representative firm is governed by the process

$$f(A_t, X_t) = A_t X_t$$

where $A$ is an aggregate default shock, and $X$ is the level of output produced by the emerging firm only accounting for uncertainty in firm-level productivity. The aggregate shock $A$ determines the state of the economy and takes only two values, $A_H$ and $A_L$ with $0 < A_L < A_H$. The transition of the state into the recession regime is characterized by a fall in the production of the emerging good, which is consistent with the empirical evidence that defaulting on external debt induces significant costs for economic activity.\(^8\)

While Hackbarth, Miao, and Morellec (2006), Bhamra, Kuehn, and Streuballaev (2009a,b), and Chen (2009) assume the regime is Markov, I model the change of the regime as an endogenous decision of the government. That is, the normal regime $H$ prevails until the government of the emerging country defaults on its debt. I presume that $X$ is independent of $A$ and evolves according to the process

$$\frac{dX_t}{X_t} = \theta_x dt + \sigma_{x,x} dW^x_t + \sigma_{x,y} dW^y_t$$

where $W^x_t$ is a Brownian motion independent of $W^y_t$, which generates idiosyncratic shocks specific to the emerging firm, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The standard filtration of $W^x_t$ is $F = \{ \mathcal{F}_t : t \geq 0 \}$. The representative firm’s idiosyncratic volatility and systematic volatility are denoted by $\sigma_{x,x}$ and $\sigma_{x,y} \equiv \kappa \sigma_y$, respectively. Finally, it is assumed that greater sovereign borrowing enhances economic growth through higher productivity growth. The growth rate of the firm is defined by

$$\theta_x = \tilde{\theta}_x + \theta_{x,c} C, \quad \theta_{x,c} > 0$$

The government thus plays an important role in the path of the emerging economy through its decision to issue and default on its debt. On the one hand, the issuance of greater sovereign debt allows for the

\(^8\)See, for example, Reinhart et al. (2003), De Paoli, Hoggarth, and Saporta (2006), and Sturzenegger and Zettelmeyer (2006).
fostering of economic growth. On the other hand, the increase in indebtedness raises the risk of entering recession through a sovereign default crisis and thus the risk of experiencing a contraction in output.\footnote{Pattillo, Poirson, and Ricci (2004) analyze 61 developing countries over the period 1969-1998 and find strong empirical support for this nonlinear impact of debt on growth (the “debt Laffer curve”): at low levels of debt, debt has positive effects in total factor productivity growth; but for high levels of debt, additional debt has negative impact on growth through the increased likelihood that in the future debt will be larger than the country’s repayment ability (the “debt overhang” effect).}

### 2.1.2 Investor Preferences and Consumption

There is a large number of infinitely-living households in both economies. I assume that the representative household has logarithmic preferences, which allow for closed-form solutions for consumption allocations and the terms of trade, as well as ensure a constant marginal rate of substitution between goods. Following Pavlova and Rigobon (2007, 2008), there is heterogeneity in consumer tastes to capture the possible home bias in the consumption baskets. The weights of the emerging good in the utility function of the emerging country and the developed country are expressed by $a_x$ and $a_y$, respectively.

I determine the equilibrium allocation by solving the world social planner’s problem to ensure Pareto optimality. The initial wealth of the representative household of each country is such that the central-planning welfare function allocates weights of $\lambda_x$ and $\lambda_y \equiv 1 - \lambda_x$ to the utility levels of the households of the emerging and the developed country, respectively. Accordingly, the planner chooses country consumption so as to maximize the weighted sum of the utilities of the representative agents:

$$U \equiv \max_{E_t} \int_0^\infty e^{-\rho t} \lambda_x \left( a_x \log(C_{xx,i,t}) + (1 - a_x) \log(C_{xy,t}) \right) dt$$

$$+ \int_0^\infty e^{-\rho t} \lambda_y \left( a_y \log(C_{yx,i,t}) + (1 - a_y) \log(C_{yy,t}) \right) dt, \quad i = \{L,H\}$$

subject to the resource constraints

$$C_{xx,i,t} + C_{yx,i,t} = A_i X_t$$

$$C_{yy,t} + C_{xy,t} = Y_t$$

where $\rho$ is the rate of time preference, and $C_{kl,i}$ denotes consumption of good $l$ by the representative agent of country $k$ in regime $i$. The optimal allocation of consumption is determined by

$$C_{xx,i,t} = \frac{\lambda_x a_x}{\lambda_y a_y + \lambda_x a_x} A_i X_t, \quad C_{yx,i,t} = \frac{\lambda_y a_y}{\lambda_y a_y + \lambda_x a_x} A_i X_t, \quad i = \{L,H\}$$
The prices per unit of the emerging good $X$ and the developed good $Y$ are denoted by $P_x$ and $P_y$, respectively. I fix the world numéraire basket as $P_1^{1-\alpha}P_\alpha$ with $\alpha \in (0,1)$ and normalize the price of this basket as equal to unity.\footnote{As an alternative world numéraire basket, one could use $\alpha P_x + (1-\alpha) P_y$ with $\alpha \in (0,1)$, and normalize the price of this basket to be equal to unity (see Pavlova and Rigobon, 2007). However, this basket appears to be much less tractable when computing asset prices.}

### 2.1.3 Pricing Kernels and Risk-free Interest Rates

The stochastic discount factor $\xi_y^t$ that prevails in the above competitive equilibrium for the developed economy is given by (see Appendix 5.1.1)

$$\frac{d\xi_y^t}{\xi_y^t} = -r_y dt - \sigma_y dW_y^t$$

with

$$r_y = \rho + \theta_y - \sigma_y^2$$

where $r_y$ is the interest rate on the default-free government bond, and $\sigma_y$ is the risk price for systematic Brownian shocks $W_y^t$. The stochastic discount factor is driven by the same set of shocks that drive aggregate output in the developed economy. As systematic shocks affect the marginal utility of investors through today’s consumption levels, the risk price of these shocks rises with economic volatility. A higher level of uncertainty, or a lower economic growth rate, then induces greater demand for the risk-free government bond. This flight-to-quality response lowers the risk-free interest rate in periods of distress.

Similarly, the stochastic discount factor $\xi_x^t$, which applies in the emerging economy and is denominated in the good of that country, follows the process

$$\frac{d\xi_x^t}{\xi_x^t} = -r_x dt - \sigma_{x,x} dW_x^t - \sigma_{x,y} dW_y^t$$

with

$$r_x = \rho + \theta_x - \sigma_x^2$$

where $r_x$ is the risk-free interest rate in terms of the emerging good, and $\sigma_x^2 = \sigma_{x,y}^2 + \sigma_{x,x}^2$. The pricing
kernels $\xi^x_t$ and $\xi^y_t$ can be used to compute prices of any contingent asset, irrespective of the good in which the asset is denominated. There exists only one risk-free asset, namely, the developed country’s government bond denominated in the developed good. As such, the risk-free rate $r_x$ represents the rate of return on the investment in this risk-free asset from the perspective of an emerging country’s investor. As both $r_x$ and $r_y$ are returns on the same asset, $r_x$ refers to the risk-free rate measured in the emerging good, and $r_y$ refers to the one denominated in the developed good.

2.2 The Exchange Rate

Within the model, the exchange rate is equal to the terms of trade and is defined in terms of the developed country’s good per unit of the emerging country’s good. Following Dumas (1992), the real exchange rate $S$ is then expressed by the ratio of either country’s marginal utilities of the emerging and developed goods:

$$S_{i,t} = \frac{P_{y,t}}{P_{x,t}} = \frac{\xi^y_{i,t}}{\xi^x_{i,t}} = \frac{A_i X_t}{Y_t}, \quad i = \{L, H\}$$

(15)

with

$$S = \frac{\lambda_y (1 - a_y) + \lambda_x (1 - a_x)}{\lambda_x a_x + \lambda_y a_y}$$

(16)

In competitive equilibrium, the prices $P_x$ and $P_y$ per unit of the good $X$ and $Y$ are equal to the pricing kernels (or state-price densities) $\xi^x$ and $\xi^y$, respectively. The pricing kernels $\xi^x$ and $\xi^y$ essentially capture the same intertemporal price expressed with a different good unit. They represent the probability-weighted cost of receiving a state-contingent payoff sometime in the future. Therefore, consistent with Backus et al. (2001), Brandt et al. (2006), and Bakshi et al. (2008), the exchange rate can also be expressed as the ratio of the pricing kernels in the two economies. Given the preferences of agents, these pricing kernels are unique, as is the ratio of the two. From Itô’s lemma, the exchange rate $S$ follows the process (see Appendix 5.1.2)

$$\frac{dS_t}{S_t} = \theta_s dt + \sigma_{x,x} dW^x_t - (\sigma_y - \sigma_{x,y}) dW^y_t$$

(17)

with

$$\theta_s = r_x - r_y + \sigma_x^2 - \sigma_y \sigma_{x,y}$$

(18)

The mean appreciation rate $\theta_s$ is the difference between the emerging risk-free interest rate and the
developed risk-free interest rate $r_x$ and $r_y$, respectively, augmented by some compensation for bearing aggregate output risk. When a country enjoys an output shock, the exchange rate adjusts exactly to offset any net payoff. This exchange rate satisfies the no-arbitrage conditions, which prove the redundancy of having a risk-free bond in each country. The exchange rate plays an important role in linking asset prices in the two economies. While the key drivers of the level of the exchange rate are the relative preferences for goods and the central planner’s welfare weights, the dynamics (i.e., time-variation) of the exchange rate solely depend on the dynamics of macroeconomic fundamentals.

2.3 The Government of the Emerging Economy

The government of the emerging economy raises fiscal revenues by taxing the value of the emerging firm’s earnings (EBIT) at the tax rate $\tau$ net of the tax-deductible debt service of the firm. I assume that government debt is denominated in the world basket. The capacity to service this debt depends on the dynamics of the government revenues $R = \tau (Z - K - C_f)_{t \geq 0}$, where $Z \equiv P_x AX$ denotes the firm’s revenues; $K$ is the firm’s operating costs per unit of time; and $\tau C_f$ is the firm’s tax-shield. All variables are measured in units of the world basket. From Itô’s formula, the process $Z$ satisfies (see Appendix 5.1.3)

$$\frac{dZ_t}{Z_t} = \theta_z dt + \sigma_{z,x} dW^x_t + \sigma_{z,y} dW^y_t$$

At the time when the government defaults on its debt, the emerging firm’s revenues fall by

$$dZ_t = -\left(1 - \left(A_L/A_H\right)^{1-\alpha}\right) Z_t = -\gamma_z Z_t$$

As the emerging good becomes relatively scarce after the sovereign default, the relative price of the emerging good increases, which partially offsets the initial decline in production. Entering recession also induces firms to reduce their operating costs from $K$ to $K_L = (1 - \gamma_z) K$ in response to the reduction in revenues.

If the government never issues any debt, the value of its assets is the expected value of future fiscal revenues discounted with the corresponding stochastic discount factor. Equivalently, the value of its assets is the expected value of these revenues discounted with the world risk-free rate under the risk-neutral probability measure. The risk-neutral measure $Q$ adjusts for risks by changing the distributions of shocks. Cash flows are risky for an investor when they are positively correlated with its marginal
utility, which is accounted for by lowering the expected growth rate under \( \mathbb{Q} \). Under \( \mathbb{Q} \) (see Appendix 5.1.5), the expected growth rate of government revenues is

\[
\hat{\theta}_z = \theta_z - \sigma^2_{z,x} - \sigma^2_{z,y} \tag{21}
\]

The sovereign strategically declares that it is defaulting on its debt obligation when the fiscal revenues are such that the firm’s revenues fall below an endogenous default boundary \( Z^D < Z_0 \) at time \( T(Z^D) = \inf \{ t \geq 0 \mid Z_t \leq Z^D \} \). However, it is assumed that the sovereign can always meet the required coupon payment \( C \), such that \( R |_{T(Z^D)} = \tau (Z^d - K - C_f) \geq C \) is always satisfied for the parameter values under consideration. In contrast to corporations, sovereigns are unable to issue additional financial claims to cover a revenue shortage.

### 2.3.1 The Price of Sovereign Debt

All terms in the debt contract are denominated in units of the world basket. The lenders require a risk-free rate of return \( r_z \) per unit of time, which corresponds to the return on the developed economy’s default-free bond measured in the world basket (see Appendix 5.1.4):

\[
r_z = \rho + \theta_z - (\sigma^2_{z,x} + \sigma^2_{z,y}) \tag{22}
\]

The sovereign pays a constant total coupon \( C \) at each moment in time. Upon defaulting, the sovereign and its lenders restructure the terms of the debt contract and agree on a reduction \( 0 \leq \phi \leq 1 \) of the debt service.\(^{12}\) I assume that the sovereign cannot scale up its debt after default.

The value of sovereign debt is (see Appendix 5.2.1)

\[
D(Z) = \mathbb{E}_0^\mathbb{Q} \left[ \int_0^{T(Z^D)} C e^{-r_z t} dt \right] + \mathbb{E}_0^\mathbb{Q} \left[ \int_{T(Z^D)}^\infty (1 - \phi) C e^{-r_z t} dt \right] \tag{23}
\]

The first term is equal to the present value of the promised cash flows \( C \) to debtholders until default. Default occurs when the level of the firm’s revenues reaches the default boundary at default time \( T(Z^D) \).

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\(^{11}\)Sovereigns do not tend to default once but several times (Reinhart et al., 2003). Generalizing the framework to account for multiple defaults is left for future research.

\(^{12}\)For simplicity, the recovery rate is assumed to be exogenous. Alternatively, Yue (2006) and Jeanneret (2009) develop a model that accounts for endogenous renegotiation upon default. Once default occurs, the sovereign country and its lenders renegotiate the terms of their debt contracts, which determines the recovery rate. The outcome of the restructuring process involves a Nash bargaining solution. However, the consideration of an endogenous recovery rate would not change the results of this paper.
The likelihood of defaulting thus increases when the firm’s level of production decreases and/or when the terms of trade depreciate. The second term corresponds to the present value of the recovered value of the debt after the government has defaulted. Lenders anticipate the opportunistic behavior of the sovereign and reflect the associated wealth loss in the pricing of the sovereign debt.

2.3.2 The Decision to Default and Sovereign Credit Spread

The sovereign uses external debt to foster its future revenues by fostering economic growth. On the other hand, defaulting on external debt induces significant costs for economic activity. In the event of default, the emerging economy enters the recession regime, thereby triggering a fall in output and, thus, in the firm’s earnings. Avoidance of this default cost in terms of economic performance, in particular for future fiscal revenues, is the sovereign country’s motivation not to default.

Define the sovereign’s wealth \( W \) as the present value of future fiscal revenues net of debt; for given levels of debt service \( C \) and default boundary \( Z^D \), this satisfies the following:

\[
W(Z) = \mathbb{E}_0^Q \left[ \int_0^{T(Z^D)} \tau (Z_{H,t} - K - C_f) e^{-\tau_z t} dt \right] + \mathbb{E}_0^Q \left[ \int_{T(Z^D)}^{\infty} \tau (Z_{L,t} - K_L - C_f) e^{-\tau_z t} dt \right] - D(Z^D)
\]

The first part of the right-hand side of Eq. (24) is the present value of the sovereign’s revenues until default. The second term accounts for the revenue loss in the aftermath of default. The sovereign’s default policy is characterized by the default boundary \( Z^D \), which is chosen to maximize the present value of future fiscal revenues. This value would be maximized by setting \( Z^D \) as low as possible to avoid a change in the macroeconomic regime. However, the requirement that the sovereign wealth’s value \( W \) must be positive prevents \( Z^D \) from being arbitrarily small. Thus, the lowest possible value for \( Z^D \) consistent with \( W(Z) \geq 0 \) for all \( Z > Z^d \), maximizes the sovereign wealth’s value \( W \), such that the smooth-pasting condition \( \frac{\partial W(Z)}{\partial Z} |_{Z=Z^D} = \frac{\tau (1-\gamma_z)}{\tau_z - \theta_z} \) is satisfied (see Appendix 5.2.2 for the formulae).

The incentive of the sovereign to default rises with the level of indebtedness \( C \), the firm’s operating costs \( K \), the non-repaid fraction of debt upon default \( \phi \), and the volatility of output in either economy \( \sigma_x \) and \( \sigma_y \). However, it decreases with the revenue loss \( \gamma_z \) upon a change in regime, the tax rate \( \tau \), and the growth rate of output in either economy \( \theta_x \) and \( \theta_y \). The incentive to default early, expressed within the model by raising \( Z^D \), is to lower the expected value of debt repayment. However, debtholders take this opportunistic behavior into account in the market valuation of debt. The market yield spread, which
is a measure of the market’s perception of default risk, is thus determined by

\[ CS(Z, C) = \frac{C}{D(Z, C)} - r_z \]  

(25)

2.4 The Firm in the Emerging Economy

The emerging firm is linked to the government because, on the one hand, the government raises its fiscal revenues by taxing this firm’s earnings (EBIT), while on the other hand, the government’s decision to default affects the probability of experiencing a contraction of output in a recession. I first model the emerging firm’s financing decision and then determine the value of its assets. I assume that the management acts in the best interests of the shareholders. Because the firm pays taxes on its earnings, it thus has an incentive to issue debt. I consider an exogenous infinite-maturity debt structure in a stationary environment. One the one hand, the perpetuity feature is shared with numerous other models, including those presented in Fischer, Heinkel, and Zechner (1989), Leland (1994), and Strebulaev (2007). On the other hand, the level of debt is assumed to be exogenous because most of the time firm leverage deviates from “optimal leverage”.

Once debt has been issued, the only decision for shareholders is to select the default policy that maximizes equity value. The shareholders strategically declare default on their debt obligation when the firm’s revenues fall below the default boundary \( Z_f^D < Z_0 \) at time \( T(Z_f^D) = \inf\{t \geq 0 \mid Z_t \leq Z_f^D \} \). I presume that the firm is liquidated when it defaults on its debt obligations. In contrast to the case of sovereign debt (Section 2.3.1), debt contracts are assumed to be non-renegotiable upon default. At that time, a new representative firm with identical value and level of debt emerges, which ensures continuity in production and consumption. In the event of default, the value of the firm is \( (1 - \eta)V_u(Z_{t=T(Z_f^D)}) \), where \( \eta \in (0, 1) \) is the fraction of asset value lost in default, and \( V_u(Z_{t=T(Z_f^D)}) \) is the value of the unlevered firm’s assets. The bankruptcy costs \( \eta \left(Z_f^D - K\right) \) consist of liquidation fees paid to a third party (e.g., lawyers) that are subject to taxes. Therefore, the government raises taxes from the new firm’s earnings \( \tau (1 - \eta) \left(Z_f^D - K\right) \) and from the third party’s gain \( \tau \eta \left(Z_f^D - K\right) \) after the firm’s default. The government’s level of fiscal revenues thus remains identical before and after the firm’s default.

\[ ^{13} \text{For reference, see Strebulaev (2007) and Bhamra et al. (2009b). Because of issuance costs, most firms optimally refinance only periodically. Hence, as shown by Strebulaev (2007), if leverage deviates from its target substantially, then the response of firms to changes in economic conditions will not be in line with the predictions of comparative statics at refinancing points.} \]
2.4.1 Valuation of Firm Securities

I first derive the values of corporate debt and equity and then determine the default thresholds selected by shareholders. I start by determining the value of corporate debt for a given default boundary. The debt is denominated in the world basket and has value equal to the sum of the present value of the earnings that accrue to debtholders until the default time and the change in this present value that arises in default. I follow Mello and Parsons (1992) and Leland (1994) by presuming that the value of the firm upon default equals the value of unlevered assets, i.e., the unlimited liability value of a perpetual claim to the current flow of after-tax earnings. The flow of earnings is given by $\text{EBIT}_H = A_H P_x X - K$ in the normal regime and by $\text{EBIT}_L = A_L P_x X - K_L = (1 - \gamma_z) (P_x X - K)$ in the recession regime.

The value of corporate debt depends on whether the sovereign country has defaulted on its debt as well as whether it is optimal for the firm to default before or after the government defaults. The value of the firm’s debt consistent with these two cases is (see Appendix 5.3.1-2)

$$D_f(Z) \mid T^- \leq T^+ = \mathbb{E}_0^Q \left[ \int_0^{T^+} C_f e^{-r_z t} dt + e^{-r_z T} D_{f,L}(Z \mid t=T^-) \right]$$

with

$$D_{f,L}(Z \mid t=T^-) = \mathbb{E}_T^- \left[ \int_{T^-}^{T^+} C_f 1_{[T(Z_D^p) > T(Z_D^r)]} e^{-r_z t} dt \right]$$

$$+ \mathbb{E}_T^- \left[ \int_{T^-}^{T^+} (1 - \eta)(1 - \tau) (Z_{H,t} - K) 1_{[T(Z_D^p) \leq T(Z_D^r)]} e^{-r_z t} dt \right]$$

$$+ \mathbb{E}_T^- \left[ \int_{T^+}^{-\infty} (1 - \eta)(1 - \tau) (Z_{L,t} - K_L) e^{-r_z t} dt \right]$$

where $T^+ = T \left( Z_f^D \right) \lor T \left( Z_D^p \right)$, $T^- = T \left( Z_f^D \right) \land T \left( Z_D^l \right)$, and $1[a]$ is an indicator function equals to one if the function $a$ is true and zero, otherwise.

I now turn to the total value of the levered firm. The firm value equals the unlimited liability value of a perpetual claim to the current flow of after-tax earnings, plus the present value of a perpetual claim to the current flow of tax benefits of debt, minus the change in those present values arising in default.
Thus, the levered firm value $V(Z)$ satisfies (see Appendix 5.3.3-4)

$$V(Z) |_{T^- \leq T^+} = E_Q^0 \left[ \int_0^{T^-} ((1 - \tau) (Z_{H,t} - K) + \tau C_f) e^{-r_z t} dt \right]$$

$$+ E_Q^0 \left[ e^{-r_z T^-} V_L(Z | t=T^-) \right]$$

with

$$V_L(Z | t=T^-) = E_Q^T \left[ \int_{T^-}^{T^+} ((1 - \tau) (Z_{L,t} - K_L) + \tau C_f) 1_\{[T(Z_f) \leq T(Z_D)]\} e^{-r_z t} dt \right]$$

$$+ E_Q^{T^-} \left[ \int_{T^-}^{T^+} ((1 - \tau) (Z_{L,t} - K_L) + \tau C_f) e^{-r_z t} dt \right]$$

$$+ E_Q^{T^-} \left[ \int_{T^+}^{\infty} (1 - \eta) (1 - \tau) (Z_{L,t} - K_L) e^{-r_z t} dt \right]$$

In the absence of arbitrage, the levered firm value equals the sum of debt and equity values. Hence, the value of the firm’s equity $E(Z)$ is determined by

$$E(Z) |_{T^- \leq T^+} = V(Z) |_{T^- \leq T^+} - D_f(Z) |_{T^- \leq T^+}$$

The volatility of the firm’s equity returns is given by (see Appendix 5.3.7)

$$\sigma_{E(Z)} = \frac{\partial E(Z)}{\partial Z} \sqrt{\sigma_{Z,x}^2 + \sigma_{Z,y}^2}$$

Equity return volatility is predicted to depend negatively on the level of economic output in either country $X$ and $Y$, the growth rate of output in either economy $\theta_x$ and $\theta_y$, and the corporate tax rate $\tau$. However, equity return volatility is predicted to increase with macroeconomic volatility in either country $\sigma_{x}$ and $\sigma_{y}$, financial leverage $C_f$, operational costs $K$, and sovereign indebtedness $C$.

### 2.4.2 The Decision to Default

I assume that the only decision for shareholders is to select the default policy that maximizes the value of equity. Within the model, markets are frictionless, and default is triggered by the shareholder decision to optimally cease injecting funds into the firm; see also Leland (1998) and Morellec (2004). The firm’s default policy is characterized by the default boundary $Z_f^D |_{T^- \leq T^+}$ and maximizes the shareholder
value such that the smooth-pasting condition \( \frac{\partial[E(Z)\mid_T \leq T+]}{\partial T} \bigg|_{Z = Z^D_T \mid_T \leq T+} = 0 \) is satisfied (see Appendix 5.3.5-6 for the value of \( Z^D_f \)). The decision to default before or after the government is determined \textit{ex ante} to maximize the shareholder value. The optimal default boundary thus satisfies

\[
Z^D_f = \begin{cases}
Z^D_f \mid_T (Z^D_f) \leq T(D) & \text{if } E(Z) \mid_T (Z^D_f) \leq T(D) \\
Z^D_f \mid_T (Z^D_f) \leq T(D) & \text{if } E(Z) \mid_T (Z^D_f) > T(D)
\end{cases}
\] (37)

The above rule determines the conditions under which the firm defaults before or after the government (see Appendix for the formulae). The model predicts that the firm tends to default first when i) the firm is relatively more leveraged than the government (i.e., high \( C_f \) and low \( C \)); ii) the firm has large operating costs (i.e., high \( K \)); iii) the economic loss upon the change of regime from \( A_H \) to \( A_L \) is important (i.e., high \( \gamma_z \)); iv) volatility in either country’s economic fundamentals is low (i.e., low \( \sigma_x \) and \( \sigma_y \)); v) either economy grows rapidly (i.e., high \( \theta_x \) and \( \theta_y \)); and finally, vi) when the corporate tax burden is severe (i.e., high \( \tau \)).

2.5 The Firm in the Developed Economy

The value of the assets of the representative firm in the developed economy is obtained under the same assumptions as for the firm of the emerging economy. This firm’s revenues are equal to \( Y_{p_y} \equiv SZ \); meanwhile the operating costs are \( K_y \), and the debt coupon is \( C_{f_y} \). All variables are measured in units of the world basket. The debt value of the developed country’s firm is given by (see Appendix 5.4.1-2)

\[
D_{f_y}(Z) \mid_T = \mathbb{E}_0^Q \left[ \int_T^T C_{f_y} e^{-r_+ t} dt + e^{-r_+ T} D_{f_y,L}(Z \mid t = T^-) \right]
\] (38)

with

\[
D_{f_y,L}(Z \mid t = T^-) = \mathbb{E}_T^Q \left[ \int_T^{T^+} C_{f_y} 1_{[T(Z^D_f) > T(Z^D_L)]} e^{-r_+ t} dt \right] + \mathbb{E}_T^Q \left[ \int_T^{T^+} (1 - \eta)(1 - \tau) (SZ_{H,t} - K_y) 1_{[T(Z^D_f) \leq T(Z^D_L)]} e^{-r_+ t} dt \right] + \mathbb{E}_T^Q \left[ \int_T^{\infty} (1 - \eta)(1 - \tau) (SZ_{L,t} - K_y,L) e^{-r_+ t} dt \right]
\] (39)

where \( T^+ = T \left( Z^D_{f,y} \right) \vee T \left( Z^D_L \right) \) and \( T^- = T \left( Z^D_{f,y} \right) \wedge T \left( Z^D_L \right) \).
The total value of the levered firm, denoted by $V_y(Z)$, satisfies (see Appendix 5.4.3-4)

$$V_y(Z) |_{T^- \leq T^+} = \mathbb{E}^Q_T \left[ \int_{T^-}^{T^+} \left( (1 - \tau) (\overline{S}Z_{H,t} - K_y) + \tau C_{fy} \right) e^{-r t} dt \right]$$

$$+ \mathbb{E}^Q_T \left[ e^{-r T^-} V_{y,L}(Z |_{t=T^-}) \right]$$

(42)

with

$$V_{y,L}(Z |_{t=T^-}) = \mathbb{E}^Q_{T^-} \left[ \int_{T^-}^{T^+} \left( (1 - \eta) (1 - \tau) (\overline{S}Z_{H,t} - K_y) 1_{[T(Z_{fy}^p) > T(Z_D)]} e^{-r t} dt \right]$$

$$+ \mathbb{E}^Q_{T^-} \left[ \int_{T^-}^{T^+} \left( (1 - \tau) (\overline{S}Z_{L,t} - K_y) + \tau C_{fy} \right) 1_{[T(Z_{fy}^p) \leq T(Z_D)]} e^{-r t} dt \right]$$

$$+ \mathbb{E}^Q_{T^-} \left[ \int_{T^+}^{\infty} (1 - \eta) (1 - \tau) (\overline{S}Z_{L,t} - K_y) e^{-r t} dt \right]$$

(44)

(45)

(46)

The firm’s default policy is characterized by the default boundary $Z_{fy}^D$ and maximizes shareholder value such that the smooth-pasting condition $\frac{\partial [\mathbb{E}^Q_y(Z) |_{T^- \leq T^+}]}{\partial Z} |_{Z = Z_{fy}^D |_{T^- \leq T^+}} = 0$ is satisfied (see Appendix 5.4.5-6). The asset prices of the developed firm also depend on whether this firm chooses to default before or after the government of the emerging economy. This decision is determined ex ante to maximize the shareholder value. The conditions under which the developed firm defaults before, or after, the government of the emerging economy are similar to those obtained for the emerging firm (see Appendix 5.4.5-6 for the formulae). Namely, the developed firm tends to default before the government when i) the firm is relatively more leveraged than the government of the emerging economy (i.e., high $C_{fy}$ and low $C$); ii) the firm has large operating costs (i.e., high $K_y$); iii) the economic loss upon the change of regime from $A_H$ to $A_L$ greatly impacts the emerging economy (i.e., high $\gamma_z$); iv) the volatility of either country’s economic fundamentals is low (i.e., low $\sigma_x$ and $\sigma_y$); v) both economies grow rapidly (i.e., high $\theta_x$ and $\theta_y$); and finally, vi) when the corporate tax burden is severe (i.e., high $\tau$).

2.6 Probability of Defaulting

The consumption-based model allows one to derive the probability of defaulting on sovereign and corporate debts under both the physical and the risk-neutral measures. The model endogenously links unobservable risk-neutral probabilities to observable actual probabilities via the market price of consumption risk. The probability of defaulting corresponds to the likelihood that the default boundary $Z^*$ is
reached within time period $T$. Under the risk-neutral measure, this probability is defined by

$$
Q\left( \inf_{0 \leq t \leq T} Z_t \leq Z^* \mid Z_0 > Z^* \right) = \phi \left( \frac{\ln \left( \frac{Z^*}{Z_0} \right) - \left( \hat{\theta}_z - \frac{\sigma^2}{2} \right) T}{\sigma_z \sqrt{T}} \right) 
$$

$$
+ \left( \frac{Z^*}{Z_0} \right)^{\frac{2\hat{\sigma}_z}{\sigma_z^2} - 1} \phi \left( \frac{\ln \left( \frac{Z^*}{Z_0} \right) + \left( \hat{\theta}_z - \frac{\sigma^2}{2} \right) T}{\sigma_z \sqrt{T}} \right)
$$

(47)

where $\phi(\cdot)$ is the cumulative density of a standard normal distribution, and

$$
Z^* = \begin{cases} 
Z^D & \text{for the government in the emerging country} \\
Z^D_f & \text{for the firm in the emerging country} \\
Z^D_{fy} & \text{for the firm in the developed country}
\end{cases}
$$

(48)

The first term of Equation (47) denotes the probability that $Z_T \leq Z^*$ at the horizon $T$, whereas the second term is the probability that $Z_t \leq Z^*$ between $t = 0$ and time $T$, conditional on $Z_T > Z^*$ at $T$. The physical probability of defaulting is given by the above expression when the physical growth rate $\theta_z$ replaces the risk-neutral one $\hat{\theta}_z$.

3 A Structural Estimation of the Model

This section provides a formal test of the model. It investigates whether the presence of the risk of a change in macroeconomic regime triggered by a sovereign default crisis helps explain the level and volatility of international equity returns. I first describe the data, discuss the rationale behind testing this hypothesis, and finally present the results.

3.1 Data and Calibration

The analysis uses monthly observations on a country’s level of industrial production for Brazil and the U.S. over the period from June 1994 through December 2008 to generate each country’s monthly equity prices as predicted by the model. The Figure 1 displays the dynamics of each country’s industrial production.
Figure 1: Industrial Production in the U.S. and Brazil, 1994-2008. This figure compares industrial production in the U.S. with industrial production in Brazil over the period from June 1994 through December 2008. Both series are normalized to unity.

I calibrate the model so that the mean and the standard deviation of the developed and the emerging country’s output growth equal the U.S. and Brazilian annual growth rates of industrial production, respectively. The parameter values related to firms are chosen to match the characteristics of a representative firm in the U.S. and Brazil, and those related to sovereign debt match the indebtedness level of the Brazilian’s government. The parameter values are presented in Table I.

Table I: Parameter Choices. This table presents the parameter values adopted for the estimation and simulation. All variables are annualized when applicable.
<table>
<thead>
<tr>
<th>Variable</th>
<th>Symbol</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preferences</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time preference</td>
<td>$\rho$</td>
<td>0.05</td>
<td>Author’s assumption</td>
</tr>
<tr>
<td>Preference of the emerging agents for the emerging good</td>
<td>$a_x$</td>
<td>0.75</td>
<td>Author’s assumption</td>
</tr>
<tr>
<td>Preference of the developed agents for the emerging good</td>
<td>$a_y$</td>
<td>0.75</td>
<td>Author’s assumption</td>
</tr>
<tr>
<td>Central planner’s weight for the emerging economy</td>
<td>$\lambda_x$</td>
<td>0.25</td>
<td>Author’s assumption</td>
</tr>
<tr>
<td>Developed economy</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Growth rate</td>
<td>$\theta_y$</td>
<td>0.01</td>
<td>Average growth rate of industrial production in the U.S. (1998-2008)</td>
</tr>
<tr>
<td>Volatility</td>
<td>$\sigma_y$</td>
<td>0.02</td>
<td>Growth rate volatility of industrial production in the U.S. (1998-2008)</td>
</tr>
<tr>
<td>Initial level of production</td>
<td>$Y$</td>
<td>100</td>
<td>[Normalization]</td>
</tr>
<tr>
<td>Emerging economy</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fixed growth rate</td>
<td>$\bar{\theta}_x$</td>
<td>0.02</td>
<td>Match average growth rate of industrial production in Brazil (1998-2008)</td>
</tr>
<tr>
<td>Variable growth rate</td>
<td>$\theta_{x,c}$</td>
<td>0.001</td>
<td>Match average growth rate of industrial production in Brazil (1998-2008)</td>
</tr>
<tr>
<td>Systematic component ($\sigma_y/\sigma_{x,y}$)</td>
<td>$\kappa$</td>
<td>0</td>
<td>Author’s assumption</td>
</tr>
<tr>
<td>Idiosyncratic volatility</td>
<td>$\sigma_{x,x}$</td>
<td>0.07</td>
<td>Growth rate volatility of industrial production in Brazil (1998-2008)</td>
</tr>
<tr>
<td>Initial level of production</td>
<td>$X$</td>
<td>100</td>
<td>[Normalization]</td>
</tr>
<tr>
<td>Government debt</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Debt service</td>
<td>$C$</td>
<td>15</td>
<td>Match Debt/GDP for Brazil (Sturzenegger and Zettelmeyer, 2006)</td>
</tr>
<tr>
<td>Haircut</td>
<td>$\phi$</td>
<td>0.66</td>
<td>Moody’s (2006)</td>
</tr>
<tr>
<td>Firms</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Debt service in emerging country</td>
<td>$C_{f}$</td>
<td>20</td>
<td>Match leverage ratio in Brazil (Lins, 2003)</td>
</tr>
<tr>
<td>Debt service in developed country</td>
<td>$C_{fy}$</td>
<td>8</td>
<td>Match leverage ratio in U.S. (Morrellec et al., 2009)</td>
</tr>
<tr>
<td>Fixed costs in emerging country</td>
<td>$K$</td>
<td>160</td>
<td>Match leverage ratio in Brazil (Lins, 2003)</td>
</tr>
<tr>
<td>Fixed costs in developed country</td>
<td>$K_y$</td>
<td>40</td>
<td>Match leverage ratio in U.S. (Morrellec et al., 2009)</td>
</tr>
<tr>
<td>Bankruptcy costs</td>
<td>$\eta$</td>
<td>0.5</td>
<td>Morrellec et al. (2009)</td>
</tr>
<tr>
<td>Tax rate</td>
<td>$\tau$</td>
<td>0.3</td>
<td>Morrellec et al. (2009)</td>
</tr>
</tbody>
</table>

Financial data for this section consist of the S&P500 for the U.S. equity price index, MSCI Brazil for the Brazilian equity price index (measured in U.S. dollars), the Citigroup Big A Corporate 10y Bond index for prices on U.S. corporate bonds, the JPMorgan EMBI Total Return Brazil index for prices on sovereign debt, and finally the JPMorgan EMBI+ spreads Brazil for sovereign credit spreads in Brazil. Data are taken from two sources, namely, i) *Datastream* for equity and bond market indices and the U.S.
dollars/Brazilian Reals exchange rate, and ii) Bloomberg for the EMBI+ spreads. All series consist of daily or monthly observations from June 1, 1994, to December 31, 2008.

### 3.2 Rationale for the Testing Hypothesis

Thus far, the existing international asset pricing literature has largely ignored the presence of either defaultable debt in a firm’s balance sheet, operating costs, or the risk of a sovereign default crisis. For example, the model developed within this paper is essentially that of Pavlova and Rigobon (2007) when none of these elements is considered. In the absence of at least one of these features, the volatility of an unlevered firm’s equity return is equal the volatility of this firm’s revenues. In addition, when international trade is allowed, the terms of trade response to output shocks mitigates the effect of these initial shocks. Thus, the volatility of this unlevered firm’s equity return is predicted to be lower than the volatility of this firm’s output. This prediction is unfortunate; the data suggests that the volatility of equity returns is far greater than the volatility of output (see Table II). In this paper, we propose to resolve this issue by accounting for financial leverage, operational leverage, and macroeconomic regimes.

<table>
<thead>
<tr>
<th>Table II: Statistics on Industrial Production and Equity Markets, 1994-2008. This table compares the mean and the standard deviation (volatility) of industrial production’s growth with the mean and the volatility of returns on equity market indices for Brazil and the U.S. All values are annualized over the period 1994-2008.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Industrial Production Growth</td>
</tr>
<tr>
<td>Mean (%)</td>
</tr>
<tr>
<td>Brazil</td>
</tr>
<tr>
<td>United States</td>
</tr>
</tbody>
</table>

First, the presence of defaultable corporate debt generates the financial leverage introduced by Black (1976) and Christie (1982). When hit by a negative output shock, the value of a firm declines, which raises the probability of defaulting and lowers the value of equity (i.e., the junior claim) relative to the value of debt (i.e., the senior claim). This increase in the firm’s financial leverage raises the volatility of equity returns. Alternatively, the introduction of portfolio constraints can also increase equity return volatility. For example, Pavlova and Rigobon (2008) show that the presence of a constraint that limits the fraction of wealth at a country’s agents may invest in the assets of the other country amplifies the asset price reaction to economic shocks.
equity returns. Then, as illustrated in Figure 1, the sensitivity of a firm’s equity price to economic shocks is greater when the firm is levered than when it is not. Second, the introduction of constant production costs borne by a firm generates the operational leverage effect. While financial leverage is related to the firm’s capital structure, operational leverage is related to the asset structure. Lev (1974) demonstrated early on that operating leverage raises the volatility of a firm’s earnings, thereby increasing a firm’s equity return volatility (see Figure 1).

Figure 2: Equity Price, Economic Shocks, and Leverage Effects. This figure shows the model’s predicted effect of production shocks in the developed economy on the emerging firm’s equity price. The figure decomposes sensitivity to economic shocks depending on the various assumptions underlying the model (i.e., presence or absence of financial leverage, operational leverage, and sovereign default risk). The parameters of the models are those presented in Table I with $\gamma_z = 0.08$.

However, little is known about the influence of the risk of a sovereign default crisis on asset return dynamics. This paper argues that the presence of sovereign default risk can additionally raise the sensitivity of equity returns to economic shocks (Figure 1), which is due to the risk of a drop in firm earnings upon the change of macroeconomic regime. This effect is observed in both countries. The intuition behind this prediction is that the propagation of shocks from one country to another arises from a Ricardian response to economic shocks. To see this, consider a negative shock in the emerging economy. This shock is accompanied by a deterioration of the terms of trade $S = \frac{P_y}{P_x}$. Because one unit of production in this economy becomes relatively more valuable (i.e., $P_x$ increases), the terms of trade move in favor of the country experiencing a negative shock. However, the deterioration in the terms of trade implies a
decrease in the relative prices of the developed good $P_y$, leading to a fall in the value of the developed country’s output $P_yY$, although output $Y$ remains constant. When there is a risk that the emerging economy may enter a recession, which is triggered in the model through a sovereign default crisis in that country, firms in both countries then experience a decrease in the present value of their revenues, thereby depressing the value of their equity and increasing the volatility of their equity returns.

As a result, the model predicts that when financial leverage, operating leverage, and the presence of macroeconomic regimes are accounted for, small economic shocks in a given country can translate into large movements in international equity returns. The effect of financial and operational leverage is not new. I thus focus on the marginal influence of the risk of entering recession, as triggered by an endogenous sovereign default’s crisis, on the level and volatility of international asset returns.

3.3 GMM Estimation and the Choice of Moments

This section describes the econometric approach used in estimating the parameter of interest, $\gamma_z$. The goal is thus to examine whether the risk of a sovereign default crisis in emerging markets has an international influence, ceteris paribus, on the level and volatility of equity returns. The econometric approach consists of testing a set of overidentifying restrictions on a system of moment equations using the generalized method of moments (GMM) developed by Hansen (1982). The moments under consideration are the mean and variance of the equity returns in both the developed country and the emerging country. The GMM technique is particularly attractive for an estimation of this type of asset pricing model. First, the GMM approach does not require that the distribution of equity returns or equity return volatility be normal\textsuperscript{15}; second, the GMM estimators and their standard errors are consistent even if the assumed disturbances are conditionally heteroskedastic.

The GMM estimation procedure chooses the parameter estimates that minimize the quadratic form

$$J(\gamma_z) = m'(\gamma_z)W(\gamma_z)m(\gamma_z)$$

(49)

where $m(\gamma_z)$ is a vector of orthogonality conditions, and $W(\gamma_z)$ is a positive-definite symmetric weighting matrix. Because I consider more moment conditions than parameters, not all of the moment restrictions

\textsuperscript{15}The asymptotic justification for the GMM procedure requires only that the distribution of equity return, equity return volatility, and credit spread be stationary and ergodic and that the relevant expectations exist.
are satisfied. The weighting matrix $W(\gamma_z)$ determines the relative importance of the various moment conditions so as to give more weight to the moment conditions with less uncertainty. Following Hansen (1982), when equal to the inverse of the asymptotic covariance matrix, the weighting matrix $W(\gamma_z) = S^{-1}(\gamma_z)$ is optimal since $\gamma_z$ is determined with the smallest asymptotic variance. I estimate the covariance matrix using the Newey and West (1987) approach to account for heteroskedasticity and serial correlation with a correction for small samples. This covariance matrix is used to test the significance of the parameter.

The optimal weighting matrix $W(\gamma_z)$ requires an estimate of the parameter $\gamma_z$, yet at the same time, estimating the parameter $\gamma_z$ requires the weighting matrix. To solve this dependency, I account for a two-stage estimation method. I first set the initial weighting matrix to be equal to the identity matrix $W_0 = I$ and then calculate the parameter estimate. I then compute a new weighting matrix with the parameter estimate obtained at the first stage. The parameter $\gamma_z$ is obtained by matching the moments of the model to those of the data as closely as possible.

### 3.4 Empirical Results

In this section, I present the empirical results and examine the explanatory power of the asset-pricing model developed in this paper. The Table III reports the parameter estimate, asymptotic standard deviations and their associated $p$-values, and GMM minimized criterion ($\chi^2$) values.

#### Table III: Results of the GMM Estimation

<table>
<thead>
<tr>
<th>Moment conditions</th>
<th>Value</th>
<th>$p$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Developed economy</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average equity return</td>
<td>$-0.00736$</td>
<td>0.3476</td>
</tr>
<tr>
<td>$\frac{1}{N} \sum_{t=2}^{N} (r_{us,t} - r_{EF,t})$</td>
<td>(0.00372)</td>
<td></td>
</tr>
<tr>
<td>Equity return volatility</td>
<td>$0.00157$</td>
<td>0.5209</td>
</tr>
<tr>
<td>$\frac{1}{N-1} \sum_{t=2}^{N} \left[ (r_{us,t} - \bar{r}<em>{us})^2 - (r</em>{EF,t} - \bar{r}_{EF})^2 \right]$</td>
<td>(0.00245)</td>
<td></td>
</tr>
<tr>
<td><strong>Emerging economy</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average equity return</td>
<td>$0.00482$</td>
<td>0.6170</td>
</tr>
<tr>
<td>$\frac{1}{N} \sum_{t=2}^{N} (r_{br,t} - r_{EF,t})$</td>
<td>(0.00965)</td>
<td></td>
</tr>
<tr>
<td>Equity return volatility</td>
<td>$0.00333$</td>
<td>0.3486</td>
</tr>
<tr>
<td>$\frac{1}{N-1} \sum_{t=2}^{N} \left[ (r_{br,t} - \bar{r}<em>{br})^2 - (r</em>{EF,t} - \bar{r}_{EF})^2 \right]$</td>
<td>(0.00356)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>GMM parameter estimates and $J$-test</th>
<th>Value</th>
<th>$p$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Parameter estimate</strong></td>
<td>$\gamma_z$</td>
<td>0.075</td>
</tr>
<tr>
<td>$J(\Phi_1)$</td>
<td>8.421</td>
<td>0.3197</td>
</tr>
<tr>
<td><strong>Test of over-identifying restrictions</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Observations</td>
<td>$N = 176$</td>
<td></td>
</tr>
</tbody>
</table>

Before studying the magnitude and the significance of the estimated parameter, let us first analyze
how well the model fits the data. As the model is over-identified, it is not possible to set every moment to zero. So the key concern is the distance from zero. The minimized value of the quadratic form $J(\gamma_z)$ is $\chi^2$-distributed under the null hypothesis that the model is true with degrees of freedom equal to the number of orthogonality conditions net of the number of parameters to be estimated. This $\chi^2$ measure thus provides a goodness-of-fit test for the model.

The $\chi^2$ tests for goodness-of-fit suggest that the model cannot be rejected at the 99% confidence level (see Table III). The table uses the covariance matrix of the moments to test the significance of individual moments and provide the corresponding $p$-values. We cannot reject that the moment conditions are not satisfied at the 99% confidence level. Thus, the model can simultaneously satisfy all moments. The model is thus successful in explaining the level of international equity risk premium and the equity return volatility. I now discuss in more detail the magnitude of the cost $\gamma_z$ of sovereign default for the economy.

### 3.4.1 Anticipated Economic Costs of a Sovereign Default Crisis: Size and Significance

Existing theoretical models on sovereign credit risk must assume the presence of the economic cost of defaulting on sovereign debt when explaining why sovereigns tend to repay their debt.\(^\text{16}\) While there exists a large body of literature empirically analyzing the ex post economic cost of a sovereign default,\(^\text{17}\) these costs have never been structurally estimated ex ante. This paper fills this gap. As suggested by Table III, the estimate of the fraction $\gamma_z$ of output lost upon sovereign default is statistically different from zero. It suggests that, once the offsetting terms of trade effect is accounted for, the present value of output loss due to sovereign default is 7.5% of the current level of output. Consider now the following back-of-the-envelope computation; for an economy that grows at 1% per annum with a risk-free interest rate equals to 3% and using the simple Gordon’s formula, the perpetual loss in output growth rate per annum that corresponds to 7.5% in present terms is equal to 0.16%. This estimate captures the loss in output due to the default event in excess of the economic losses that triggered this event. The magnitude of the estimated economic costs of defaulting appears to be close to those measured by De Paoli, Hoggarth, and Saporta (2006). Analyzing 45 sovereign default crises over the period 1970-2000, these authors find a loss in GDP growth of 15.1% per annum.\(^\text{18}\)

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\(^{16}\)See, for example, Westphalen (2001), Gibson and Sundaresan (2001), Guimares (2009), and Jeanneret (2009).

\(^{17}\)See, Borensztein and Panizza (2008) and the references therein.

\(^{18}\)They look at the difference per year between potential and actual output, where potential output is based on the country’s pre-crisis (HP filter) trend.
4 Model Implications

In this section, I show that the model can generate new theoretical predictions that help explain some of the recent observations provided by the empirical literature. This analysis starts with the examination of the diversification property of investing across different asset classes (equity, corporate debt, and sovereign debt), then analyzes how equity return volatility co-move across countries, and finally offers some insights on the relationship between sovereign credit risk in emerging economies and equity return volatility in the U.S.

4.1 Co-movement across Asset Prices

**Prediction 1:** There is positive co-movement across equity returns, corporate debt returns, and sovereign debt returns. Thus, in equilibrium, there are no benefits to investing across assets.

The co-movement in international equity markets is certainly one of the most studied phenomena in international finance. In line with the work of Helpman and Razin (1978), Cole and Obstfeld (1991), Zapatero (1995), and Pavlova and Rigobon (2007, 2008), the model predicts that international equity returns move in the same direction in response to an economic shock in one of the countries despite the independence of the output innovations of a country (see Figure 3, left panels). The foreign exchange market acts as the channel through which shocks are propagated internationally. However, little is known about the theoretical diversification benefits of investing internationally across different asset classes. In this section, I aim to fill this gap and analyze how the equity, corporate bond, and sovereign bond markets relate to each other.

The model developed in this paper allows for a breakdown of the liability structure of a firm. We can thus disentangle the effect of economic shocks on corporate debt and the equity markets, in addition to the effect of economic shocks on the sovereign debt market. The Figure 3 shows that a negative shock in one country simultaneously depresses the value of corporate debt and equity in both countries. Kwan (1996) finds strong empirical support for co-movement between corporate debt and equity returns. Co-movement in the debt and equity markets arises because (i) the value of debt and equity are linked

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by a common state-price density, and (ii) they are affected by the same endogenous default decision. In addition, we can see from Figure 3 that a positive shock benefits equities much more than bonds because equityholders receive all of the residual profits, while corporate bond holders receive no more than the promised payments of interest and, upon liquidation, the principal.

### Figure 3: Equity, Corporate Debt, and Sovereign Debt

This figure shows the effect of production shocks in the developed economy (upper panels) on equity value (left panels) as well as on corporate and sovereign debt prices (right panels). The figure also provides the effects when shocks originate from the emerging economy (lower panels).

Sovereign bond prices also appear to respond negatively to a bad shock in one of the countries (Figure 3, right panels). The co-movement between corporate asset returns and sovereign bond returns arises because the government and the firm are closely linked in the economy: On the one hand, the capacity to service sovereign debt and thus to avoid defaulting depends on the emerging firm’s earnings through corporate income taxes; on the other hand, the present value of the firm’s earnings depends on the likelihood of a change in economic regime, which itself depends on the sovereign country’s creditworthiness. As a result, the model predicts that these interactions generate co-movements across the corporate bond and sovereign bond markets, which is consistent with the data presented in the Figure 4.
Figure 4: Corporate and Sovereign Bonds, 1994-2008. The figure illustrates the relationship between the daily value of the Citigroup Large A U.S. Corporate Bond 10y index and the JPMorgan EMBI Total Return Bond Index for Brazil over the period 1994-2008. The right panel includes the linear regression relationship between both series.

As a corollary, sovereign credit risk tends to rise dramatically when equities perform poorly. Figure 5 (left panel) illustrates the model’s predicted relationship between the emerging firm’s equity value and the sovereign credit spread, which captures the non-linearity in the relationship observed in the data (right panel). We can thus explain the close interdependence between the sovereign credit and equity markets that we have witnessed during the recent credit crisis.

Figure 5: Sovereign Credit Risk and Equity. The left panel of the figure plots the relationship between the emerging firm’s equity value and sovereign credit spread, as obtained from the model. The right panel illustrates the relationship between the daily equity market and the EMBI+ spreads for Brazil over the period 1994-2008.
Most importantly, the model predicts that the portfolio diversification property, with respect to corporate bonds, sovereign bonds, and equity markets seems rather limited. While previous research has highlighted the absence of benefits to investing internationally, the model suggests a complimentary prediction. That is, in equilibrium, there are no benefits to investing across assets.

4.2 Countercyclical Equity Return Volatility and Co-movement

**Prediction 2**: *Equity return volatility is time-varying, countercyclical, and co-move across countries.*

As previously noted (see Section 2.4.1), the level of equity return volatility depends on current macroeconomic conditions. The time-variation of equity return volatility calls for an analysis of two important recent empirical observations, namely, i) the countercyclical nature of equity return volatility and ii) the relationship between equity return volatilities at the international level.
Figure 6: Equity Return Volatility, Production, and Asset Prices. This figure shows the effect of production shocks in the developed economy on equity return volatility in the same country (upper-left panel) and the relationship between the price and the volatility of the developed firm’s equity (upper-right panel). This figure also illustrates co-movement between equity return volatility in the two countries in response to shocks in either country (lower-left panel). The lower-right panel displays the co-movement between equity return volatility and sovereign credit risk in response to shocks in either country.

First, equity return volatility is predicted to be countercyclical; a negative economic shock raises equity return volatility, whereas a positive economic shock lowers it (Figure 6, upper-left panel). Equity return volatility is also negatively, and non-linearly related to a firm’s equity value and, thus, present an asymmetric leverage effect (upper-right panel). Figure 7 offers empirical evidence in line with these model’s predictions; the volatility of S&P500 return computed using a GARCH(1,1) model increases when the U.S. economy performs poorly, as measured either with the level of industrial production (upper panels) or with equity market prices in a non-linear fashion (lower panels). Schwert (1989), Forbes and Rigobon (2002), Bae et al. (2003), Engle and Rangel (2008), and Engle, Ghysels, and Sohn (2009) also confirm the countercyclical nature of equity return volatility.

The countercyclical nature of equity return volatility arises from the leverage effects described in Section 3. Alternative explanations of the countercyclical nature of equity return volatility include Bansal and Yaron (2004) and Tauchen (2005). These authors argue that investors with a preference for early resolution of uncertainty require compensation, thereby inducing negative co-movements between ex-post returns and volatility. Some models on limited equity market participation such as Basak and Cuoco (1998) are also able to generate asymmetric equity return volatility movements.

20 The countercyclicality of equity return volatility arises from the leverage effects described in Section 3. Alternative explanations of the countercyclical nature of equity return volatility include Bansal and Yaron (2004) and Tauchen (2005). These authors argue that investors with a preference for early resolution of uncertainty require compensation, thereby inducing negative co-movements between ex-post returns and volatility. Some models on limited equity market participation such as Basak and Cuoco (1998) are also able to generate asymmetric equity return volatility movements.
Second, the model offers insights on how equity return volatility is related internationally. Numerous studies have shown that equity return volatility moves across countries in a coordinated fashion.\footnote{See, for example, Hamao, Masulis, and Ng (1990), Lin, Engle, and Ito (1994), Edwards and Susmel (2001), Forbes and Rigobon (2002).} In particular, Bae et al. (2003) find that the conditional volatility in the U.S. equity market explains a substantial fraction of the conditional volatility in international equity markets. Figure 8 shows that equity return volatility in Brazil is closely related to the volatility of equity returns in the U.S., both of which are measured using a GARCH(1,1) model.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{Equity, Equity Return Volatility, and Industrial Production in the U.S., 1998-2008. This figure shows the relationship between volatility on the S&P500 return as measured using a GARCH(1,1) model and industrial production in the U.S. (upper panels). The lower panels present the relationship between volatility on the S&P500 return and the S&P index price. The right panels include a linear regression line. The figure reports data for the 1998-2008 period, instead of the 1994-2008 period, to avoid a non-stationarity issue related to the S&P500 series.}\end{figure}
Figure 8: Equity Return Volatility in Brazil and the U.S., 1994-2008. This figure shows the dynamics of return volatility on the MSCI Brazil index and the S&P500 as measured using a GARCH(1,1) model. The series presented in the left panel are smoothed with a 20-day moving average. The right panel shows the relationship between the raw data, as well as the linear regression trend.

In line with this empirical evidence, the model predicts that a shock in either country affects the equity return volatility of both countries in the same direction (Figure 6, left panel). The co-movement in equity return volatility is the result of two factors, namely, (i) the presence of financial leverage, operational leverage, and sovereign default risk, which generates time-variation in equity return volatility as well as volatility persistence and clustering, and (ii) the co-movement in equity returns, which is due to the Ricardian effect that propagates shocks internationally and drives the movements of volatility in the same direction.

4.3 Equity Return Volatility and Sovereign Credit Risk

Prediction 3: Equity return volatility in the developed economy is positively correlated with sovereign credit risk in the emerging economy.

Measures of equity return volatility in the U.S. have attracted much interest in explaining sovereign credit risk in emerging markets. Longstaff et al. (2008), Pan and Singleton (2008), Remolona et al. (2008), and Jeanneret (2009) find uncertainty in the U.S. to be a key factor in explaining sovereign credit risk movements. As displayed in Figure 8, there is a positive relationship between sovereign credit risk and equity return volatility in the U.S.
Figure 9: Equity Return Volatility and Sovereign Credit Risk, 1998-2008. This figure compares S&P500 return volatility and the EMBI+ sovereign credit spread for Brazil. The figure breaks down the relationship between these series into two subsamples: from June 1, 1994 through December 31, 2006 and from January 1, 2007 through December 31, 2008. The right panel also provides the average linear relationship between the two series for each subsample.

The model generates a positive relationship between equity return volatility in the developed economy and sovereign credit risk (Figure 9, right panel). The reason for the positive co-movement between these two measures is as follows. A negative economic shock on the developed country lowers firm revenues in both countries due to the Ricardian effect. With lower earnings, the firm in the developed economy is closer to its distress level, thereby lowering the equity value and increasing the firm’s financial leverage. The volatility of equity return then increases because of the asymmetric volatility effect. Regarding the emerging economy, the same economic shock decreases the level of fiscal revenues necessary to service the sovereign debt, which raises the likelihood of defaulting and, thus, the sovereign credit spread.

Longstaff et al. (2008), Pan and Singleton (2008), and Remolona et al. (2008) suggest that variation in equity return volatility in the U.S. explains movements in sovereign credit spreads in emerging markets because it affects the risk-premium required by investors and, thus, the pricing of sovereign debt. In contrast, the model developed in this paper may partially change the interpretations of these authors. The predicted co-movement between equity return volatility in the developed economy and sovereign credit spread in the emerging economy does not arise from a causal relationship between these two measures; they are both subject to the same initial exogenous economic shocks. Therefore, as the model suggests, co-movement does not necessarily imply causality.

\footnote{This result obtains even though macroeconomic uncertainty in the emerging country is independent of macroeconomic uncertainty in the developed economy, as both of them are kept constant in the analysis.}
References


5 Appendix

5.1 Stochastic Processes and Pricing Kernels

5.1.1 The Stochastic Discount Factor

The pricing kernel of the emerging economy that prevails in a competitive equilibrium is equal to

\[ \xi_{x,t} = \lambda_x e^{-\rho t} \frac{\partial u_x}{\partial C_{xx,t}, C_{xy,t}} \]

\[ = \lambda_x e^{-\rho t} \frac{\partial [a_x \log (C_{xx,t}) + (1 - a_x) \log (C_{xy,t})]}{\partial C_{xx,t}} \]

\[ = \frac{\lambda_x e^{-\rho t} a_x}{C_{xx,t}} = \frac{e^{-\rho t} (\lambda_y a_y + \lambda_x a_x)}{X_t} \]

(50)

(51)

(52)

(53)

(54)

(55)

(56)

Dropping the time and the regime subscript and applying Itô’s formula to \( \xi_{x,t} \) yields,

\[ df(t, X) = f_t dt + f_x dX + \frac{1}{2} f_{xx} dX dX \]

\[ = -\rho f dt - \frac{f}{X} (\theta_x X dt + \sigma_{x,x} X dW^x + \sigma_{x,y} X dW^y) \]

\[ + \frac{f}{X^2} \left[ (\sigma_{x,x} X)^2 dt + (\sigma_{x,y} X)^2 dt \right] \]

\[ = f \left[ (-\rho - \theta_x + \sigma_{x,x}^2 + \sigma_{x,y}^2) dt + \sigma_{x,x} X dW^x + \sigma_{x,y} X dW^y \right] \]

The pricing kernel of the emerging economy thus follows the process defined by

\[ \frac{d\xi_{x,t}}{\xi_{x,t}} = -r_x dt - \sigma_{x,y} dW^y_t - \sigma_{x,x} dW^x_t \]

(57)

(58)

where \( r_x \) is the risk-free rate prevailing in the emerging economy, given by

\[ r_x = \rho + \theta_x - (\sigma_{x,x}^2 + \sigma_{x,y}^2) \]

Using a similar approach to obtain the pricing kernel of the developed economy \( \xi_{y,t} = \frac{\lambda_y e^{-\rho t} (1 - a_y)}{\xi_{y,t}} \), we obtain

\[ \frac{d\xi_{y,t}}{\xi_{y,t}} = -r_y dt - \sigma_y dW^y_t \]

where \( r_y \) is the risk-free rate prevailing in the developed economy, given by

\[ r_y = \rho + \theta_y - \sigma_y^2 \]

5.1.2 The Exchange Rate

The exchange rate is defined by \( S_t = f(t, \xi_{y,t}, \xi_{x,t}) = \frac{\xi_{y,t}}{\xi_{x,t}} \), with

\[ \frac{d\xi_{y,t}}{\xi_{y,t}} = -r_y dt - \sigma_y dW^y_t \]

(59)

(60)
and
\[ \frac{d\xi_{x,t}}{\xi_{x,t}} = -r_x dt - \sigma_{x,y} dW_t^y - \sigma_{x,x} dW_t^x \] (61)

From Itô’s formula, dropping the time subscript,
\[ df(t, \xi_y, \xi_x) = f_t dt + f_{\xi_y} d\xi_y + f_{\xi_x} d\xi_x + \frac{1}{2} \left( f_{\xi_y \xi_y} d\xi_y d\xi_y + f_{\xi_x \xi_x} d\xi_x d\xi_x + 2 f_{\xi_x \xi_y} d\xi_y d\xi_x \right) \] (62)
\[ = 0 + \frac{1}{\xi_x} (-r_y \xi_y dt - \sigma_{y,y} dW^y) \] (63)
\[ - \frac{\xi_y}{(\xi_y)} (-r_x \xi_x dt - \xi_x (\sigma_{x,y} dW^y + \sigma_{x,x} dW^x)) + 0 \]
\[ + \frac{\xi_y}{(\xi_x)^2} \left( (\xi_y \sigma_{x,y})^2 + (\xi_x \sigma_{x,x})^2 \right) dt - \frac{1}{(\xi_x)^2} (\xi_y \sigma_{y} \xi_x \sigma_{x,y}) dt \]
\[ = \xi_y \xi_x \left[ \left( r_x - r_y + (\sigma_{x,y})^2 + (\sigma_{x,x})^2 - \sigma_{y} \sigma_{x,y} \right) dt \right] \] (64)

The exchange rate \( S \) thus follows the process defined by
\[ \frac{dS_t}{S_t} = \theta_s dt + \sigma_{x,x} dW_t^x - (\sigma_{y} - \sigma_{x,y}) dW_t^y \] (65)

with
\[ \theta_s = r_x - r_y + \sigma_{x,y}^2 + \sigma_{x,x}^2 - \sigma_{y} \sigma_{x,y} \] (66)

5.1.3 The Cash-flows \( Z \)

Let’s define a new process \( f = Z_t \equiv P_{x,t} X_t = (S_t)^{-\alpha} X_t \) (from the world basket numeraire) with
\[ \frac{dX_t}{X_t} = \theta_x dt + \sigma_{x,x} dW_t^x + \sigma_{x,y} dW_t^y \] (67)

and
\[ \frac{dS_t}{S_t} = \theta_x dt + \sigma_{x,x} dW_t^x - (\sigma_{y} - \sigma_{x,y}) dW_t^y \] (68)

From Itô’s formula, dropping the time and regime substricts,
\[ df(t, X_t, S_t) = f_t dt + f_x dX + f_s dS + \frac{1}{2} (f_{xx} dXdX + f_{ss} dSdS + 2 f_{xs} dXdS) \] (69)
\[ = 0 + \tau S^{-\alpha} X (\theta_x dt + \sigma_{x,x} dW^x + \sigma_{x,y} dW^y) \] (70)
\[ - \alpha \tau S^{-\alpha} X (\theta_s dt + \sigma_{x,x} dW^x - (\sigma_{y} - \sigma_{x,y}) dW^y) \]
\[ + 0 + \frac{\alpha(1+\alpha)}{2} \tau S^{-\alpha} X \left( \sigma_{x,x}^2 + (\sigma_{y} - \sigma_{x,y})^2 \right) dt \]
\[ - \alpha \tau S^{-\alpha} X \left( \sigma_{x,x}^2 + \sigma_{x,y} (\sigma_{y} - \sigma_{x,y}) \right) dt \]
\[ = \tau S^{-\alpha} X \left[ \left( \theta_x - \alpha \theta_s + \frac{\alpha(1+\alpha)}{2} \left( \sigma_{x,x}^2 + (\sigma_{y} - \sigma_{x,y})^2 \right) - \alpha \sigma_{x,x}^2 + \sigma_{x,y} (\sigma_{y} - \sigma_{x,y}) \right) dt \right] \] (72)

The dynamics of \( Z_t \) is thus characterized by the process defined by
\[ \frac{dZ_t}{Z_t} = \theta_z dt + \sigma_{x,x} dW_t^x + \sigma_{x,y} dW_t^y \] (73)
with
\[
\theta_z = \theta_x - \alpha \theta_s + \frac{\alpha (1 + \alpha)}{2} \left( \sigma_{x,x}^2 + (\sigma_y - \sigma_{x,y})^2 - \alpha \left( \sigma_{x,x}^2 + \sigma_{x,y} (\sigma_y - \sigma_{x,y}) \right) \right) - \alpha \left( \sigma_{x,x}^2 + \sigma_{x,y} (\sigma_y - \sigma_{x,y}) \right)
\]
\[
\sigma_{x,x} = (1 - \alpha) \sigma_{x,x}
\]
\[
\sigma_{x,y} = (1 - \alpha) \sigma_{x,y} + \alpha \sigma_y
\]

5.1.4 The State-Price Density of the World Basket Numeraire

The pricing kernel \( \xi_t \) under the world basket numeraire is equal to

\[
\xi_t = e^{-\rho t} \left( \frac{\lambda_y a_y + \lambda_x a_x}{P_{x,t} X_t} \right) = \frac{e^{-\rho t} \left( \lambda_y (1 - a_y) + \lambda_x (1 - a_x) \right)}{P_{y,t} Y_t}
\]

(77)

Substituting out consumption goods prices, we obtain

\[
\xi_t = e^{-\rho t} \left( \frac{\lambda_y a_y + \lambda_x a_x}{P_{x,t} X_t} \right) = \frac{e^{-\rho t} \left( \lambda_y (1 - a_y) + \lambda_x (1 - a_x) \right)}{P_{y,t} Y_t}
\]

(78)

\[
\xi_t = (\xi_t P_{x,t})^{1-\alpha} (\xi_t P_{y,t})^\alpha
\]

(79)

\[
= e^{-\rho t} \left( \frac{\lambda_y a_y + \lambda_x a_x}{X_t} \right)^{1-\alpha} \left( \frac{\lambda_y (1 - a_y) + \lambda_x (1 - a_x)}{Y_t} \right)^\alpha
\]

(80)

\[
= \frac{e^{-\rho t} \left( \lambda_y a_y + \lambda_x a_x \right)^{1-\alpha}}{Z_t \left( \lambda_y (1 - a_y) + \lambda_x (1 - a_x) \right)^{-\alpha}}
\]

(81)

Dropping the time and the regime subscript and applying Itô's formula to \( \xi_t \) yields,

\[
df(t, Z) = df_t dt + f_z dZ + \frac{1}{2} f_{zz} dZ dZ
\]

(82)

\[
= -\rho f_t dt - \frac{f}{Z} (\theta_z Z dt + \sigma_{z,x} Z dW^x + \sigma_{z,y} Z dW^y) + \frac{f}{Z^2} \left[ (\sigma_{z,x} Z)^2 dt + (\sigma_{z,y} Z)^2 dt \right]
\]

(83)

\[
= f \left[ (-\rho - \theta_z + \sigma^2_{z,x} + \sigma^2_{z,y}) dt + \sigma_{z,x} Z dW^x + \sigma_{z,y} Z dW^y \right]
\]

(84)

The pricing kernel under the world basket numeraire thus follows the process defined by

\[
\frac{d\xi_t}{\xi_t} = -r_z dt - \sigma_{z,y} dW^y_t - \sigma_{z,x} dW^x_t
\]

(85)

where \( r_z \) is the risk-free rate prevailing under the world basket numeraire, given by

\[
r_z = \rho + \theta_z - \left( \sigma^2_{z,x} + \sigma^2_{z,y} \right)
\]

(86)

5.1.5 The Risk-neutral Measure

Let be \((\Omega, \mathcal{F}, \mathbb{P})\) the probability space on which the Brownian motions are defined. The corresponding information filtration is \( F = \{ \mathcal{F}_t : t \geq 0 \} \).
The emerging and the developed firm’s revenues

First, we define the risk-neutral measure associated with the pricing kernel under the world basket numeraire \( \xi_t \) by specifying the density process \( \hat{\varphi}_t \),

\[
\hat{\varphi}_t = \mathbb{E}_t \left[ \frac{dQ}{dP} \right]
\]

(87)

which evolves according to the process

\[
\frac{d\hat{\varphi}_t}{\hat{\varphi}_t} = -\sigma_{z,y} dW_t^y - \sigma_{z,x} dW_t^x
\]

(88)

Applying the Girsanov theorem, we obtain new Brownian motions under \( Q \), \( \tilde{W}_t^y \) and \( \tilde{W}_t^x \), which solve

\[
dW_t^y = d\tilde{W}_t^y - \sigma_{z,y} dt
\]

(89)

\[
dW_t^x = d\tilde{W}_t^x - \sigma_{z,x} dt
\]

(90)

Under the risk-neutral probability measure \( Q \), the developed and the emerging firm’s revenues then follow the process

\[
\frac{dZ_t}{Z_t} = \hat{\theta}_z dt + \sigma_{z,y} d\tilde{W}_t^y + \sigma_{z,x} d\tilde{W}_t^x
\]

(91)

with

\[
\hat{\theta}_z = \theta_z - (\sigma_{z,x}^2 + \sigma_{z,y}^2)
\]

(92)

5.2 The Government of the Emerging Economy

5.2.1 Sovereign Debt

The price of the debt is determined subject to a number of conditions. First, when the firm’s revenues \( Z_t \) tend to infinity (and so do the revenues \( R_t \)), the value of the sovereign debt \( D_t \) tends to the value of the risk-free debt

\[
\lim_{Z \to \infty} D(Z) = \mathbb{E}_0^Q \left[ \int_0^\infty C e^{-r_x t} dt \right] = \frac{C}{r_z}
\]

(93)

Second, lenders value this debt upon default, which depends on the recovery rate. Upon default, the sovereign and its lenders restructure the terms of the debt contract and agree on a reduction of the debt service. I determine the value matching conditions that impose equality between the value of the sovereign debt and the value of the restructured debt in default. At default time \( T(Z^D) \), the value of the sovereign debt is

\[
\lim_{Z \to Z^D} D(Z) = \frac{C (1 - \phi)}{r_z}
\]

(94)

where \( 0 \leq 1 - \phi \leq 1 \) denotes the recovery rate on the debt service \( C \). The stochastic discount factor is defined as the Arrow-Debreu price of default \( \mathbb{E}_0^Q \left[ e^{-r_x T(Z^D)} \right] = \left( \frac{Z}{Z^D} \right)^\beta \), where \( \beta \) is the negative root of the quadratic equation

\[
\frac{1}{2} \sigma_z^2 \beta (\beta - 1) + \theta_z \beta - r_z = 0,
\]

(95)

defined by

\[
\beta = \frac{1}{2} \frac{\hat{\theta}_z}{\sigma_z^2} - \sqrt{\left( \frac{1}{2} \frac{\hat{\theta}_z}{\sigma_z^2} \right)^2 + \frac{2r_z}{\sigma_z^2}} < 0
\]
with $\sigma_z = \sqrt{\sigma_{z,x}^2 + \sigma_{z,y}^2}$. The value of the sovereign debt can then be rewritten as

\[
D(Z) = \mathbb{E}_Q^0 \left[ \int_0^{T(Z^D)} e^{-r_z t} dW_t + \mathbb{E}_Q^0 \left[ \int_{T(Z^D)}^\infty e^{-r_z t} dW_t \right] \right] + \mathbb{E}_Q^0 \left[ \int_{T(Z^D)}^\infty (1 - \phi) e^{-r_z t} dW_t \right]
\]

where the default time on sovereign debt can be written as

\[
T(R^D) = \inf \{ t \geq 0 | R_t \leq R^D \} = \inf \{ t \geq 0 | Z_t \leq Z^D \} = T(Z^D)
\]  

### 5.2.2 Sovereign Wealth and Default Boundary

Sovereign’s wealth $W$ satisfies for given levels of debt service $C$ and default boundary $Z^D$,

\[
W(Z) = \mathbb{E}_Q^0 \left[ \int_0^{T(Z^D)} \tau (Z_t - K - C_f) e^{-r_z t} dW_t \right] + \mathbb{E}_Q^0 \left[ \int_{T(Z^D)}^\infty \tau [(1 - \gamma_z) (Z_t - K) - C_f] e^{-r_z t} dW_t \right] - D(Z)
\]

\[
= \mathbb{E}_Q^0 \left[ \int_0^{T(Z^D)} \tau (Z_t - K) e^{-r_z t} dW_t \right] - \mathbb{E}_Q^0 \left[ \int_{T(Z^D)}^\infty \tau \gamma_z (Z_t - K) e^{-r_z t} dW_t \right] - D(Z)
\]

\[
= \frac{\tau Z}{r_z - \theta_z} - \tau \gamma_z \left[ (\frac{Z^D}{r_z - \theta_z} - K) \left( \frac{Z}{Z^D} \right)^\beta - \frac{\tau (C_f + K)}{r_z} - \frac{C}{r_z} \left[ 1 - \phi \left( \frac{Z}{Z^D} \right)^\beta \right] \right]
\]

where

\[
Z_{H,T(Z^D)} - Z_{L,T(Z^D)} = [X_H P_{x,H} - X_L P_{x,L}] |_{t=T(Z^D)}
\]

\[
= [X_H P_{x,H} - (1 - \gamma) X_L^{1-\alpha} P_{x,H}] |_{t=T(Z^D)}
\]

\[
= [(1 - (1 - \gamma)^{1-\alpha}) X_H P_{x,H}] |_{t=T(Z^D)}
\]

\[
= [\gamma_z X_H P_{x,H}] |_{t=T(Z^D)} = \gamma_z Z_{H,T(Z^D)}
\]

The sovereign’s default policy is characterized by the default boundary $Z^D$. It is chosen to maximize the sovereign wealth $W(Z)$, such that the smooth-pasting condition $\frac{\partial W(Z)}{\partial Z} |_{Z = Z^D} = \frac{\tau (1 - \gamma_z)}{r_z - \theta_z}$ is satisfied. The first-order maximization yields
\[ \frac{\partial W(Z)}{\partial Z} = \frac{\tau}{r_z - \hat{\theta}_z} + \beta \left( \frac{\phi C + K \tau \gamma_z}{Z^D r_z} - \frac{\gamma_z \tau}{r_z - \hat{\theta}_z} \right) \left( \frac{Z}{Z^D} \right)^{\beta - 1} \]  

(110)

Using the smooth-pasting condition \( \frac{\partial W(Z)}{\partial Z} \big|_{Z=Z_D} = \tau \frac{(1-\gamma_z)}{r_z, L - \hat{\theta}_z} \), we have

\[ \frac{\tau}{r_z - \hat{\theta}_z} = \frac{\tau}{r_z - \hat{\theta}_z} + \beta \left( \frac{\phi C + K \tau \gamma_z}{Z^D r_z} - \frac{\gamma_z \tau}{r_z - \hat{\theta}_z} \right) \]  

(111)

Finally,

\[ Z^{D_s} = \frac{\beta (\phi C + K \tau \gamma_z) (r_z - \hat{\theta}_z)}{(\beta - 1) \tau \gamma_z r_z} \]  

(112)

### 5.3 The Emerging Firm

#### 5.3.1 Debt Evaluation of the Emerging Firm if \( T(Z^D_f) > T(Z^D) \)

Should the emerging firm default after the government, the value of the emerging firm’s debt is

\[ D_f(Z) \big|_{T(Z^D_f) > T(Z^D)} = \mathbb{E}_\theta^Q \left[ \int_{0}^{T(Z^D_f)} C_f e^{-r_s t} dt + e^{-r_s T(Z^D_f)} D_{f,L}(Z \mid t = T(Z^D_f)) \right] \]  

(113)

\[ = \frac{C_f}{r_z} \left[ 1 - \left( \frac{Z}{Z^D} \right)^{\beta} \right] + D_{f,L}(Z \mid t = T(Z^D_f)) \left( \frac{Z}{Z^D} \right)^{\beta} \]  

(114)

with

\[ D_{f,L}(Z \mid t = T(Z^D_f)) = \mathbb{E}_{T(Z^D_f)}^Q \left[ \int_{T(Z^D_f)}^{\infty} C_f e^{-r_s t} dt \right] \]  

(115)

\[ + \mathbb{E}_{T(Z^D_f)}^Q \left[ \int_{T(Z^D_f)}^{\infty} (1-\eta)(1-\tau)(1-\gamma_z)(Z_t - K) e^{-r_s t} dt \right] \]  

(116)

\[ = \frac{C_f}{r_z} \left[ 1 - \left( \frac{(1-\gamma_z)Z^D}{Z^D_f} \right)^{\beta} \right] \]  

(117)

\[ + (1-\eta)(1-\tau)(1-\gamma_z) \left( \frac{Z^D_f}{r_z - \hat{\theta}_z} - \frac{K}{r_z} \right) \left( \frac{(1-\gamma_z)Z^D}{Z^D_f} \right)^{\beta} \]  

(118)

Finally, substituting \( D_{f,L}(Z \mid t = T(Z^D_f)) \) by the above expression, the debt value can be written as

\[ D_f(Z) = \frac{C_f}{r_z} \left[ 1 - \left( \frac{(1-\gamma_z)Z}{Z^D_f} \right)^{\beta} \right] + (1-\eta)(1-\tau)(1-\gamma_z) \left( \frac{Z^D_f}{r_z - \hat{\theta}_z} - \frac{K}{r_z} \right) \left( \frac{(1-\gamma_z)Z}{Z^D_f} \right)^{\beta} \]  

(119)
5.3.2 Debt Evaluation of the Emerging Firm if $T \left( Z_f^D \right) \leq T \left( Z^D \right)$

Should the emerging firm default before the government, the value of this firm’s debt is

$$D_f(Z) \mid_{T(Z_f^p) \leq T(Z^o)} = E^D_Q \left[ \int_0^{T(Z_f^p)} C_f e^{-r_z t} dt + e^{-r_z T(Z_f^p)} D_f, L(Z \mid_{t=T(Z_f^p)}) \right]$$

$$= \frac{C_f}{r_z} \left[ 1 - \left( \frac{Z}{Z_f^p} \right)^\beta \right] + D_f, L(Z \mid_{t=T(Z_f^p)}) \left( \frac{Z}{Z_f^p} \right)^\beta$$

with

$$D_f, L(Z \mid_{t=T(Z_f^p)}) = +E^D_Q \left[ \int_{T(Z_f^p)}^{T(Z^o)} (1 - \eta)(1 - \tau)(Z_t - K) e^{-r_z t} dt \right]$$

$$+E^D_Q \left[ \int_{T(Z^D)}^{\infty} (1 - \eta)(1 - \gamma_z)(Z_t - K) e^{-r_z t} dt \right]$$

$$= (1 - \eta)(1 - \tau) \left( \frac{Z_f^D}{r_z - \theta_z} - \frac{K}{r_z} \right) \left[ 1 - \gamma_z \left( \frac{Z_f^D}{Z_f^D} \right)^\beta \right]$$

Finally, substituting $D_f, L(Z \mid_{t=T(Z_f^p)})$ by the above expression, the debt value can be written as

$$D_f(Z) \mid_{T(Z_f^p) \leq T(Z^o)} = \frac{C_f}{r_z} \left[ 1 - \left( \frac{Z}{Z_f^p} \right)^\beta \right]$$

$$(1 - \eta)(1 - \tau) \left( \frac{Z_f^D}{r_z - \theta_z} - \frac{K}{r_z} \right) \left[ 1 - \gamma_z \left( \frac{Z}{Z_f^p} \right)^\beta \right]$$

5.3.3 The Emerging Firm’s Value if $T \left( Z_f^D \right) > T \left( Z^D \right)$

The levered firm value $V(Z)$ satisfies

$$V(Z) \mid_{T(Z_f^p) > T(Z^o)} = E^D_Q \left[ \int_0^{T(Z_f^p)} ((1 - \tau)(Z_t - K) + \tau C_f) e^{-r_z t} dt \right]$$

$$+E^D_Q \left[ e^{-r_z T(Z_f^p)} V_L(Z \mid_{t=T(Z_f^p)}) \right]$$

$$= \left( \frac{1 - \tau}{r_z - \theta_z} + \frac{\tau C_f - (1 - \tau)K}{r_z} \right) \left[ 1 - \left( \frac{Z}{Z_f^D} \right)^\beta \right]$$

$$+V_L(Z \mid_{t=T(Z_f^p)}) \left( \frac{Z}{Z_f^D} \right)^\beta$$
with

\[ V_L(Z | t=T(Z^D)) = \mathbb{E}_T^{Q(T(Z^D))} \left[ \int_{T(Z^D)}^{T(Z^P)} ((1-\tau)(1-\gamma_z)(Z_t-K) + \tau C_f) e^{-r_z t} dt \right] \] (131)

\[ + \mathbb{E}_T^{Q(T(Z^D))} \left[ \int_{T(Z^P)}^{\infty} (1-\eta)(1-\tau)(1-\gamma_z)(Z_t-K) e^{-r_z t} dt \right] \] (132)

\[ = \left(1-\gamma_z(1-\tau) \left(\frac{Z^D}{r_z-\theta_z} - \frac{K}{r_z} \right) + \tau C_f \right) \left[ 1 - \left(\frac{(1-\gamma_z)Z^D}{Z^D_f}\right)^{\beta} \right] \] (133)

\[ + (1-\eta)(1-\tau)(1-\gamma_z) \left(\frac{Z^D}{r_z-\theta_z} - \frac{K}{r_z} \right) \left(\frac{(1-\gamma_z)Z^D}{Z^D_f}\right)^{\beta} \] (144)

5.3.4 The Emerging Firm’s Value if \( T \left( Z^D_f \right) \leq T \left( Z^D \right) \)

Should the emerging firm default before the government, the levered firm value \( V(Z) \) satisfies

\[ V(Z) |_{T(Z^P) \leq T(Z^D)} = \mathbb{E}_T^{Q(T(Z^D))} \left[ \int_{0}^{T(Z^D)} ((1-\tau)(Z_t-K) + \tau C_f) e^{-r_z t} dt \right] \] (135)

\[ + \mathbb{E}_T^{Q(T(Z^D))} \left[ e^{-r_z T(Z^D)} V_L(Z | t=T(Z^P)) \right] \] (136)

\[ = \left(1-\gamma_z(1-\tau) \left(\frac{Z^D}{r_z-\theta_z} + \frac{\tau C_f - (1-\tau)K}{r_z} \right) \right) \left[ 1 - \left(\frac{Z^D}{Z^D_f}\right)^{\beta} \right] \] (137)

\[ + V_L(Z | t=T(Z^P)) \left(\frac{Z^D}{Z^D_f}\right)^{\beta} \] (138)

with

\[ V_L(Z | t=T(Z^P)) = \mathbb{E}_T^{Q(T(Z^P))} \left[ \int_{T(Z^P)}^{T(Z^D)} (1-\eta)(1-\tau)(Z_t-K) e^{-r_z t} dt \right] \] (139)

\[ + \mathbb{E}_T^{Q(T(Z^P))} \left[ \int_{T(Z^P)}^{\infty} (1-\eta)(1-\tau)(1-\gamma_z)(Z_t-K) e^{-r_z t} dt \right] \] (140)

\[ = (1-\eta)(1-\tau) \left(\frac{Z^D}{r_z-\theta_z} - \frac{K}{r_z} \right) \left[ 1 - \gamma_z \left(\frac{Z^D}{Z^D_f}\right)^{\beta} \right] \] (141)

Finally, substituting \( V_L(Z | t=T(Z^P)) \) by the above expression, the firm value can be written as

\[ V(Z) |_{T(Z^P) \leq T(Z^D)} = \left(\frac{(1-\tau)Z}{r_z-\theta_z} + \frac{\tau C_f - (1-\tau)K}{r_z} \right) \left[ 1 - \left(\frac{Z^D}{Z^D_f}\right)^{\beta} \right] \] (142)

\[ + (1-\eta)(1-\tau) \left(\frac{Z^D}{r_z-\theta_z} - \frac{K}{r_z} \right) \left[ \left(\frac{Z^D}{Z^D_f}\right)^{\beta} - \gamma_z \left(\frac{Z}{Z^D}\right)^{\beta} \right] \] (143)
5.3.5 The Emerging Firm’s Default Boundary if $T\left(Z^D_f\right) > T\left(Z^D\right)$

The default threshold is selected by shareholders. As the value of the firm until sovereign default is, by assumption, independent from the default policy, the optimal default policy is the one that maximizes equity value at time $T\left(Z^D\right)$

$$E(Z |_{t=T(Z^D)}) |_{T(Z^D_f)>T(Z^D)} = V(Z |_{t=T(Z^D)}) |_{T(Z^D_f)>T(Z^D)} - D_f(Z |_{t=T(Z^D)}) |_{T(Z^D_f)>T(Z^D)}$$

$$= (1 - \gamma_z)(1 - \tau) \left(\frac{Z}{r_z - \theta_z} - \frac{K}{r_z}\right) \left[1 - \left(\frac{(1 - \gamma_z)Z}{Z^D_f}\right)^\beta\right]$$

$$- \frac{(1 - \tau) C_f}{r_z} \left[1 - \left(\frac{(1 - \gamma_z)Z}{Z^D_f}\right)^\beta\right]$$

The first-order maximization yields

$$\frac{\partial E(Z)}{\partial Z} = \frac{(1 - \gamma_z)(1 - \tau)}{r_z - \theta_z} \left[1 - \left(\frac{(1 - \gamma_z)Z}{Z^D_f}\right)^\beta\right]$$

$$- \frac{\beta}{Z^D_f} \left(1 - \gamma_z\right)^2 (1 - \tau) \left(\frac{Z}{r_z - \theta_z} - \frac{K}{r_z}\right) - \frac{(1 - \gamma_z)(1 - \tau) C_f}{r_z} \left(\frac{(1 - \gamma_z)Z}{Z^D_f}\right)^{\beta-1}$$

Using the smooth-pasting condition $\frac{\partial [E(Z) |_{t=T(Z^D)}]}{\partial Z} |_{Z=Z^D_f} = 0$, we have

$$0 = \frac{1}{r_z - \theta_z} \left[1 - (1 - \gamma_z)^\beta\right] - \left(\frac{(1 - \gamma_z)}{r_z - \theta_z} - \frac{(C_f + (1 - \gamma_z) K)}{Z^D_f r_z}\right) \beta (1 - \gamma_z)^{\beta-1}$$

Finally,

$$Z^D_f |_{T(Z^D_f)>T(Z^D)} = \frac{(C_f + (1 - \gamma_z) K) \left(\frac{r_z - \theta_z}{r_z (1 - \gamma_z) \left(1 - \frac{1-(1-\gamma_z)^\beta}{\beta(1-\gamma_z)}\right)}\right)}$$

The above default boundary is obtained under the assumption that $T\left(Z^D\right) < T\left(Z^D_f\right)$, which is verified when the government’s level of debt service $C$ satisfies

$$C > (C_f + (1 - \gamma_z) K) k$$

with

$$k = \frac{(\beta - 1) \gamma_z \tau \left(1 - \frac{1-(1-\gamma_z)^\beta}{\beta(1-\gamma_z)}\right)}{\beta (1 - \gamma_z)^\phi}$$
5.3.6 The Emerging Firm’s Default Boundary if $T(Z_f^D) \leq T(Z_D)$

Should the emerging firm default before the government, the optimal default policy is the one that maximizes equity value at time $t = 0$

$$E(Z | t=0) |_{T(Z_f^D) \leq T(Z_D)} = V(Z | t=0) |_{T(Z_f^D) \leq T(Z_D)} - D_f(Z | t=0) |_{T(Z_f^D) \leq T(Z_D)}$$

$$= \left( \frac{(1-\tau)Z}{r_z-\theta_z} - \frac{(1-\tau)(C_f + K)}{r_z} \right) \left[ 1 - \left( \frac{Z}{Z_f^D} \right)^\beta \right]$$

(153)

The first-order maximization yields

$$\frac{\partial E(Z)}{\partial Z} = \frac{(1-\tau)}{r_z-\theta_z} \left[ 1 - \left( \frac{Z}{Z_f^D} \right)^\beta \right]$$

(155)

$$- \frac{\beta}{Z_f^D} \left( \frac{(1-\tau)Z}{r_z-\theta_z} - \frac{(1-\tau)(C_f + K)}{r_z} \right) \left( \frac{Z}{Z_f^D} \right)^{\beta-1}$$

(156)

Using the smooth-pasting condition $\frac{\partial [E(Z) | t=0]}{\partial Z} |_{Z=Z_f^D} = 0$, we have

$$0 = \frac{1}{r_z-\theta_z} - \left( \frac{1}{r_z-\theta_z} - \frac{(C_f + K)}{Z_f^D} \right) \beta$$

(157)

Finally,

$$Z_f^D |_{T(Z_f^D) \leq T(Z_D)} = \frac{\beta (C_f + K) (r_z-\theta_z)}{(\beta-1) r_z}$$

(158)

The above default boundary is obtained under the assumption that $T(Z_D) \geq T(Z_f^D)$, which is verified when the government’s level of debt service $C$ satisfies

$$C \leq (C_f + K) \frac{\gamma_z}{\phi}$$

(159)

5.3.7 The Emerging Firm’s Equity Return Volatility

Let’s determine the volatility of the emerging firm’s equity return, which is denoted by $\frac{dE_t}{E_t}$. Dropping the time and the regime subscript and applying Itô’s formula to $E_t$ yields,

$$dE_t = E_t dt + E_z dZ + \frac{1}{2} E_{zz} dZdZ$$

(160)

$$= E_z (\theta_z Z dt + \sigma_{z,x} Z dW^x + \sigma_{z,y} Z dW^y)$$

$$+ E_{zz} \left[ (\sigma_{z,x} Z)^2 dt + (\sigma_{z,y} Z)^2 dt \right]$$

$$= [\theta_z Z + (\sigma_{z,x}^2 + \sigma_{z,y}^2) E_{zz} Z^2] dt + E_z Z (\sigma_{z,x} dW^x + \sigma_{z,y} dW^y)$$

(161)

Hence, the dynamics of the equity return is
\[
\frac{dE}{E} = \frac{1}{E} \left[ \theta_z Z + (\sigma_{z,x}^2 + \sigma_{z,y}^2) E_{zz} Z^2 \right] dt + \frac{E_z Z}{E} (\sigma_{z,x} dW^x + \sigma_{z,y} dW^y) \quad (163)
\]

where \( E_z \) and \( E_{zz} \) denote the first and second derivatives of \( E \) with respect to the state variable \( Z \), respectively. Finally, the equity return volatility is given by

\[
\sigma_E = \frac{E_z Z}{E} \sqrt{\sigma_{z,x}^2 + \sigma_{z,y}^2} \quad (164)
\]

where

\[
E = \left( \frac{(1-\tau)Z}{r_z - \theta_z} + \frac{\tau C_f - (1-\tau)K}{r_z} \right) \left[ 1 - \left( \frac{Z}{Z_D} \right)^\beta \right] + V_L \left( \frac{Z}{Z_D} \right)^\beta \quad (165)
\]

\[
- \frac{C_f}{r_z} \left[ 1 - \left( \frac{(1-\gamma_z)Z}{Z_D^f} \right)^\beta \right] - (1-\eta)(1-\tau)(1-\gamma_z) \left( \frac{Z_D^f}{r_z - \theta_z} - \frac{K}{r_z} \right) \left( \frac{(1-\gamma_z)Z}{Z_D^f} \right)^{\beta-1} \quad (166)
\]

and

\[
E_z = \frac{(1-\tau)}{r_z - \theta_z} \left[ 1 - \left( \frac{Z}{Z_D} \right)^\beta \right] + \frac{\beta}{Z_D} \left( V_L - \frac{(1-\tau)Z}{r_z - \theta_z} - \frac{\tau C_f - (1-\tau)K}{r_z} \right) \left( \frac{Z}{Z_D} \right)^{\beta-1} \quad (167)
\]

\[
+ \frac{(1-\gamma_z)\beta}{Z_D^f} \left( C_f - (1-\eta)(1-\tau) - \frac{\tau C_f - (1-\tau)K}{r_z} \right) \left( \frac{(1-\gamma_z)Z}{Z_D^f} \right)^{\beta-1} \quad (168)
\]

with

\[
V_L = \left( (1-\tau)(1-\gamma_z) \left( \frac{Z_D^f}{r_z - \theta_z} - \frac{K}{r_z} \right) + \frac{\tau C_f}{r_z} \right) \left[ 1 - \left( \frac{(1-\gamma_z)Z_D^f}{Z_D^f} \right)^\beta \right] \quad (169)
\]

\[
+ (1-\eta)(1-\tau)(1-\gamma_z) \left( \frac{Z_D^f}{r_z - \theta_z} - \frac{K}{r_z} \right) \left( \frac{(1-\gamma_z)Z_D^f}{Z_D^f} \right)^\beta \quad (170)
\]

### 5.4 The Developed Firm

#### 5.4.1 Debt Evaluation of the Developed Firm if \( T(Z_{Dy}) > T(Z_D) \)

Should the developed firm default after the government of the emerging economy, the value of this firm’s debt is

\[
D_{fy}(Z) |_{T(Z_{Dy}) > T(Z_D)} = \mathbb{E}^D \left[ \int_0^{T(Z_D)} C_{fy} e^{-r_s t} dt + e^{-r_s T(Z_D)} D_{fy,L}(Z |_{t=T(Z_D)}) \right] \quad (171)
\]

\[
= \frac{C_{fy}}{r_z} \left[ 1 - \left( \frac{Z}{Z_D} \right)^\beta \right] + D_{fy,L}(Z |_{t=T(Z_D)}) \left( \frac{Z}{Z_D} \right)^\beta \quad (172)
\]

with
\[ D_{fy,L}(Z \mid t = T(Z^D_f)) = \mathbb{E}^Q_{T(Z^D_f)} \left[ \int_{T(Z^D_f)}^{T(Z^D_f)} C_{fy} e^{-r_s t} dt \right] \]  
\[ + \mathbb{E}^Q_{T(Z^D_f)} \left[ \int_{T(Z^D_f)}^{\infty} (1 - \eta)(1 - \tau)(1 - \gamma_z) (SZ_t - K_y) e^{-r_s t} dt \right] \]  
\[ = \frac{C_f}{r_z} \left[ 1 - \left( \frac{(1 - \gamma_z)Z}{Z^D_{fy}} \right)^\beta \right] \]  
\[ + (1 - \eta)(1 - \tau)(1 - \gamma_z) \left( \frac{\overline{SZ}^D_{fy}}{r_z - \theta_z} - \frac{K_y}{r_z} \right) \left( \frac{(1 - \gamma_z)Z}{Z^D_{fy}} \right)^\beta \]  

Finally, substituting \( D_{fy,L}(Z \mid t = T(Z^D_f)) \) by the above expression, the debt value can be written as

\[ D_{fy}(Z) \mid_{T(Z^D_f) > T(Z^D_f)} = \frac{C_{fy}}{r_z} \left[ 1 - \left( \frac{(1 - \gamma_z)Z}{Z^D_{fy}} \right)^\beta \right] \]  
\[ + (1 - \eta)(1 - \tau)(1 - \gamma_z) \left( \frac{\overline{SZ}^D_{fy}}{r_z - \theta_z} - \frac{K_y}{r_z} \right) \left( \frac{(1 - \gamma_z)Z}{Z^D_{fy}} \right)^\beta \]  

### 5.4.2 Debt Evaluation of the Developed Firm if \( T(Z^D_f) \leq T(Z^D_f) \)

Should the developed firm default before the government of the emerging economy, the value of this firm’s debt is

\[ D_{fy}(Z) \mid_{T(Z^D_f) \leq T(Z^D_f)} = \mathbb{E}^Q_{0} \left[ \int_{0}^{T(Z^D_f)} C_{fy} e^{-r_s t} dt + e^{-r_s T(Z^D_f)} D_{fy,L}(Z \mid t = T(Z^D_f)) \right] \]  
\[ = \frac{C_{fy}}{r_z} \left[ 1 - \left( \frac{Z}{Z^D_{fy}} \right)^\beta \right] + D_{fy,L}(Z \mid t = T(Z^D_f)) \left( \frac{Z}{Z^D_{fy}} \right)^\beta \]  

with

\[ D_{fy,L}(Z \mid t = T(Z^D_f)) = \mathbb{E}^Q_{T(Z^D_f)} \left[ \int_{T(Z^D_f)}^{T(Z^D_f)} (1 - \eta)(1 - \tau) (SZ_t - K_y) e^{-r_s t} dt \right] \]  
\[ + \mathbb{E}^Q_{T(Z^D_f)} \left[ \int_{T(Z^D_f)}^{\infty} (1 - \eta)(1 - \tau)(1 - \gamma_z) (SZ_t - K_y) e^{-r_s t} dt \right] \]  
\[ = (1 - \eta)(1 - \tau) \left( \frac{\overline{SZ}^D_{fy}}{r_z - \theta_z} - \frac{K_y}{r_z} \right) \left[ 1 - \gamma_z \left( \frac{Z^D_{fy}}{Z^D} \right)^\beta \right] \]

Finally, substituting \( D_{f,L}(Z \mid t = T(Z^D_f)) \) by the above expression, the debt value can be written as
\[ D_{fy}(Z) \mid_{T(Z^D_{fy}) \leq T(Z^D)} = \frac{C_{fy}}{r_z} \left[ 1 - \left( \frac{Z}{Z^D_{fy}} \right)^\beta \right] \]

\[ + (1 - \eta)(1 - \tau) \left( \frac{SZ_{fy}}{r_z - \theta_z} - \frac{K_y}{r_z} \right) \left[ \left( \frac{Z}{Z^D_{fy}} \right)^\beta - \gamma_z \left( \frac{Z}{Z^D} \right)^\beta \right] \]  

(184)

(185)

5.4.3 The Developed Firm’s Value if \( T(Z^D_{fy}) \geq T(Z^D) \)

Should the developed firm default after the government of the emerging economy, the levered firm value \( V_y(Z) \) satisfies

\[ V_y(Z) \mid_{T(Z^D_{fy}) > T(Z^D)} = \mathbb{E}^Q_T \left[ \int_{0}^{T(Z^D)} (1 - \tau) \left( \frac{SZ_t - K_y}{r_z - \theta_z} + \tau C_{fy} \right) e^{-r_z t} dt \right] \]  

\[ + \mathbb{E}^Q_T \left[ e^{-r_z T(Z^D)} V_{y,L} \left( Z \mid_{t=T(Z^D)} \right) \right] \]  

\[ = \left( \frac{1 - \tau}{r_z - \theta_z} + \frac{\tau C_{fy}}{r_z} \right) \left( 1 - \left( \frac{Z}{Z^D} \right)^\beta \right) \]  

\[ + V_{y,L} \left( Z \mid_{t=T(Z^D)} \right) \left( \frac{Z}{Z^D} \right)^\beta \]  

(186)

(187)

(188)

(189)

with

\[ V_{y,L} \left( Z \mid_{t=T(Z^D)} \right) = \mathbb{E}^Q_T \left[ \int_{T(Z^D)}^{T(Z^D_{fy})} (1 - \tau) (1 - \gamma_z) \left( \frac{SZ_t - K_y}{r_z - \theta_z} + \tau C_{fy} \right) e^{-r_z t} dt \right] \]  

\[ + \mathbb{E}^Q_T \left[ \int_{T(Z^D_{fy})}^{\infty} (1 - \eta) (1 - \tau) (1 - \gamma_z) \left( \frac{SZ_t - K_y}{r_z - \theta_z} + \tau C_{fy} \right) e^{-r_z t} dt \right] \]  

\[ = \left( 1 - \tau \right) (1 - \gamma_z) \left( \frac{SZ_{fy}}{r_z - \theta_z} - \frac{K_y}{r_z} \right) \left( 1 - \left( \frac{1 - \gamma_z}{Z^D_{fy}} \right)^\beta \right) \]  

\[ + (1 - \eta) (1 - \tau) (1 - \gamma_z) \left( \frac{SZ_{fy}}{r_z - \theta_z} - \frac{K_y}{r_z} \right) \left( \frac{1 - \gamma_z}{Z^D_{fy}} \right)^\beta \]  

(190)

(191)

(192)

(193)
5.4.4 The Developed Firm’s Value if \( T(Z_D^{fy}) \leq T(Z^D) \)

Should the developed firm default after the government of the emerging economy, the levered firm value \( V_y(Z) \) satisfies

\[
V_y(Z) \mid_{T(Z_D^{fy}) \leq T(Z^D)} = \mathbb{E}_0^Q \left[ \int_0^{T(Z_D^{fy})} \left( (1 - \tau) \left( S_Z - K_y \right) + \tau C_{fy} \right) e^{-r_s t} dt \right] + \mathbb{E}_0^Q \left[ e^{-r_s T(Z_D^{fy})} V_{y,L}(Z \mid T(Z_D^{fy})) \right] = \left( \frac{(1 - \tau) S_Z}{r_z - \theta_z} + \frac{\tau C_{fy} - (1 - \tau) K_y}{r_z} \right) \left[ 1 - \left( \frac{Z}{Z_D^{fy}} \right)^\beta \right] + V_{y,L}(Z \mid T(Z_D^{fy})) \left( \frac{Z}{Z_D^{fy}} \right)^\beta
\]

with

\[
V_{y,L}(Z \mid T(Z_D^{fy})) = \mathbb{E}_0^Q \left[ \int_{T(Z_D^{fy})}^{T(Z^D)} \left( (1 - \eta) \left( 1 - \tau \right) \left( S_Z - K_y \right) e^{-r_s t} dt \right) \right] + \mathbb{E}_0^Q \left[ \int_{T(Z_D^{fy})}^{\infty} \left( (1 - \eta) \left( 1 - \tau \right) (1 - \gamma_z) \left( S_Z_t - K_y \right) e^{-r_s t} dt \right) \right] = \left( 1 - \eta \right) \left( 1 - \tau \right) \left( \frac{S_Z^{fy}}{r_z - \theta_z} - \frac{K_y}{r_z} \right) \left[ 1 - \gamma_z \left( \frac{Z_D^{fy}}{Z_D^{fy}} \right)^\beta \right]
\]

Finally, substituting \( V_{y,L}(Z \mid T(Z_D^{fy})) \) by the above expression, the debt value can be written as

\[
V_y(Z) \mid_{T(Z_D^{fy}) \leq T(Z^D)} = \left( \frac{(1 - \tau) S_Z}{r_z - \theta_z} + \frac{\tau C_{fy} - (1 - \tau) K_y}{r_z} \right) \left[ 1 - \left( \frac{Z}{Z_D^{fy}} \right)^\beta \right] + (1 - \eta) \left( 1 - \tau \right) \left( \frac{S_Z^{fy}}{r_z - \theta_z} - \frac{K_y}{r_z} \right) \left[ \left( \frac{Z}{Z_D^{fy}} \right)^\beta - \gamma_z \left( \frac{Z}{Z_D^{fy}} \right)^\beta \right]
\]

5.4.5 The Developed Firm’s Default Boundary if \( T(Z_D^{fy}) > T(Z^D) \)

The default threshold is selected by shareholders. As the value of the firm until sovereign default is, by assumption, independent from the default policy, the optimal default policy is the one that maximizes
equity value at time $T(Z^D)$

$$E_y(Z \mid t=T(z^D) \mid T(z^D_y) > T(z^D)) = V_y(Z \mid t=T(z^D) \mid T(z^D_y) > T(z^D))$$

(203)

$$-D_{fy}(Z \mid t=T(z^D) \mid T(z^D_y) > T(z^D))$$

(204)

$$= (1 - \gamma_z)(1 - \tau) \left( \frac{SZ}{r_z - \theta_z} - \frac{K_y}{r_z} \right) \left[ 1 - \left( \frac{(1 - \gamma_z)Z}{Z_fy} \right)^\beta \right]$$

(205)

$$- (1 - \tau) C_{fy} \left[ 1 - \left( \frac{(1 - \gamma_z)Z}{Z_fy} \right)^\beta \right]$$

(206)

The first-order maximization yields

$$\frac{\partial E_y(Z)}{\partial Z} = \frac{(1 - \gamma_z)(1 - \tau)SZ}{r_z - \theta_z} \left[ 1 - \left( \frac{(1 - \gamma_z)Z}{Z_fy} \right)^\beta \right]$$

(207)

$$- \frac{\beta}{Z_fy} \left( 1 - \gamma_z \right)^2 (1 - \tau) \left( \frac{SZ}{r_z - \theta_z} - \frac{K_y}{r_z} \right) - \frac{(1 - \gamma_z)(1 - \tau) C_{fy}}{r_z} \left( \frac{(1 - \gamma_z)Z}{Z_fy} \right)^\beta - 1$$

(208)

Using the smooth-pasting condition $\frac{\partial [E_y(Z) \mid t=T(z^D)]}{\partial Z} \mid z=Z^D_y = 0$, we have

$$0 = \frac{\bar{S}}{r_z - \theta_z} \left[ 1 - (1 - \gamma_z)^\beta \right] - \frac{(1 - \gamma_z)\bar{S}}{r_z - \theta_z} - \frac{(C_{fy} + (1 - \gamma_z)K_y)}{Z_fy} \left( \frac{(1 - \gamma_z)Z}{Z_fy} \right)^\beta - 1$$

(209)

Finally,

$$Z^D_{fy} \mid T(z^D_y) > T(z^D) = \frac{(C_{fy} + (1 - \gamma_z)K_y) (r_z - \theta_z)}{\bar{S} r_z (1 - \gamma_z) \left( 1 - \frac{1 - (1 - \gamma_z)^\beta}{\beta (1 - \gamma_z)^\beta} \right)}$$

(210)

The above default boundary is obtained under the assumption that $T(Z^D) < T(Z^D_fy)$, which is verified when the following condition is satisfied:

$$C < (C_{fy} + (1 - \gamma_z)K_y) k_y$$

(211)

with

$$k_y = \frac{(\beta - 1) \tau \gamma_z \left( 1 - \frac{1 - (1 - \gamma_z)^\beta}{\beta (1 - \gamma_z)^\beta} \right)}{\beta \bar{S} (1 - \gamma_z) \phi}$$

(212)
5.4.6 The Developed Firm’s Default Boundary if $T(Z_{fy}) \leq T(Z^D)$

Should the developed firm default before the government of the emerging country, the optimal default policy is the one that maximizes equity value at time $t = 0$

$$E_y(Z |_{t=0}) |_{T(Z_{fy}) \leq T(Z^D)} = V_y(Z |_{t=0}) |_{T(Z_{fy}) \leq T(Z^D)} - D_{fy}(Z |_{t=0}) |_{T(Z_{fy}) \leq T(Z^D)}$$

(213)

$$\left(\frac{(1 - \tau)S}{r_z - \theta_z} - \frac{(1 - \tau) (C_{fy} + K_y)}{r_z} \right) \left[ 1 - \left( \frac{Z}{Z_{fy}^D} \right)^{\beta} \right]$$

(214)

The first-order maximization yields

$$\frac{\partial E_y(Z)}{\partial Z} = \left( \frac{(1 - \tau)S}{r_z - \theta_z} - \frac{(1 - \tau) (C_{fy} + K_y)}{r_z} \right) \left[ 1 - \left( \frac{Z}{Z_{fy}^D} \right)^{\beta} \right]$$

(215)

$$-\beta \frac{Z_{fy}^D}{Z_{fy}^D} \left( \frac{(1 - \tau)S}{r_z - \theta_z} - \frac{(1 - \tau) (C_{fy} + K_y)}{r_z} \right) \left( \frac{Z}{Z_{fy}^D} \right)^{\beta - 1}$$

(216)

Using the smooth-pasting condition $\frac{d[E_y(Z)|_{t=0}]}{dz} |_{z = Z_{fy}^D} = 0$, we have

$$0 = \frac{S}{r_z - \theta_z} - \left( \frac{S}{r_z - \theta_z} - \frac{(C_{fy} + K_y)}{Z_{fy}^D r_z} \right) \beta$$

(217)

Finally,

$$Z_{fy}^D |_{T(Z_{fy}) \leq T(Z^D)} = \frac{\beta (C_{fy} + K_y) (r_z - \theta_z)}{(\beta - 1) S \phi_0}$$

(218)

The above default boundary is obtained under the assumption that $T(Z^D) \geq T(Z_{fy}^D)$, which is verified when the government’s level of debt service $C$ satisfies

$$C \leq (C_{fy} + K_y) \frac{r_z}{S \phi_0}$$

(219)

5.4.7 The Developed Firm’s Equity Return Volatility

Let’s determine the volatility of the developed firm’s equity return, which is denoted by $\frac{dE_y}{E_y}$. Dropping the time and the regime subscript and applying Itô’s formula to $E_{y,t}$ yields,

$$dE_y(t, Z) = E_{y,t} dt + E_{y,z} dZ + \frac{1}{2} E_{y,zz} dZ dZ$$

(220)

$$= E_{y,z} \left( \theta_z Z dt + \sigma_{z,x} Z dW^x + \sigma_{z,y} Z dW^y \right)$$

(221)

$$+ E_{y,zz} \left[ (\sigma_{z,x} Z)^2 dt + (\sigma_{z,y} Z)^2 dt \right]$$

$$= \left[ \theta_z Z + (\sigma_{z,x}^2 + \sigma_{z,y}^2) \frac{E_{y,z} Z^2}{2} \right] dt + E_{y,z} Z \left( \sigma_{z,x} dW^x + \sigma_{z,y} dW^y \right)$$

(222)

Hence, the dynamics of the equity return is
\[
\frac{dE_y}{E_y} = \frac{1}{E_y} \left[ \theta_y Z + (\sigma_{z,x}^2 + \sigma_{z,y}^2) E_{y,z} Z^2 \right] dt + \frac{E_{y,z}}{E_y} \left( \sigma_{z,x} dW^x + \sigma_{z,y} dW^y \right)
\]  

(223)

where \(E_{y,z}\) and \(E_{y,zz}\) denote the first and second derivatives of \(E_y\) with respect to the state variable \(Z\), respectively. Finally, the equity return volatility is given by

\[
\sigma_{E_y} = \frac{E_{y,z} Z}{E_y} \sqrt{\sigma_{z,x}^2 + \sigma_{z,y}^2}
\]

(224)

where

\[
E_y = \left( \frac{(1 - \tau)\overline{S} Z}{r_z - \theta_z} + \frac{\tau C_{fy} - (1 - \tau) K_y}{r_z} \right) \left[ 1 - \left( \frac{Z}{Z_D} \right)^{\beta} \right] + \frac{Z}{Z_D} + V_{y,L} \left( \frac{Z}{Z_D} \right)^{\beta} - \frac{C_{fy}}{r_z} \left[ 1 - \left( \frac{(1 - \gamma_z)Z}{Z_D} \right)^{\beta} \right] - (1 - \eta)(1 - \tau)(1 - \gamma_z) \left( \frac{\overline{S} Z D}{r_z} \right) \left( \frac{K_y}{r_z} \right) \left( \frac{(1 - \gamma_z)Z}{Z_D} \right)^{\beta}
\]

(225)

and

\[
E_{y,z} = \left( \frac{(1 - \tau)\overline{S}}{r_z - \theta_z} \right) \left[ 1 - \left( \frac{Z}{Z_D} \right)^{\beta} \right] + \frac{\beta}{Z_D} \left( \frac{V_{y,L} - (1 - \tau)\overline{S} Z}{r_z - \theta_z} - \frac{\tau C_{fy} - (1 - \tau)K_y}{r_z} \right) \left( \frac{Z}{Z_D} \right)^{\beta-1} + \frac{(1 - \gamma_z)\beta}{Z_D} \left( \frac{C_{fy}}{r_z} - (1 - \eta)(1 - \tau)(1 - \gamma_z) \left( \frac{\overline{S} Z D}{r_z} \right) \left( \frac{K_y}{r_z} \right) \left( \frac{(1 - \gamma_z)Z}{Z_D} \right)^{\beta-1} \right)
\]

(226)

with

\[
V_{y,L} = \left( (1 - \gamma_z)(1 - \tau) \left( \frac{\overline{S} Z D}{r_z - \theta_z} - \frac{K_y}{r_z} \right) \right) \left[ 1 - \left( \frac{(1 - \gamma_z)Z}{Z_D} \right)^{\beta} \right] + \frac{(1 - \gamma_z)(1 - \tau)}{r_z - \theta_z} \left( \frac{\overline{S} Z D}{r_z} \right) \left( \frac{(1 - \gamma_z)Z}{Z_D} \right)^{\beta}
\]

(227)

(228)

(229)

(230)