The Optimal Composition of Public Spending in a Deep Recession*

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Abstract

We study optimal monetary and fiscal policy under commitment in an economy where monetary policy is constrained by the zero lower bound on the nominal interest rate, and where the government can allocate spending to public consumption and public investment. We show that the optimal response to an adverse shock that precipitates the economy into a liquidity trap entails a small and short-lived increase in public consumption but a large and persistent increase in public investment, which lasts well after the natural rate of interest has ceased to be negative. During this period, the optimal composition of public spending is therefore heavily skewed towards public investment. Contrary to the literature that abstracts from public investment, we find that the optimal increase in total public spending in a deep recession is sizable. However, we show that this fiscal expansion has little to do with a stabilization motive and is instead warranted by the intertemporal allocation of resources that efficiency dictates even in the absence of an output gap.

JEL classification: E4, E52, E62, H54
Key words: Public spending, Public investment, Time to build, Ramsey policies, Zero lower bound.

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1 Introduction

What is the optimal composition of a fiscal expansion in a depressed economy? Despite the widespread interest in the stimulative effects of fiscal policy generated by the unprecedentedly large stimulus plans enacted in most industrialized economies at the onset of the Great Recession, this question has remained largely overlooked by the literature. Existing studies indeed mostly focus on the size of the spending multiplier when the economy is plunged in a liquidity trap, that is, when nominal interest rates are at their zero lower bound (ZLB), and on the desirability of public spending from a welfare standpoint in such circumstances.

Two main conclusions emerge from that literature. First, the spending multiplier can be substantially large when the ZLB binds.\(^1\) Second, it is optimal to temporarily increase public spending while the economy is in a liquidity trap.\(^2\) The intuition for why public spending improves welfare is the following. When monetary policy is unconstrained—so that it can replicate the flexible-price allocation—and to the extent that government spending provides utility to households, optimality requires that the marginal utilities of private and public spending be equated, a condition commonly referred to as the Samuelson rule. The latter thus implies that government spending co-moves with consumption: if an adverse shock causes a fall in consumption, it will also lead to a fall in public spending. When the ZLB binds, however, nominal interest rates cannot be used to stabilize the economy, and an undesired (negative) output gap emerges, creating an additional motive for varying public spending. In response to an adverse shock, public spending rises to help close the output gap.

Bilbiie et al. (2014), however, argue that the optimal increase in public spending in response to a typical recession is tiny—less than 0.5 percent of steady-state output. Essentially, this result reflects the fact that optimal public spending needs to strike a balance between stabilizing the output gap and meeting the Samuelson condition. Under empirically plausible scenarios about the size of the adverse shock, optimal public spending rises but only by a small amount in order not to deviate too far from the Samuelson condition. To obtain large optimal levels of public spending, Bilbiie et al. (2014) need to assume implausibly severe recessions that take the economy close to the starvation point (the point where private consumption is zero). Furthermore, Sims & Wolff (2013) argue that the welfare multiplier of public spending, defined as the change in aggregate welfare for a one unit change in government spending, albeit positive, is procyclical. That is, it tends to

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\(^1\)See Christiano et al. (2011), Eggertsson (2011) and Woodford (2011) among many others.

be low during recessions and high during expansions. Together, these findings cast doubt on the usefulness of public spending from a normative perspective.

A common characteristic of all of these studies is the assumption that public spending consists exclusively of purchases of consumption goods, so that there is no scope for public investment. This assumption is unlikely to be innocuous when it comes to determining the optimal level of public spending and its welfare consequences. But perhaps more importantly, it precludes the analysis of the optimal composition of a fiscal expansion, an issue that was at the center of policy debates during the Great Recession. The various fiscal plans that have been implemented worldwide in 2008-2009 assigned a significant fraction of the additional spending to public investment in infrastructure, but to our knowledge, there has not been any formal attempt to determine whether this allocation scheme was warranted from a welfare standpoint.

The objective of this paper is to study optimal monetary and fiscal policy in an economy where monetary policy is constrained by the ZLB on the nominal interest rate, and where the government can allocate spending to public consumption and public investment. More specifically, we study the optimal policy response under commitment to an adverse shock that precipitates the economy into a deep recession characterized by a liquidity trap. The main difference of our model with respect to those studied in the literature cited above is the possibility for the government to accumulate public capital, which is an external input in the firms’ production technology. As in Leeper et al. (2010), Leduc & Wilson (2013) and Bouakez et al. (2015), we assume that the accumulation of public capital is subject to lengthy time-to-build delays, a distinctive feature of public infrastructure projects. As a benchmark, we compute the first-best (efficient) allocation, i.e., the welfare-maximizing allocation chosen by a benevolent central planner.

We find that the optimal policy response to an adverse shock that makes the ZLB bind is to initially raise public consumption and public investment above their steady-state levels. The increase in public consumption is negligible and short-lived, followed by a prolonged cut that persists even after the natural rate of interest has ceased to be negative. In contrast, the increase in public investment is large—reaching 1.7 percent of steady-state output at the peak—and persistent, lasting well after the natural rate of interest has become positive again. These patterns are subsequently reversed, as the optimal plan eventually entails raising public consumption and decreasing public investment relative to their steady-state levels.

\[3\text{For instance, the American Recovery and Reinvestment Act and the European Economic Recovery Plan allocated, respectively, 40 and 71 percent of the additional public spending to investment in infrastructure.}\]

\[4\text{Leeper et al. (2010) and Leduc & Wilson (2013) do not consider the case of a binding ZLB, and none of the three papers studies optimal policy.}\]
Our findings have two salient implications. First, the optimal size of a fiscal expansion in a severe economic downturn is sizable. The cumulative increase in total public spending amounts to roughly 12 percent of steady-state output. This result stands in sharp contrast with the conclusion based on optimal fiscal plans in which only public consumption is adjusted, as is the case in existing studies. When we exclude public investment from the set of policy instruments that are available to the policymaker, the optimal plan is such that the cumulative increase in public spending is virtually zero. Second, the optimal plan features a change in the composition of public spending in a way that assigns a larger weight to public investment for a prolonged period of time after the shock. At the peak, the fraction of public investment in total public spending exceeds its steady-state value by 6 percentage points in our baseline simulation.

We then ask: how much of this fiscal expansion is due to the fact that the ZLB is binding, or, equivalently, to the fact that the economy is producing below its potential? We refer to this spending component as *stimulus spending* and we compute it as the difference between the spending level obtained under the Ramsey allocation and that obtained under fully flexible prices. The latter, which we label *neoclassical spending*, would occur if monetary policy were able to fully eliminate any output gap. We find that the stimulus components of public consumption and public investment are frontloaded but negligible, summing to less than 0.1 percent of steady-state output at the time of the shock and cumulating to approximately −0.1 percent of steady-state output. This result highlights an important point: the desirability of a fiscal expansion and the larger weight assigned to public investment in a response to an adverse shock have very little to do with a stabilization motive. Instead, they mostly reflect the role of public capital in enabling the intertemporal allocation of resources that efficiency dictates even in the absence of an output gap.

We also evaluate the welfare gains associated with the optimally designed fiscal plan relative to a scenario in which only the nominal interest rate is chosen optimally. When both public consumption and public investment are chosen optimally, this gain is two orders of magnitude larger than the gain achieved by only adjusting public consumption while keeping public investment constant.

[To be Completed]

2 A New Keynesian Model with Public Capital

We consider a simple new-Keynesian economy without private capital as in Bouakez et al. (2015). The economy is composed of infinitely lived households, firms, a government, and a monetary au-
thority. The key feature of the model is that a fraction of government spending is invested in public capital subject to a time-to-build requirement. The stock of public capital enters as an external input in the production of intermediate goods, which are used to produce an homogenous final good. The latter is used for consumption and investment purposes. Intermediate-good producers are monopolistically competitive and set their prices subject to a Rotemberg (1982)-type adjustment cost, whereas final-good producers are perfectly competitive. The monetary authority sets the nominal interest rate according to a Taylor-type rule subject to a non-negativity constraint.

2.1 Households

The economy is populated by a large number of identical households who have the following lifetime utility function:

$$E_t \sum_{s=0}^{\infty} \beta^s \xi_{t+s} U (C_{t+s}, N_{t+s}, G_{C,t+s})$$, (1)

where $\beta$ is the discount factor, $C_t$ is consumption, $N_t$ denotes hours worked, $G_C$ is utility enhancing government spending as in Woodford (2011) and $\xi_t$ is a preference shock that evolves according to the following process:

$$\log(\xi_t) = \rho \log(\xi_{t-1}) + \epsilon_t, \quad \epsilon_t \sim N(0, 1)$$,

where $\rho \in (0, 1)$. The representative household enters period $t$ with $B_{t-1}$ units of one-period riskless nominal bonds. During the period, it receives a nominal wage payment, $W_t N_t$, and dividends, $D_t$, from the monopolistically competitive firms. This income is used to pay a lump-sum tax, $T_t$, to the government, to consumption, and to the purchase of new bonds. The household’s budget constraint is therefore

$$P_t C_t + T_t + \frac{B_t}{1 + R_t} \leq W_t N_t + D_t + B_{t-1}$$, (2)

where $P_t$ is the price of the final good, $W_t$ is nominal wage rate, and $\frac{1}{1 + R_t}$ is the price of a nominal bond purchased at time $t$, $R_t$ being the nominal interest rate. The household maximizes (1) subject to (2) and to no-Ponzi-game condition. The first order conditions for this problem are given by

$$W_t = -\frac{U_{N,t}}{U_{C,t}}$$, (3)

$$\frac{1}{1 + R_t} = \beta E_t \left( \frac{\xi_{t+1} U_{C,t+1}}{\xi_t U_{C,t}} \frac{P_t}{P_{t+1}} \right)$$, (4)

where $W_t = \frac{W_t}{P_t}$ is the real wage rate and $U_{X,t} = \partial U (C_t, N_t, G^c_t) / \partial X_t$. 

4
2.2 Firms

The final good is produced by perfectly competitive firms using the following constant-elasticity-of-substitution technology:

\[ Y_t = \left( \int_0^1 Y_t(z) \, dz \right)^{\frac{\theta}{\theta - 1}}, \quad (5) \]

where \( Y_t(z) \) is the quantity of intermediate good \( z \) and \( \theta \geq 1 \) is the elasticity of substitution between intermediate goods. Denoting by \( P_t(z) \) the price of intermediate good \( z \), demand for \( z \) is given by

\[ Y_t(z) = \left( \frac{P_t(z)}{P_t} \right)^{-\theta} Y_t. \quad (6) \]

Firms in the intermediate-good sector are monopolistically competitive, each producing a differentiated good using labor as a direct input and public capital as an external input

\[ Y_t(z) \leq F(N_t(z), K_{G,t}), \quad (7) \]

where \( F(.) \) is increasing and concave in its both arguments. We assume that firms set prices subject to a Rotemberg (1982)-type adjustment cost. That is, in each period, a given firm pays a quadratic adjustment cost to resets its nominal price \( P_t(z) \), measured in (real) terms of the final good and given by

\[ \Xi_t(z) = \frac{\psi}{2} \left( \frac{P_t(z)}{P_{t-1}(z)} - 1 \right)^2 Y_t, \quad (8) \]

where \( \psi \geq 0 \) governs the magnitude of the price adjustment cost.

Real dividends paid by firm \( z \) is given by

\[ D_t(z) = (1 + \tau) \frac{P_t(z)}{P_t} Y_t(z) - W_t N_t(z) - \Xi_t(z), \quad (9) \]

where \( \tau = 1/(\theta - 1) \) is a subsidy that corrects the steady-state distortion stemming from monopolistic competition in the goods market.

Price-adjustment costs make the firm’s problem dynamic. More specifically, the firm chooses \( P_t(z) \) for all \( t \) to maximize its total real market value

\[ \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \xi_{t+s} \frac{U_{C,t+s}}{U_{C,t}} D_{t+s}(z), \quad (10) \]

where \( D_t(z) \) is given by (9), and subject to the production technology (7), and the Hicksian demand function (6).
Since all the firms face an identical problem, the optimal price will satisfy the following condition:

\[
0 = \theta (MC_t - 1) - \psi \left[ (1 + \pi_t) \pi_t - \beta \mathbb{E}_t \frac{\xi_{t+1} U_{C,t+1}^t Y_{t+1}^t}{\xi_t U_{C,t}^t Y_t^t} (1 + \pi_t) \pi_{t+1} \right],
\]  

(11)

where \( \pi_t = \frac{P_t}{P_{t-1}} - 1 \) is the inflation rate, and \( MC_t \) is the marginal cost of production, defined by

\[
MC_t = \frac{W_t}{F_N(N_t, K_{G,t})}.
\]

(12)

### 2.3 Fiscal and monetary authorities

The government levies lump-sum taxes to finance its expenditures and the subsidy given to firms in the intermediate-good sector. Its budget constraint is given by

\[
G_t + \tau Y_t = \frac{T_t}{P_t},
\]

(13)

where \( G_t \) is government spending, which is composed of two parts, public consumption and public investment

\[
G_t = G_t^c + G_t^i.
\]

(14)

Public investment increases the stock of public capital according to the following accumulation equation:

\[
K_{t+T} = (1 - \delta) K_{t+T-1} + \left( 1 - S \left( \frac{G_t^i}{G_t^i} \right) \right) G_t^i,
\]

(15)

where \( T \geq 0 \) and the function \( S(\cdot) \) satisfies \( S'(\cdot) \geq 0, S''(\cdot) \geq 0, S(1) = 0, \) and \( S'(1) = 0. \) Equation (15) allows for the possibility that several periods may be required to build new productive capital, i.e., time to build (see Kydland & Prescott (1982)). This feature reflects the implementation delays typically associated with the different stages of public investment projects (planning, bidding, contracting, construction, etc.). The function \( S(\cdot) \) captures adjutment costs of public investment, as an additional unit of investment at time \( t \) increases the stock of capital at time \( t + T \) by less than one unit.

To close the model, we need three equations to pin down the path of i) utility enhancing government spending, ii) public investment and iii) the nominal interest rate. Following much of the literature on optimal fiscal policy (add references), we will assume that the government is able to commit to a certain path for these policy variables. Thus, the latter will be the solution of a Ramsey problem.
2.4 Market clearing

Since households are identical and there is no private capital, the net supply of bonds must be zero in equilibrium ($B_t = 0$). Plugging the definition of profits in the budget constraint of the representative household, one gets the resource constraint of this economy

\[ \Delta_t F(N_t, K_{G,t}) = \Delta_t Y_t = C_t + G_t, \tag{16} \]

where $\Delta_t = \left(1 - \frac{\psi}{2} \pi_t^2 \right)$.

3 The First-Best Allocation

In order to gain some intuition into the welfare implications of varying public consumption and investment over the business cycle, it is useful to have as a benchmark the efficient allocation of resources, i.e., the allocation that would be chosen by a benevolent central planner. Throughout the paper we will refer to this equilibrium allocation as the first best. In what follows, we characterize the efficient allocation in response to a negative preference shock.

3.1 Maximization program and solution

The planner’s problem is to choose the sequence of allocations that maximize households’ lifetime utility given the sequence of the economy’s resource constraints and the production technology. Formally,

\begin{align*}
\max_{Z_t} & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \xi_t U(C_t, N_t, G_t^c) \\
& + \lambda_{1,t} \left[ F(N_t, K_{G,t}) - C_t - G^c_t - G_i^c \right] \\
& + \lambda_{2,t} \left[ (1 - \delta) K_{G,t+T-1} + \left(1 - S \left( \frac{G_i^c}{G_{t-1}} \right) \right) G_i^c \right] - K_{G,t+T} \right\},
\end{align*}

where $Z_t = [C_t, G_t^c, N_t, G_i^c, K_{G,t+T}]$ is the vector of choice variables and the $\lambda$’s are Lagrange multipliers. Note that, due to the time to build delay, the planner does not have any control over $K_{t+i}$ ($i < T$) at time $t$, and that the true choice variable is thus $K_{t+T}$. The efficient allocation is
the solution to the following set of equations:

\[ U_{C,t} = \lambda_{1,t}/\xi_t, \quad (17) \]
\[ U_{G,t} = \lambda_{1,t}/\xi_t, \quad (18) \]
\[ -\frac{U_{N,t}}{U_{C,t}} = F_N(N_t, K_{G,t}), \quad (19) \]

\[ \lambda_{1,t} = \left[ 1 - S \left( \frac{G_t^i}{G_{t-1}^i} \right) - S' \left( \frac{G_t^i}{G_{t-1}^i} \right) \right] \lambda_{2,t} + \beta \mathbb{E}_t \lambda_{2,t+1} S' \left( \frac{G_{t+1}^i}{G_t^i} \right) \left( \frac{G_{t+1}^i}{G_t^i} \right)^2 \quad (20) \]

\[ \lambda_{2,t} = \beta (1 - \delta) \mathbb{E}_t \lambda_{2,t+1} + \beta^T \mathbb{E}_t \lambda_{1,t+T} F_{K_G}(N_{t+T}, K_{G,t+T}), \quad (21) \]

\[ F(N_t, K_{G,t}) = C_t + G_t^C + G_t^i, \quad (22) \]

\[ K_{G,t+T} = (1 - \delta) K_{G,t+T-1} + \left( 1 - S \left( \frac{G_t^i}{G_{t-1}^i} \right) \right) G_t^i \quad (23) \]

Equation (30) defines the Lagrangian multiplier \( \lambda_1 \) as the marginal utility of consumption scaled by the preference shock. Equations (30) and (31) imply the so-called Samuelson condition:

\[ U_{G,t} = U_{C,t}, \quad (24) \]

which equates the marginal utilities of private and public consumption. This condition states that since final output can be transformed into public as well as private consumption goods, the marginal rate of substitution between \( C_t \) and \( G_t^C \) must be equal to their marginal rate of transformation, which is 1. Equation (32) equates the marginal rate of substitution between consumption and labor to their marginal rate of transformation, which is equal to the marginal product of labor. Equation (33) is the first order condition with respect to public investment, relating the shadow value of private consumption to that of public capital. Without investment adjustment cost, we have \( \lambda_{1,t} = \lambda_{2,t} \). The Euler equation (34) equates the costs and benefits of an additional unit of capital in period \( t \). The cost is given by the left-hand side of the equation and is equal to the shadow value of capital in utils (\( \lambda_{2,t} \)). The two terms in the right-hand side of the equation are interpreted as follows. At time \( t + 1 \), the invested unit of capital is worth \( \beta (1 - \delta) \mathbb{E}_t \lambda_{2,t+1} \) in utils as of time \( t \). Furthermore, investing in public capital today will generate higher output \( T \) periods in the future by a factor of \( F_{K_G,t+T} \), which is valued at \( \beta^T \mathbb{E}_t U_{C,t+T} \) in utils as of time \( t \). Finally, equations (35) and (36) are, respectively, the resource constraint and the accumulation equation for public capital.
3.2 Functional forms and calibration

In order to study the way in which the efficient allocation changes in responses to a negative shock to $\xi_t$, we need to specify functional forms for the utility and production functions, and to assign values to the model parameters. We assume that preferences are given by

$$U(C_t, N_t, G_t^c) = \frac{(C_t^\gamma(1-N_t)^{1-\gamma})^{1-\sigma}}{1-\sigma} + \chi \frac{(G_t^c)^{1-\sigma}}{1-\sigma}$$

if $\sigma \neq 1$

$$= \gamma \ln C_t + (1-\gamma) \ln(1-N_t) + \chi \ln (G_t^c)$$

if $\sigma = 1$,  

where $\sigma > 0$ and $0 < \gamma \leq 1$, and that the production function takes the form

$$F(N_t, K_{G,t}) = N_t^a K_{G,t}^b,$$  

where $0 \leq a, b \leq 1$. This specification nests the linear technology assumed by Christiano et al. (2011) and Woodford (2011) as a special case in which $a = 1$ and $b = 0$. We also assume that the adjustment-cost function, $S$, is given by

$$S \left( \frac{G_t^i}{G_{t-1}^i} \right) = \frac{\varpi}{2} \left( 1 - \frac{G_t^i}{G_{t-1}^i} \right)^2,$$

where $\varpi > 0$.

Our calibration closely follows Bouakez et al. (2015), and is summarized in Table 1. The values of $\beta$, $\sigma$, and $\gamma$ are based on Christiano et al. (2011). The value of $b$ is based on the meta-regression results of Bom & Ligthart (2013). The elasticity of substitution between domestic goods, $\theta$, is chosen so as to yield a steady-state markup of 20 percent. The price-cost-adjustment parameter, $\psi$, is set such that, conditional on the chosen value of $\theta$, it implies a slope of the (linearized Phillips curve) equal to 0.03. Consistent with the evidence discussed in Leeper et al. (2010), Leduc & Wilson (2013), and Bouakez et al. (2015) regarding the delays associated with the completion of public investment projects, we set $T = 16$. We also follow Leeper et al. (2010) and set the depreciation rate, $\delta$, to 0.02. The investment-adjustment-cost parameter, $\varpi$, is more difficult to pin down, as empirical estimates are only available for private investment. We set $\varpi = 2.5$, which is very close to the macro estimate of 2.48 obtained by Christiano et al. (2005) and to the micro estimate of 1.86 obtained by Eberly et al. (2012) for private investment. Finally, to calibrate $\chi$, we exploit the steady-state properties of the model. More specifically, evaluating (24) at the steady state, and
using (25) we obtain
\[ \gamma C^{\gamma(1-\sigma)-1}(1-N)(1-\gamma)(1-\sigma) = \chi (G^c)^{-\sigma}. \] (27)

Moreover, evaluating (34) at the steady state, and using (26), we obtain
\[ \frac{G^i}{Y} = \frac{\delta K}{Y} = \frac{b\beta T}{1 - \beta(1 - \delta)}. \] (28)

Using this result, equations (32)–(36) and (27) can be used to obtain the steady-state values of \( C, N, K, \) and \( G \) given \( \beta, b, \delta, \chi, \) and \( T. \) Alternatively, one can choose the value of \( \chi \) that is consistent with a given steady-state level of public spending. We choose to calibrate \( \chi \) such that the steady-state ratio of total government spending to output, \( g \equiv \frac{G^c + G^i}{Y} = \frac{G^c + \delta K}{Y}, \) is equal to 0.2. Given our calibration, the implied share of public investment in total public spending, \( \alpha \equiv \frac{G^i}{G^c + G^i}, \) is equal to 0.2286, which is very close to the average of 0.23 that we observe in U.S data.

<table>
<thead>
<tr>
<th>Table 1: Parameter values</th>
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<tbody>
<tr>
<td>Discount factor ( \beta = 0.99 )</td>
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<tr>
<td>Preference parameter ( \sigma = 2 )</td>
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<tr>
<td>Preference parameter ( \gamma = 0.29 )</td>
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<tr>
<td>Elasticity of output w.r.t public capital ( b = 0.08 )</td>
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<tr>
<td>Elasticity of output w.r.t hours worked ( \alpha = 1 )</td>
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<tr>
<td>Elasticity of substitution between intermediate goods ( \theta = 6 )</td>
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<tr>
<td>Time-to-build delay ( T = 16 )</td>
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<tr>
<td>Price-adjustment-cost parameter ( \psi = 200 )</td>
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<tr>
<td>Depreciation rate of public capital ( \delta = 0.02 )</td>
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<tr>
<td>Investment-adjustment-cost parameter ( \varpi = 0.25 )</td>
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<tr>
<td>Steady-state ratio of public spending to output ( g = 0.2 )</td>
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<tr>
<td>Autocorrelation of the preference shock ( \rho = 0.9 )</td>
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</table>

3.3 The efficient response to a negative preference shock

Assume that the economy is initially at the steady state when a negative preference shock hits. The shock is normalized to 1 percent and is assumed to be persistent, with an autocorrelation coefficient of 0.9. Using a linearized version of the equilibrium conditions (30)–(34), we compute the economy’s response to the shock. The results are depicted in Figure 1. All the responses, except that of hours worked, are expressed as percentage deviations from steady-state output.

The negative preference shock increases households’ desire to save. Because the accumulation of (public) physical capital allows the intertemporal substitution of consumption while raising future production capacities, current consumption falls and public investment rises in response to the
Figure 1: First best allocation after a negative preference shock

shock. Due to the time-to-build delays, public investment increases the stock of capital – and thus the marginal productivity of labor – 16 quarters later. Public investment falls, however, below its steady-state level prior to $t = 16$ in order to enable consumption to increase before public capital becomes productive (i.e., consumption smoothing). Note that due to investment-adjustment costs, the response of public investment is hump shaped. In the absence of these costs, the maximum increase in investment would take place at the time of the shock, and the response would be less persistent. The figure also shows that the optimal path of government consumption follows closely that of private consumption, in accordance with the Samuelson condition.

Hours worked respond in an opposite way to consumption, at least as until $t < 16$. This follows from (32): since public capital is predetermined for $t < 16$, and given our assumptions that the production function is concave in labor, that preferences are quasi-concave, and that consumption and leisure are assumed to be normal goods, an increase in the marginal utility of consumption ($U_{C,t}$) requires an increase in the marginal disutility of labor ($-U_{N,t}$) to restore the equilibrium. This in turn implies that hours worked must increase following the shock.

The main take-away from these results is that the efficient response of public investment to a (negative) preference shock differs drastically from that of public consumption. At least during the first quarters following the shock, the optimal allocation of resources calls for a simultaneous increase in public investment and a cut in public consumption. In other words, the efficient response to the
shock requires a substantial change in the composition of public spending, assigning a substantially larger weight on public investment compared with the steady-state allocation.

4 Optimal Monetary and Fiscal Policy

The policy maker has three tools: i) the nominal interest rate ii) government consumption and iii) government investment. The optimal path for these instruments following a demand shock is the solution to the following Ramsey problem, expressed as a Lagrangian:

\[
\mathcal{L}_0 \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \xi_t U(C_t, N_t, G_t^c) + \phi_{1,t} \left[ \beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{1 + \pi_{t+1}} (1 + R_t) - U_{C,t} \right] + \phi_{2,t} \left[ \psi \Lambda_{t+1} \frac{F(N_{t+1}, K_{G,t+1})}{F(N_t, K_{G,t})} d(\pi_{t+1}) - \psi d(\pi_t) - \theta \frac{U_{N,t}}{U_{C,t}} F_{N,t} - \theta \right] + \phi_{3,t} \left[ F(N_t, K_{G,t}) \left( 1 - \frac{\psi}{2} \pi_t^2 \right) - C_t - G_t^c - G_t^i \right] + \phi_{4,t} \left[ (1 - \delta) K_{G,t+1} - \left( 1 - S \left( \frac{G_t^i}{G_{i-1}^i} \right) \right) G_t^i - K_{G,t+1} \right] + \phi_{5,t} [R_t - 0] \right\},
\]

where we have defined \( d(\pi_t) \equiv \pi_t (1 + \pi_t), \ F_{N,t} = F_N(N_t, K_{G,t}) \) and \( \Lambda_{t+1} = \beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{U_{C,t}}. \)

After deriving the (non-linear) first order conditions (see the appendix for details) with respect to the control variables, which are grouped in the vector \( Z_t = [Z_t, R_t, \pi_t] \), the system of equilibrium conditions is solved up to a first-order approximation around the steady state. In what follows, we study the Ramsey-optimal policy under two distinct cases: one in which the demand shock is relatively small, such that the economy never hits the ZLB, and one in which the demand shock is large enough to make the ZLB binds, sending the economy in a liquidity trap. We use the piecewise linear perturbation algorithm developed by Guerrieri & Iacoviello (2014) to deal with the ZLB constraint.\(^5\) In what follows, we will refer to the level of public spending that would be optimal under fully flexible prices as neoclassical spending and label as stimulus spending the difference between the spending level obtained under the Ramsey plan and neoclassical spending.

\(^5\)Guerrieri & Iacoviello (2014) show that their algorithm approximates reasonably well the global solution in models with an occasionally binding ZLB constraint. In section 4.4.3, we study the sensitivity of our results to the use of a non-linear solution method.
Stimulus spending would therefore solely be due to the presence of price stickiness (or, equivalently, the presence of a non-zero output gap).

4.1 Non-binding ZLB

Consider first the case in which monetary policy is not constrained by the ZLB. Since the only distortion in this economy stems from price rigidity in the goods market, monetary policy can, in the absence of (inefficient) supply shock and abstracting from public spending, replicate the efficient allocation by equating the nominal interest rate to the (efficient) natural rate of interest. This policy fully stabilizes inflation and the output gap in every period, a result that has come to be known as the divine coincidence (Blanchard & Galí (2007)). The optimal levels of government consumption and investment are therefore obvious: they must coincide with those obtained under the efficient allocation, discussed in Section 3. In other words, to the extent that monetary policy can manoeuvre freely without hitting the ZLB, the Ramsey allocation replicates the first best. In this case, Keynesian stimulus is obviously nil.

4.2 Binding ZLB

We now consider the scenario in which the demand shock is large enough to precipitate the economy into a liquidity trap. To calibrate the size of this shock in a realistic manner, we proceed as follows. Consider a version of our economy in which monetary policy is set according to the following Taylor rule

\[ R_t = \max \{0; \phi \pi_t - \ln(\beta)\}, \]

where \( \phi > 1 \). We select the size of the shock such that the resulting decline in output when \( \phi = 1.5 \) and in the absence of any fiscal-policy response matches the observed decline in U.S. GDP from peak to trough during the Great Recession, which amounted to 5.65%. With an autocorrelation coefficient of 0.9, such a shock generates a liquidity trap of 8 periods. Below, we discuss three different sets of policy responses: In the first, only the nominal interest rate is used as an instrument; in the second, both the nominal interest rate and public consumption are used; and in the third, government investment is added. Studying the optimal choice of public investment as a tool to stimulate an economy plunged in a liquidity trap is the main novelty of this paper with respect to the existing literature, which has focused exclusively on optimal public consumption.

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\(^6\)Recall that the monopolistic competition distortion is corrected using per-unit subsidy given to monopolistically competitive producers.
4.2.1 Optimal monetary policy

We start with the case in which the Ramsey planner chooses the nominal interest optimally while keeping public consumption and investment at their (optimal) steady-state levels, a scenario that we will refer to as scenario A. Figure 2 shows that, in response to the negative demand shock, the nominal interest rate falls until it reaches the ZLB floor where it remains for 6 quarters before gradually reverting to its steady-state level (see Figure 2). In line with the results obtained by Werning (2011), Nakata (2013), and Nakata (2015), consumption and inflation fall initially but rise subsequently during a few quarters before falling again below their steady-state levels, to which they ultimately converge from below. The Ramsey allocation deviates from the flexible-price allocation, which would be efficient in the absence of fiscal instruments. The reason is that the natural rate of interest., i.e, the nominal rate that implements the flexible-price allocation is negative during the first 4 periods. As is well known by now, the optimal policy in this case is to commit to keeping the nominal rate at zero even after the natural rate has become positive.\(^7\) By doing so, the Ramsey planner commits to generating a boom in consumption at some point in the future, which raises inflation expectations. Although the negative output gap is not fully eliminated (−0.35 percent), it is much smaller than that obtained under the Tylor rule above.

4.2.2 Optimal monetary and fiscal policy without public investment

Next, consider the case where both the nominal interest rate and public consumption are chosen optimally, while public investment is kept constant at its steady-state level, a plan that we will refer to as scenario B. Under this scenario, depicted in Figure 3, the nominal interest rate, inflation, and consumption exhibit very similar responses to those obtained under scenario A. In particular, the nominal interest rate remains at the ZLB for 6 quarters and consumption falls initially before temporarily overshooting its steady-state level in the subsequent periods. In contrast, optimal public consumption rises during the first three quarters after the shock, then falls during the subsequent 4 quarters before eventually returning to its steady-state value. In other words, the optimal fiscal plan is front-loaded, and eventually entails a fiscal consolidation, a pattern also shown by Werning (2011), Nakata (2013), and Nakata (2015). Note that, under flexible prices, optimal public consumption remains equal to its steady-state level, which means that public spending under the Ramsey plan is entirely a stimulus spending.

\(^7\)See Eggertsson & Woodford (2003), Jung et al. (2005) and Adam & Billi (2006).
the Ramsey plan, differentiate the households’ lifetime utility with respect to government consumption and evaluate it at the maximum. This yields

$$U_{C,t} \frac{dC_t}{dG_t^c} + U_{N,t} \frac{dN_t}{dG_t^c} + U_{G,t} + \mathbb{E}_t \sum_{s=t+1}^{\infty} \beta^{s-t} \left\{ \xi_s \frac{dU(C_s, N_s, G_s^c)}{dG_t^c} \right\} = 0.$$  

For simplicity, let us abstract from the effects of $G_t^c$ on future variables. Since $\frac{dC_t}{dG_t^c} = \Delta_t \left( \frac{dY_t}{dG_t^c} + \frac{Y_t}{\Delta_t} \frac{d\Delta_t}{dG_t^c} \right) - 1$ and $\frac{dN_t}{dG_t^c} = \frac{dY_t}{dG_t^c} F_{N,t}^{-1}$, the condition above can be rewritten as

$$U_{G,t} = U_{C,t} + \left( \Delta_t U_{C,t} + \frac{U_{N,t}}{F_{N,t}} \right) \frac{dY_t}{dG_t^c} + U_{C,t} Y_t \frac{d\Delta_t}{dG_t^c}.$$  

Under flexible prices, $\Delta_t = 1$, $d\Delta_t = 0$ and $U_{C,t} = -U_{N,t} F_{N,t}^{-1}$ so that $U_{G,t} = U_{C,t}$. In the decentralized economy with sticky prices, this equation can be re-written as

$$U_{G,t} = U_{C,t} + \left[ \Delta_t - 1 + \frac{\psi}{\theta} (d(\pi_t) - \beta \mathbb{E}_t^t \Delta_t+1 d(\pi_{t+1})) \right] U_{C,t} \frac{dY_t}{dG_t^c} + U_{C,t} Y_t \frac{d\Delta_t}{dG_t^c}.$$  

Figure 2: Ramsey optimal monetary policy after a negative preference shock
The last term in the right-hand side of this equation is of second order and will therefore disappear in a log-linear version of the equilibrium conditions. Thus, to the extent that monetary policy can fully stabilize inflation (thus eliminating the output gap), we recover the Samuelson condition \( U_{G,t} = U_{C,t} \). However, when monetary policy is constrained by the ZLB so that full stabilization is unattainable, the optimal choice of \( G^c_t \) needs to strike a balance between two competing objectives: filling the output gap (or, equivalently, stabilizing inflation) and satisfying the Samuelson condition. The reason for the tradeoff is the following. On the one hand, because the negative demand shock raises the marginal utility of consumption, the Samuelson condition calls for a fall in \( G^c_t \). On the other hand, public consumption raises aggregate demand and inflation, and actually attenuates the decline in private consumption when the ZLB binds, and this motive calls for an increase in \( G^c_t \).\(^8\)

\(^8\)A well established result is that the output multiplier associated with public consumption is well above 1 when the ZLB binds. The inflationary effect of higher public spending lowers the real interest rate (since the nominal rate is constant) and raises consumption.
The optimal policy response in this case is therefore to raise $G_t^c$, but not to the extent of fully stabilizing the output gap, as this would entail too much deviation from the Samuelson condition.

In fact, the optimal increase in public consumption is very small: the largest response, which occurs immediately after the shock, is roughly 0.05 percent of steady-state output. The total size of the fiscal stimulus plan in this case is given by the cumulative variation of public spending from its steady-state level as a percentage of steady-state output, $\sum_{t=0}^{\infty}(G_t - G)/Y$, where a letter without time subscript denotes the steady-state value of the corresponding variable. Since government investment is constant in this scenario, the numerator only captures changes in public consumption. As expected from the path of public consumption, the total size of the stimulus is virtually nil: $-0.003$ percent of steady-state output. Though in a different setting, this result echoes Bilbiie et al. (2014)’s findings that the optimal level of public spending is tiny, even when that latter is utility-enhancing.

The close resemblance of the Ramsey allocations obtained under scenarios A and B in turn suggests that the welfare gain from optimally adjusting $G_t^c$ is quite small. To verify this conjecture, we compute the compensating variation in consumption, i.e., the percentage in consumption that would make households as well off under scenario A as under scenario B. The results, reported in Table 1, indicate that this quantity is indeed negligible ($2 \times 10^{-4}$ percent).

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Instrument</th>
<th>Welfare gain relative to A</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>$R, G^c$</td>
<td>$2 \times 10^{-4}$</td>
</tr>
<tr>
<td>C</td>
<td>$R, G^c, G^i$</td>
<td>0.082</td>
</tr>
</tbody>
</table>

Notes: Entries are expressed in percent. Scenario A corresponds to the Ramsey plan in which only the nominal interest rate is adjusted optimally while public spending is kept constant. Welfare gains are measured by the compensating variation in consumption, i.e., the percentage change in consumption that would make households as well off under scenario A as under the alternative scenario.

### 4.2.3 Public investment as an additional tool

Finally, consider the case in which the set of policy instruments is enlarged to include public investment. The economy’s optimal response to a negative preference shock in this case – which we label scenario C – is shown in Figure ???. The response of private consumption is characterized by a protracted decline followed by a persistent increase above its-steady state level. The initial decline

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9We truncate the summation at horizon 1000.
in consumption is larger than that occurring under the efficient allocation of resources, giving rise to a negative output gap. Public consumption increases immediately after the shock by 0.06 percent of steady-state output but falls persistently in the subsequent periods. Public investment initially increases by 0.6 percent of steady-state output and reaches its maximum response of 1.7 percent of steady-state output 3 quarters after the shock. It remains above average for more than 4 years before eventually falling below its steady-state level. Once the ZLB has ceased to bind, the Ramsey allocation tracks the first best almost exactly.

Three important observations about optimal fiscal policy are worth emphasizing. First, as is the case with the first-best response to the preference shock, the Ramsey plan entails a change in the composition of public spending in a way that assigns a relatively larger weight to public investment. Figure 5 depicts the (time-varying) optimal share of public investment in total public spending. At the steady state, this fraction is equal to 22.8 percent. Immediately after the shock, it surges to 25.2 percent, reaches a peak of roughly 29 percent, before falling steadily until it reaches
a trough of roughly 21.7 percent at around 24 quarters after the shock. It then converges towards its steady-state value from below.

Figure 5: Optimal share of public investment in total public spending (in percent).

Second, the cumulative increase in total public spending in response to the shock is substantial, amounting to 11.6 percent of steady-state output. This result contrasts sharply with that obtained under scenario B – and in the literature cited above –, in which public spending plays no productive role. Third, most of the increase in public spending involves public investment and is almost entirely “neoclassical” in nature. In other words, the stimulus component of the fiscal expansion is negligible. The latter observation is illustrated in Figure 6, which depicts the stimulus component of public consumption and investment in response to the shock. In both cases, stimulus spending – albeit positive at the time of the shock – is tiny, reaching a maximum of roughly 0.085 percent of steady-state output for public consumption and 0.015 percent of steady-state output for public investment. In cumulative terms, total stimulus spending amounts to −0.12 percent of steady-state output.

In order to understand why the stimulus component of public investment is positive when the shock hits, recall that the first-order condition associated with the optimal choice of $K_{G,t+T}$ in the efficient allocation (i.e., equation (34)) calls for an increase in public investment following the
shock. In the Ramsey problem, the analogous first-order condition is given by

$$
\phi_{4,t} = \beta (1 - \delta) E_t \phi_{4,t+1} + b \beta^T E_t \left[ \phi_{3,t+T} \frac{Y_{t+T}}{K_{G,t+T}} \left( 1 - \psi \frac{\pi_{t+T}}{2} \right) - \theta \phi_{2,t+T} \frac{MC_{t+T}}{K_{G,t+T}} \right] - \psi b \beta^T E_t \phi_{2,t+T} A_{t+T+1} d(\pi_{t+T+1}) \frac{Y_{t+T+1}}{K_{G,t+T} Y_{t+T}} + \psi \beta T^{-1} E_t \phi_{2,t+T-1} A_{t+T} d(\pi_{t+T}) \frac{F_{K_{G,t+T}}}{Y_{t+T-1}}.
$$

(29)

The main difference between this equation and equation (34) is the term between brackets. When the stock of public capital increases $T$ periods after the shock, and because prices are costly to adjust, this entails inefficient variations in future real marginal costs and thus in current inflation.

As long as monetary policy is constrained by the ZLB, prices cannot be stabilized, which causes (29) to deviate from (34). By raising public investment (and thus $K_{G,t+T}$) above and beyond its flexible-price level, the planner can help minimize the expression between brackets, which help close the output gap.

In terms of welfare, Table 1 shows that optimally adjusting public investment to the shock allows the economy to achieve a substantial welfare gain relative to the scenario in which public spending is kept constant, which amounts to 0.082 percent of steady-state consumption. This gain

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10 In a first-order approximation the two expressions in the second line of equation (29) disappear since $d(0) = 0$ at steady state.

11 In a log-linearized version of equation (11), current inflation can be expressed as a discounted sum of current and future real marginal costs.
is two orders of magnitude larger than the gain achieved by only adjusting public consumption while keeping public investment constant.

These findings highlight the following important point: the conclusion one draws about the optimal size of a fiscal expansion and its welfare implications crucially depends on the set of instruments available to the policy maker. Whenever public investment is possible, the optimal fiscal expansion is rather sizable and the welfare gains it achieves are substantially larger than those associated with a plan that abstracts from public investment.

4.3 Discussion

Bachmann & Sims (2012) present evidence that, conditional on a positive government spending shock, the ratio of U.S. public investment to public consumption rises more during recessions than during expansions. Our analysis, on the other hand, suggests that the optimal policy response to an economic downturn entails raising public spending while changing its composition in a way that assigns a larger weight to public investment relative to public consumption. To the extent that increases in U.S. public spending during recessions reflect intended policy actions by the U.S government to cushion the economy, the evidence reported by Bachmann & Sims (2012) suggests that this policy choice is consistent with the prescriptions of our model.

Our results also give credence (at least partially) to the two largest fiscal stimulus plans that were implemented in 2008–2009 to cope with the global economic downturn, namely, the American Recovery and Reinvestment Act and the European Economic Recovery Plan. Roughly 40 percent of the spending component (excluding transfers) of the ARRA was allocated to public investment projects in infrastructure (see Drautzburg & Uhlig (2013)). This fraction is nearly twice as large as the historical average share of public investment in total public spending in the U.S. (23 percent). Likewise, Coenen et al. (2013) report that the European Economic Recovery Plan allocated 71 percent of the additional government spending to public investment, whereas the historical average of this fraction is approximately 11.5 percent (see also Cwik & Wieland (2011)).

4.4 Sensitivity Analysis

4.4.1 Size of the shock

The results discussed so far have shown that the optimal policy response to a preference shock that would otherwise generate a contraction of the same magnitude as the Great Recession entails

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12See Table 5 in Coenen et al. (2013).
raising public investment significantly and persistently above normal. Yet, the fraction of this fiscal expansion intended to fill the negative output gap – stimulus spending – is negligible. Would this still be the case if the economy faced an even larger shock? To investigate this question, we follow Woodford (2011) and Bilbiie et al. (2014) and consider a scenario akin to the Great Depression, that is, one in which the preference shock is large enough to generate an output decline of 28.8 percent in the decentralized economy. Figure 7 depicts the economy’s response to the shock in the first-best and Ramsey equilibria.

Figure 7: Sensitivity analysis: Ramsey optimal monetary and fiscal policy in response to a large negative preference shock (Great Depression calibration).

Qualitatively, the results are similar to those shown in Figure ???. There are, however, important quantitative differences. Under the Great Depression scenario, the natural rate of interest remains negative for 10 quarters. In the Ramsey equilibrium, consumption falls by 10 percent and the policymaker keeps the nominal interest rate at zero for 17 quarters. Public consumption rises initially by roughly 0.6 percent of steady-state output and remains above its steady-state level for
4 quarters; it then falls below that level during the subsequent 12 quarters. Public investment also rises in a hump-shaped manner during the first 18 quarters after the shock, with a peak response of 4.2 percent of steady-state output. These results imply that, in response to the shock, the optimal composition of public spending is even more heavily skewed towards public investment than in the benchmark scenario: in the Ramsey allocation, the share of public investment in total public spending surges to roughly 29 percent on impact and reaches a peak of 37 percent 4 quarters after the shock.

Figure 7 also shows that both public consumption and public investment are larger in the Ramsey allocation than in the first-best during the first 4 quarters after the shock, thus implying that stimulus spending is initially positive. Figure 8 quantifies this component. At the time of the shock, stimulus spending amount to, respectively, 0.7 and 0.3 percent of steady-state output for public consumption and public investment. This increase, however, is almost fully offset by a drop in stimulus spending during the subsequent 4 periods.

![Figure 8: Stimulus Spending as a Share of Steady State GDP, Great Depression Calibration](image)

**4.4.2 Model with private capital**

In this section, we augment the baseline model with private capital, which is accumulated by private households and rented to firms at the nominal rental rate $R^k_t$. As is the case with public
capital, the accumulation of private capital is also subject to investment adjustment costs. For simplicity, however, we abstract from time-to-build delays. In this extended version of the model, the production function takes the following form:

$$F(N_t, K_t, K_{G,t}) = N_t^a K_t^{1-a} K_{G,t}^b,$$

where $a, b \in [0,1]$. The full description of the model, the first-best and Ramsey equilibrium conditions are relegated to the Appendix. In our simulations, we set $a$ to 2/3. We also assume that private and public capital have the same depreciation rate and adjustment-cost parameter. The preference shock is again calibrated such that the resulting fall in output matches the observed decline in U.S. GDP from peak to trough during the Great Recession. Figure 9 shows the economy’s response to the shock in the first-best and Ramsey equilibria.

In this extension of the model, the presence of private capital means that there is another way to channel savings in order to consume more in the future. Furthermore, due to the Cobb-Douglas production function, private and public capital are complimentary with one another. From these two features, one can anticipate that both types of capital will move in the same direction. Looking at Figure 9, this is exactly what happens.

Because the Ramsey planner can now use private investment to smooth consumption, he will automatically rely less on public investment. As a result, the increase in public investment following the preference shock peaks at roughly 0.7% of steady state output — compared to 1.7% in the model without private capital. That being said, the optimal policy still relies more heavily on public investment than public consumption. Indeed, if we compute the share of government investment in total government spending, the former rises to 24% on impact to reach a maximum of 25.7% before converging back to its steady state value.

From Figure 9, it is clear that the fraction of government investment that is carried out because of a stimulus motive is still small. We plot the path of stimulus spending for the two components of total government spending in Figure 10. To compare the results with the model without private capital, we plot these results alongside. Observe that the total amount of stimulus spending is now higher as a proportion of steady state GDP. This comes from the fact that stimulus spending for both public consumption and investment increase as a share of steady state GDP.

The reason for this result is two-fold. First, because of the presence of private capital, the decrease in private consumption after the preference shock is larger in the first best allocation. Because of the Samuelson condition, this means that public consumption should also decrease by
Figure 9: Ramsey monetary and fiscal policy in the model with private capital
more. But in the Ramsey allocation, public consumption should still be used to stabilize the output gap. As a result, the difference of government consumption between the Ramsey and first best allocations is greater when there is private capital in the model. The reverse happens for public investment. Because of the presence of private capital, in the first best allocation the increase in public investment is now lower than in the model without private capital. But there is still a role for public investment to close the output gap today, which is why the difference between the two allocations (which, by definition is the amount devoted to stimulus spending) is now greater.

4.4.3 Non-linear solution method

[To be Completed]

5 Concluding Remarks

[To be Completed]
References


Bom, P. R. & Ligthart, J. E. (2013). What have we learned from three decades of research on the productivity of public capital? *Journal of Economic Surveys*, (pp. n/a–n/a).


6 Appendix

7 Functional Forms

We consider the following utility function:

\[ U(C_t, N_t) = \frac{(C_t^\gamma (1 - N_t)^{1-\gamma})^{1-\sigma} - 1}{1 - \sigma} \quad \text{if } \sigma \neq 1 \]
\[ = \gamma \ln C_t + (1 - \gamma) \ln(1 - N_t) \quad \text{if } \sigma = 1, \]

This specification imply the following first and second derivatives:

\[ U_C = \gamma C_t^\gamma (1 - N_t)^{(1-\gamma)(1-\sigma)}, \]
\[ U_{CC} = -(1 + \gamma (\sigma - 1)) \frac{U_C}{C_t}, \]
\[ U_{CN} = (1 - \gamma) (\sigma - 1) \frac{U_C}{1 - N_t}, \]
\[ U_N = -\left[ (1 - \gamma) C_t^\gamma (1 - N_t)^{(1-\gamma)(1-\sigma)-1} \right], \]
\[ U_{NN} = -\left[ (1 + (1 - \gamma) (\sigma - 1)) (1 - \gamma) C_t^\gamma (1 - N_t)^{(1-\gamma)(1-\sigma)-2} \right]. \]

We consider the following production function:

\[ F(N_t, K_{G,t}) = N_t^a K_{G,t}^b K_t^c, \quad c = 1 - a \]

Adjustment costs are given by

\[ S \left( \frac{G_t}{G_{t-1}} \right) = \frac{\varpi}{2} \left( 1 - \frac{G_t}{G_{t-1}} \right)^2, \quad \varpi \geq 0. \]

8 Model without private capital (a = 1)

8.1 First-Best

\[ \max_{Z_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \xi_t U(C_t, N_t, G_t^c) + \lambda_{1,t} \left[ F(N_t, K_{G,t}) - C_t - G_t^c - G_t^i \right] + \lambda_{2,t} \left[ (1 - \delta) K_{G,t+T-1} + \left( 1 - S \left( \frac{G_t}{G_{t-1}} \right) \right) G_t^i - K_{G,t+T} \right] \right\}, \]
The efficient allocation is the solution to the following set of equations:

\[
\begin{align*}
0 &= U_{C,t} - \lambda_{1,t}/\xi_t, \\
0 &= U_{G,t} - \lambda_{1,t}/\xi_t, \\
0 &= \frac{U_{N,t}}{U_{C,t}} + F_N(N_t, K_{G,t}), \\
0 &= \lambda_{1,t} \left[ 1 - S \left( \frac{G_{i}^i}{G_{i-1}^i} \right) - S' \left( \frac{G_{i}^i}{G_{i-1}^i} \right) \right] \\
0 &= \lambda_{2,t} - \beta (1 - \delta) E_t \lambda_{2,t+1} + \beta T E_t \lambda_{1,t+T} F_{K_G} (N_{t+T}, K_{G,t+T}), \\
0 &= F(N_t, K_{G,t}) - C_t + G_{t}^C + G_{t}^i, \\
0 &= K_{G,t+T} - (1 - \delta) K_{G,t+T-1} - \left( 1 - S \left( \frac{G_{i+1}^i}{G_{i-1}^i} \right) \right) G_{i}^i
\end{align*}
\]

(30) (31) (32) (33) (34) (35) (36)

8.1.1 Steady state

Let us introduce the following notations for steady state ratios:

\[
\tilde{g} = \frac{G^C + G^i}{Y}, \quad \tilde{g}^i = \frac{G^i}{Y}, \quad \alpha = \frac{G^i}{G} = \frac{\tilde{g}^i}{\tilde{g}}
\]

From equations (30), (33), and the specification for adjustment costs at steady state, we get \( \lambda_1 = \lambda_2 = U_C \).

Using these results and the functional form of the production function, we can rewrite equation (34) as follows:

\[
\tilde{g}^i = \alpha \tilde{g} = \frac{b \beta T \delta}{1 - \beta (1 - \delta)}
\]

Given our functional form for the utility function, equation (32) can be re-written as:

\[
F_N = \frac{1 - \gamma}{\gamma} \frac{C}{1 - N} \\
\Leftrightarrow a Y N = \frac{1 - \gamma}{\gamma} \frac{Y}{1 - N} (1 - \tilde{g}) \\
\Leftrightarrow N = \frac{a \gamma}{a \gamma + (1 - \gamma) (1 - \tilde{g})}
\]

where we have used the resource constraint for the second equation. In practice, we choose \( \gamma \) so that \( N = 1/3 \) at steady state. In this case, we have

\[
\gamma = \frac{1 - \tilde{g}}{2a + 1 - \tilde{g}}
\]
The resource constraint can be re-written as:
\[
C = (1 - \bar{g})N^aK_G^b
\]
\[
\Leftrightarrow C = (1 - \bar{g})N^a \left( \frac{\bar{g}^i C}{\delta 1 - \bar{g}} \right)^b
\]
\[
\Leftrightarrow C = (1 - \bar{g}) \left[ N^a \left( \frac{\bar{g}^i}{\delta} \right) \right]^{\frac{1}{1-b}}
\]

Using the steady state value of \( C \) and \( N \) we can compute \( \lambda_i = U_c \) for \( i = 1, 2 \). We can also recover the steady state levels of output and stock of public capital:
\[
K_G = \frac{\bar{g}^i C}{\delta 1 - \bar{g}}, \quad Y = N^a K_G^b.
\]

### 8.1.2 Log-linear approximation

\[
0 = \frac{CU_{CC}}{U_c} c_t + \frac{NU_{CN}}{U_C} n_t + \hat{\xi}_t - \hat{\lambda}_{1,t}
\]
\[
0 = \frac{GU_{GG}}{U_g} g_t^i + \hat{\xi}_t - \hat{\lambda}_{1,t}
\]
\[
0 = \left( \frac{NU_{NN}}{U_N} - \frac{NU_{CN}}{U_C} \right) n_t + \left( \frac{CU_{CN}}{U_N} - \frac{CU_{CC}}{U_C} \right) c_t - \frac{NF_{NN}}{F_N} n_t - \frac{KF_N K_G}{F_K} k_{G,t}.
\]
\[
0 = \hat{\lambda}_{1,t} - \hat{\lambda}_{2,t} + S''(1)(g_{t}^i - g_{t-1}^i) + \beta S''(1)(\hat{E} g_{t+1}^i - g_{t}^i)
\]
\[
0 = \hat{\lambda}_{2,t} - \beta (1 - \delta) \hat{E}_t \hat{\lambda}_{2,t+1} - (1 - \beta (1 - \delta)) \hat{E}_t \left( \frac{NF_{K_G N}}{F_K} n_{t+T} + \frac{KF_{K_G K_G}}{F_K} k_{G,t+T} \right)
\]
\[
0 = \frac{NF_N}{F} n_t + \frac{KF_K F_{K_G}}{F} k_{G,t} - (1 - g) c_t - (1 - \alpha) g g_t^i - \alpha g_t^i
\]
\[
0 = k_{G,t} - (1 - \delta) k_{G,t-1} - \delta g_t^i
\]

### 8.2 The Ramsey problem

The policy maker has three tools: i) the nominal interest rate ii) government consumption and iii) government investment. The optimal path for these instruments following a demand shock is the solution to the
following Ramsey problem, expressed as a Lagrangian:

\[
L_0 \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \xi_t U(C_t, N_t, G_t^c) + \phi_{1,t} \left[ \frac{\beta \xi_{t+1} U_{C,t+1}}{\xi_t} \frac{U_{C,t}}{C_t} \frac{C_t + 1 + R_t}{1 + \pi_{t+1}} \right] + \phi_{2,t} \left[ \psi \Lambda_{t+1} \frac{F(N_{t+1}, K_{G,t+1})}{F(N_t, K_{G,t})} d(\pi_{t+1}) - \psi d(\pi_t) - \theta \frac{U_{N,t}}{U_{C,t}} \frac{1}{F_{N,t}} - \theta \right] + \phi_{3,t} \left[ F(N_t, K_{G,t}) \left( 1 - \frac{\psi}{2} \pi_t^2 \right) - C_t - G_t^c + G_t^i \right] + \phi_{4,t} \left[ (1 - \delta) K_{G,t+T-1} + \left( 1 - S \left( \frac{G_t^i}{G_{t+1}} \right) \right) G_t^i - K_{G,t+T} \right] + \phi_{5,t} [R_t - 0] \right\},
\]

where we have defined \( d(\pi_t) \equiv \pi_t (1 + \pi_t) \) and \( \Lambda_{t+1} = \beta \frac{\xi_{t+1} U_{C,t+1}}{\xi_t} \frac{U_{C,t}}{C_t} \). The first-order conditions with respect to \( C_t, G_t^c, N_t, K_{G,t}, G_t^i, \pi_t, R_t \) and \( \phi_{i,t} \) for \( i = 1...5 \) are
In zero-inflation steady state, the Ramsey allocation is identical to the first best. This implies that

\[ \phi_1 = \phi_2 = \phi_5 = 0. \]

and

\[ \phi_3 = \phi_4 = UC. \]
8.2.2 Log-linear approximation

The system above is linearized as follows:

\[ 0 = \frac{C U C}{U C} c_t + \frac{N U C N}{U C} n_t + \hat{\xi}_t - \hat{\phi}_{3, t} - \theta \frac{1}{U C F N} \left( \frac{U C N U C - U C C U N}{U C^2} \right) \phi_{2, t} - \frac{U C C}{U C} \phi_{1, t} + \frac{U C C}{\beta U C} \phi_{1, t-1} \]

\[ 0 = \frac{G U G G}{U G} g^c_t + \hat{\xi}_t - \hat{\phi}_{3, t} \]

\[ 0 = \frac{C U C N}{U N} c_t + \frac{N U N N}{U N} n_t + \hat{\xi}_t - \hat{\phi}_{3, t} - \theta \left[ \frac{1}{U N F N} \left( \frac{U N N U C - U C N U N}{U N^2} \right) - \frac{1}{U C F N^2} \right] \phi_{2, t} \]

\[ - \frac{K G F K G}{F N} k_{G, t} - \frac{N F N N}{F N} n_t - \frac{U C N}{U N} \phi_{1, t} + \frac{U C N}{\beta U N} \phi_{1, t-1} \]

\[ 0 = \hat{\phi}_{3, t} - \beta (1 - \delta) E_t \hat{\phi}_{3, t+1} - (1 - \beta (1 - \delta)) E_t \left( \phi_{3, t+1} + \frac{N F K G N}{F K G} n_{t+1} + \frac{K G F K G}{F K G} k_{G, t+1} - \frac{\theta}{U C F N F K G} \phi_{2, t+1} \right) \]

\[ 0 = \phi_{2, t} - \phi_{3, t} + S''(1) (g^i_t - g^i_{t-1}) + \beta S''(1) (E_t g^i_{t+1} - g^i_t) \]

\[ 0 = \phi_3 F(K, K_G) \pi_t + \phi_{2, t} - \phi_{2, t-1} + \frac{\phi_3}{\psi \beta} \hat{\phi}_{1, t-1} \]

\[ 0 = \phi_5, t + \beta \phi_3 \phi_{1, t} \]

\[ 0 = c_t - E_t c_{t+1} - \frac{U C}{C U C} \left[ R_t - E_t n_{t+1} + \log(\beta) + (\rho - 1) \hat{\xi}_t \right] - \frac{N U C N}{C U C} (E_t n_{t+1} - n_t) \]

\[ 0 = \pi_t - \beta E_t \pi_{t+1} - \frac{\theta}{\psi} \left( \frac{C U C N}{U N} - \frac{C U C C}{U C} \right) c_t + \left( \frac{N U N N}{U N} - \frac{N U C N}{U C} - \frac{N F N N}{F N} n_t \right) n_t - \frac{K G F N K G}{F N} k_{G, t} \]

\[ 0 = \frac{N F N}{F} n_t + \frac{K G F K G}{F} k_{G, t} - (1 - g) c_t - (1 - \alpha) g g^c_t - \alpha g g^i_t \]

\[ 0 = k_{G, t} - (1 - \delta) k_{G, t-1} - \delta g^i_t, \]

\[ 0 = \phi_5, t R_t \]

Note that when the ZLB binds, \( \phi_{5, t} > 0 \). Otherwise, \( \phi_{5, t} = \phi_{1, t} = 0 \).