The New Keynesian Model and the Small Open Economy RBC Model: Equivalence Results for Consumption

Dan Cao, Jean-Paul L’Huillier, Donghoon Yoo*

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Abstract

We consider a modern New Keynesian model with “bells and whistles” (Christiano, Eichenbaum, and Evans 2005) with both permanent and temporary technology shocks. We focus on the behavior of consumption and compare it to the one obtained by the Small Open Economy RBC model (Schmitt-Grohe and Uribe 2003; Aguiar and Gopinath 2007). We use the same (separable) preferences in both models. We find that, for common parametrizations used in the respective literatures, consumption behaves similarly in both models. Specifically, consumption is (almost) entirely and solely determined by the long-run level of TFP: After a permanent impulse to TFP, consumption (almost) jumps to that level and stays there. By the same logic, consumption (almost) does not react to temporary TFP shocks. These results are useful to interpret some quantitative results provided by these literatures.

Keywords: Consumption, DSGE, permanent income hypothesis, rigidities.

JEL codes: E10, E27, E37.

*Georgetown University; Einaudi Institute for Economics and Finance (EIEF); EIEF and Tor Vergata. Dan thanks EIEF for hospitality and support during the preparation of this paper.
1 Introduction

Modern DSGE models are complex. They usually feature many ingredients and parameters for which it is often difficult to derive precise results about the equilibrium behavior of endogenous variables. In this paper, we consider close versions of two widely used DSGEs in the literature and provide a simple and sharp numerical characterization of the conditional dynamics of consumption. The models we consider are the following. First, we take a modern version of the New Keynesian model with so-called “bells and whistles” introduced by Christiano, Eichenbaum, and Evans (2005) and later used by Smets and Wouters (2007) and Justiniano, Primiceri, and Tambalotti (2010), among others (henceforth NK-BW). Then, we consider the Small Open Economy RBC model, originally introduced by Mendoza (1991), and subsequently used by Schmitt-Grohe and Uribe (2003) and Aguiar and Gopinath (2007) (among others), where the model features a debt-elastic interest-rate premium (henceforth SOE-RBC). The only modification we introduce to the latter is that we use the same preferences as in the NK-BW model. Both specifications only include permanent and temporary technology shocks.

The first set of results we obtain is the following. We identify a parameter region, in both models, for which the behavior of consumption is solely determined by the long-run level of Total Factor Productivity (TFP). Indeed, after a persistent shock to TFP (which is modeled as the sum of an \( AR(1) \) process in growth rates and a similar process in levels), consumption jumps to the long-run level of TFP immediately when the initial permanent impulse hits. That is, consumption is flat and features no fluctuations around the long-run level of TFP. Also, by the same logic, consumption does not move after a temporary technology shock. For both models, the parameter region features no habit formation. Moreover, in the case of the NK-BW model, the parameter region is one where a1) prices are very sticky or the interest rate rule is very accommodative to inflation, and b1) the interest rule does not react to output. In the case of the SOE-RBC model, the parameter region is one where a2) the discount factor is close to one, b2) the elasticity of the interest premium is close to zero, and c2) the steady state level of debt is zero\(^1\), with the requirement that b2) holds more strongly than a2), i.e. the elasticity tends to zero faster than the discount factor tends to one. We refer to this parametrization (for

\(^1\)This last condition is not necessary for obtaining that consumption dynamics are flat, but they are necessary to obtain that consumption jumps to the same level than long-run productivity.
each model respectively) by “limit parametrization”.

The parameter regions identified above are relevant because they are close to standard parametrizations of both the NK-BW and SOE-RBC models in the literature. These parametrizations are usually obtained by matching these models to the data. In the case of the NK-BW model, Christiano et al. (2005) estimate this model matching VAR-obtained IRFs to a monetary shock. In another important paper, Smets and Wouters (2007) estimate the model through Bayesian methods. Last but not least, Justiniano et al. (2010) estimate a model with permanent instead of temporary shocks to TFP. In the case of the SOE-RBC model, researchers usually focus on a calibrated version of the model. The parametrization we consider is close to the calibration usually adopted, which we refer to as “standard parametrization” for each model respectively. (In the body of the paper we provide a detailed explanation of why we consider limit and standard parametrizations close to each other.)

The intuition for these results is straightforward and is given by the random-walk permanent-income consumption model. In both parameter regions identified above the real interest rate is (almost) constant, and therefore consumption is determined by permanent income at all periods. In the NK-BW model, when prices are very sticky inflation does not move, and therefore the rate determined by the nominal interest rate rule does not move either (so long as the rate does not react to output.) As a result, the real rate is constant. In the SOE-RBC model, when the elasticity of the interest premium is small, the domestic real interest rate is constant and equal to the foreign rate. Note that these findings are not entirely obvious given that there are multiple differences between the two models. To mention some of these differences, consider first goods markets. The NK-BW model features monopolistic competition and Calvo price stickiness, while the SOE-RBC model features a competitive market with flexible prices. In terms of labor markets, the NK-BW model features Calvo wage stickiness and the SOE-RBC model features, again, a competitive market. Moreover, the NK-BW model is closed economy, and the SOE-RBC is open economy. Because our limit parametrization is close to standard, our results imply that these elements of the specification (among others not described) are largely irrelevant for the conditional behavior of consumption after TFP shocks.

\footnote{Aguiar and Gopinath (2007) estimate a subset of parameters via method of moments, mainly concerned with the stochastic shock processes. The parametrization above is close to the set of fixed parameters in their paper.}

\footnote{In a related paper, Kocherlakota (2012) studied the implications of a fixed real interest rate for labor markets.}
We rely on previously obtained theoretical results which form the basis for our numerical exploration of the full-blown models mentioned above. The random walk behavior of consumption was first established (to the best of our knowledge) in Blanchard, L’Huillier, and Lorenzoni (2013) (Online Appendix Section 6.4.2) in the case of a baseline New Keynesian model (Gali 2008). Similar results were established in Cao and L’Huillier (2014b) (p. 11) for a Small Open Economy RBC model without capital and inelastic labor. These theoretical results help understand for instance why it is not required in the NK-BW model to have the discount factor to be close to 1, whereas this is a requirement in the SOE-RBC model. (We discuss these issues more in depth in the body of the paper.)

The second set of results of the paper consists in showing that, for the usually adopted parametrization of both models in the literature, the consumption implications of TFP shocks are similar in both models in the following sense. We focus on the variance decomposition of consumption. We find that consumption is almost entirely driven to permanent shocks. Given that here we are not exactly in the parameter region identified above, the Impulse Response Functions (IRFs) of consumption are no longer flat as before, but they are fairly close to flat, and they look fairly similar in both models.

Our results are useful to understand some mechanics of consumption within these models and to interpret some empirical results in the literature. We discuss these interpretations after having presented our results in the conclusion.

The rest of the paper is organized as follows. The next Section lays down both models. Section 4 presents first the optimality conditions of the models, and then all numerical results. Section 5 concludes.
2 The Models

In this Section we present the models. We start by the elements common to both models. We then specify the remaining elements of the NK-BW. Immediately after we specify the remaining elements of the SOE-RBC model.

2.1 Elements Common to Both Models: Total Factor Productivity and Preferences

Productivity $a_t$ (in logs) is the sum of two components, permanent, $x_t$, and temporary $z_t$

$$a_t = x_t + z_t$$

The permanent component follows the unit root process

$$\Delta x_t = \rho_x \Delta x_{t-1} + \varepsilon_t$$

The temporary component follows the stationary process

$$z_t = \rho_z z_{t-1} + \eta_t$$

The coefficients $\rho_x$ and $\rho_z$ are in $[0, 1)$, and $\varepsilon_t$ and $\eta_t$ are i.i.d. normal shocks with variances $\sigma^2_\varepsilon$ and $\sigma^2_\eta$.

Preferences are given by

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left( \ln(C_t - hC_{t-1}) - \frac{1}{1 + \varphi} \int_0^1 N_{jt}^{1+\varphi} dj \right) \right]$$

where $C_t$ is consumption, the term $hC_{t-1}$ captures internal habit formation, and $N_{jt}$ is the supply of specialized labor of type $j$.

2.2 Remaining Specification of the NK-BW Model

The model is standard. The household budget constraint is

$$P_tC_t + P_tI_t + P_tC(U_t)\bar{K}_{t-1} + B_t = R_{t-1}B_{t-1} + \int_0^1 W_{jt}N_{jt}dj + R^k_tK_t$$

where $P_t$ is the price level, $I_t$ is investment, $C(U_t)$ is the cost associated with capital utilization in terms of current production, $\bar{K}_t$ is the stock of capital, $B_t$
are holdings of one-period bonds, \( R_t \) is the one-period nominal interest rate, \( W_{jt} \) is the wage of specialized labor of type \( j \), and \( R^k_t \) is the capital rental rate, and \( K_t \) are capital services rented (and used).

The capital stock \( \bar{K}_t \) is owned and rented by the representative household and the capital accumulation equation is

\[
\bar{K}_t = (1 - \delta) \bar{K}_{t-1} + \left[ 1 - \chi \left( \frac{I_t}{\bar{I}_{t-1}} \right) \right] I_t
\]

where adjustment costs in investment are captured by

\[
\chi \left( \frac{I}{\bar{I}} \right) = \frac{\chi}{2} \left( \frac{I}{\bar{I}} - 1 \right)^2
\]

The model features variable capital utilization and the capital services provided by the capital stock \( \bar{K}_{t-1} \) are

\[
K_t = U_t \bar{K}_{t-1}
\]

where \( U_t \) represents the degree of capital utilization. The cost associated with capital utilization in terms of current production is given by

\[
C(U) = \frac{1}{1 + \xi} U^{1 + \xi}
\]

**Final good producers.** The final good is produced using intermediate goods with the CES production function

\[
Y_t = \left[ \int_0^1 Y_{it}^{1+\mu_p} \, di \right]^{1+\mu_p}
\]

where \( \mu_p \) captures a constant elasticity of substitution across goods.

Final good producers are perfectly competitive and maximize the profits subject to the above production function, taking intermediate good prices \( P_{it} \) as given and final good price \( P_t \) such that final good producers profit maximization problem is

\[
\max_{Y_{it}} P_t Y_t - \int_0^1 P_{it} Y_{it} \, di
\]

**Intermediate goods producers.** There is a continuum of intermediate goods producers where each producer produces good \( i \) with following production tech-
Technology

\[ Y_{it} = K_{it}^\alpha (A_t N_{it})^{1-\alpha} \]

where \( K_{it} \) and \( N_{it} \) are capital and labor services employed and \( \alpha \in (0, 1) \) represents capital’s share of output. As in the SOE-RBC model below, the parameter \( A_t = e^{a_t} \).

Intermediate good prices are assumed to be Calvo-sticky. Each period intermediate firm \( i \) can freely adjust the nominal prices with probability \( 1 - \theta \) and firms that cannot adjust prices (with probability \( \theta \)) set their price according to

\[ P_{it} = P_{it-1} \Pi_{t-1}^{1-t} \]

where \( \Pi \) is the steady state level of inflation.

**Labor markets.** A representative competitive firm hires the labor supplied by household \( j \) and aggregates the specialized labor supplied by the households with the following technology

\[ N_t = \left[ \int_0^1 N_{jt}^{1+\mu_w} dj \right]^{1+\mu_w} \]

where \( \mu_w \) is a constant elasticity of substitution among specialized labor.

Wages are Calvo-sticky. For each type of labor \( j \), the household can freely adjust the price \( W_{jt} \) with probability \( 1 - \theta_w \) and those that cannot adjust prices (with probability \( \theta_w \)) set their price according to

\[ W_{jt} = W_{jt-1} \left( \Pi_{t-1} e^{\Delta a_{t-1}} \right)^{\mu_w} \left( \Pi \right)^{1-\mu_w} \]

where \( \Pi_{t-1} \) is inflation at \( t - 1 \).

**Monetary policy.** Monetary policy follows the following interest rate rule

\[ \frac{R_t}{R} = \frac{R_{t-1}^{\rho_r}}{R} \left( \frac{\Pi_t}{\Pi} \right)^{\gamma_y} \left( \frac{Y_t/A_t}{Y/A} \right)^{\gamma_y} \]

**Market clearing.** Market clearing in the final good market requires

\[ C_t + I_t + C(U_t) \bar{K}_{t-1} = Y_t \]
and market clearing in the market for labor services requires

$$\int_0^1 N_{jt} dj = N_t$$

As Christiano et al. (2005), we define output gross of capital utilization costs.

### 2.3 Remaining Specification of the SOE-RBC Model

The specification is standard. Here, following the literature, there is no labor heterogeneity and thus $N_{jt} = N_t$.

Maximization of household utility is constrained by

$$C_t + I_t + B_{t-1} = Y_t + Q_tB_t$$

where $C_t$, $I_t$ and $Y_t$ are the consumption, investment, and output of the country, $B_t$ is the external debt of the country and $Q_t$ is the price of this debt.

Output is produced with capital and labor inputs through a Cobb-Douglas production function

$$Y_t = K_{t-1}^\alpha (A_tN_t)^{1-\alpha}$$

where $\alpha \in (0, 1)$ represents capital’s share of output. The parameter $A_t = e^{a_t}$.

The resource constraint is

$$C_t + I_t + NX_t = Y_t$$

where $NX_t$ are net exports. Following Aguiar and Gopinath (2007), the law of motion for capital is

$$K_t = (1 - \delta)K_{t-1} + I_t - \frac{\nu}{2} \left( \frac{K_t}{K_{t-1}} - 1 \right)^2 K_{t-1}$$

Capital depreciates at the rate $\delta$, and the (quadratic) capital adjustment cost is captured by

$$\frac{\nu}{2} \left( \frac{K_t}{K_{t-1}} - 1 \right)^2 K_{t-1}$$

Following Schmitt-Grohe and Uribe (2003) and Aguiar and Gopinath (2007), among others, the price of debt is sensitive to the level of debt outstanding

$$\frac{1}{Q_t} = R_t = R^* + \psi \left\{ e^{\frac{bt}{n_t}} - 1 \right\}$$
where $R^*$ denotes the world interest rate, and $b$ represents the steady state level of the debt-to-output ratio.
3 Optimality Conditions and Steady State Conditions

Following standard steps, we derive the optimality conditions (for both models) and log-linearize. Here we present the FOC and deterministic steady state relations for both models, which allows us to comment on the assumption that the steady state level of debt in the SOE-RBC needs to be set to zero in order to more easily compare across models (on p. 15). All details of the log-linearization are given in the Appendix.

3.1 NK-BW Model: Optimality Conditions

Households. The FOC for consumption is
\[
\frac{1}{C_t - hC_{t-1}} - \beta h \frac{1}{C_{t+1} - hC_t} = \Lambda_t
\]
where \( \Lambda_t \) is the multiplier of the budget constraint. The FOC for bond holdings is
\[
-\frac{\Lambda_t}{P_t} + \beta R_t \frac{\Lambda_{t+1}}{P_{t+1}} = 0
\]
The FOC for investment
\[
-\Lambda_t + \Phi_t \left[ \chi \left( \frac{I_t}{I_{t-1}} - 1 \right) \right] + \beta \Phi_{t+1} \left[ 1 - \frac{\chi}{2} \left( \frac{I_{t+1}}{I_t} - 1 \right)^2 + \frac{\chi}{2} \left( \frac{I_{t+1}}{I_t} \right)^2 \left( \frac{I_{t+1}}{I_t} - 1 \right) \right] = 0
\]
where \( \Phi_t \) is the multiplier for capital accumulation. The FOC for capital
\[
\Phi_t - \beta \Lambda_{t+1} C(U_{t+1}) - \beta (1 - \delta) \Phi_{t+1} \bar{K}_t = 0
\]
Remain the budget constraint and the capital accumulation equation
\[
\frac{B_{t-1}}{P_t} R_{t-1} - \frac{B_t}{P_t} + \frac{W_t}{P_t} N_t + \frac{R_t^k}{P_t} K_t - C_t - I_t - C(U_t) \bar{K}_{t-1} = 0
\]
\[
\bar{K}_t - (1 - \delta) \bar{K}_{t-1} - \left[ 1 - \frac{\chi}{2} \left( \frac{I_t}{I_{t-1}} - 1 \right)^2 \right] I_{t-1} = 0
\]
The optimality condition for capacity utilization \( U_t \) is the solution to
\[
\max_{U_t} \left( \frac{R_t^k}{P_t} U_t \bar{K}_{t-1} - C(U_t) \bar{K}_{t-1} \right)
\]
which yields
\[ \frac{R^k}{P_t} = C'(U_t) = \xi U_t^\xi \]

**Final good producers.** Final good producers maximize
\[
\max_{Y_{it}} P_t Y_t - \int_0^1 P_{it} Y_{it} di
\]
which gives the demand for intermediate good \( i \)
\[ Y_{it} = Y_t \left( \frac{P_{it}}{P_t} \right)^{-\frac{1+\mu_p}{\mu_p}} \]
Since final good producer derives zero profit,
\[ P_t = \left[ \int P_{it}^{\frac{1}{\mu_p}} di \right]^\mu_p \]

**Intermediate goods producers.** From the demand for intermediate good \( Y_{it}, \mu_p \) is the constant markup for monopolist \( i \). The cost minimization problem for the monopolist is
\[
\min_{K_{it}, N_{it}} R^k_{it} K_{it} + W_{it} N_{it}
\]
subject to the production technology
\[ K_{it}^\alpha (A_t N_{it})^{1-\alpha} = Y_{it} \]
From the cost minimization problem, we obtain following FOCs:
\[
R^k_{it} - MC_t \alpha K_{it}^{\alpha-1} (A_t N_{it})^{1-\alpha} = 0
\]
\[
W_{it} - MC_t (1 - \alpha) K_{it}^{\alpha} A_t^{1-\alpha} N_{it}^{-\alpha} = 0
\]
Because of constant returns to scale all intermediate good firms choose the same capital labor ratio
\[ \frac{K_t}{N_t} = \frac{\alpha}{1-\alpha} \frac{W_t}{R^k_t} \]
and the nominal cost of producing \( Y_{it} \) is \( MC_i \cdot Y_{it} \) where \( MC_i \) is the marginal cost
\[ MC_t = \frac{1}{\alpha \alpha (1-\alpha)^{1-\alpha}} (R^k_t)^{\alpha} \left( \frac{W_t}{A_t} \right)^{1-\alpha} \]
Calvo pricing implies that with probability \((1 - \theta)\) firms change price and maximize

\[
\mathbb{E} \left[ \sum_{s=0}^{\infty} \theta^s \beta^s \frac{\Lambda_{t+s}}{P_{t+s}} \left( P_t \tilde{I}_{t+s} - MC_{t+s} \right) Y_{it+s} \right]
\]

where \(\tilde{I}_{t+s}\) is the indexing function

\[
\tilde{I}_{t+s} = \Pi_{k=1}^{\theta} \left( \Pi_{t+t-1}^{\ell_p} \Pi^{1-\ell_p} \right)
\]

The optimality condition is then

\[
\mathbb{E} \left[ \sum_{s=0}^{\infty} \theta^s \beta^s \frac{\Lambda_{t+s}}{P_{t+s}} \left( \frac{1}{\mu_p} \tilde{I}_{t+s} - \frac{1 + \mu_p MC_{t+s}}{P_t} \right) Y_{it+s} \right] = 0
\]

**Labor markets.** Demand for variety \(j\) is

\[
N_{jt} = N_t \left( \frac{W_{jt}}{W_t} \right)^{-\frac{1+\mu_w}{\mu_w}}
\]

and the equilibrium wage is

\[
W_t = \left[ \int W_{jt}^{\frac{1}{\mu_w}} \right]
\]

Calvo pricing for wage implies that with probability \((1 - \theta_w)\) workers can adjust their wage and maximize

\[
\mathbb{E} \left[ \sum_{s=0}^{\infty} \theta^s w^s \beta^s \frac{\Lambda_{t+s}}{P_{t+s}} \left( P_{jt} \tilde{I}_{wt+s} W_{jt} N_{jt+s} - \frac{1}{1 + \phi} N_{jt+s}^{1+\phi} \right) \right]
\]

The optimality condition is then

\[
\mathbb{E} \left[ \sum_{s=0}^{\infty} \theta^s w^s \beta^s \frac{\Lambda_{t+s}}{P_{t+s}} \left( \frac{1}{\mu_w} \tilde{I}_{wt+s} - \frac{1 + \mu_w N_{jt+s}^{\phi}}{W_{jt}} \right) N_{jt+s} \right] = 0
\]

### 3.2 NK-BW Model: Steady State Relations

We look for a steady state of stationary variables. This means that, because of the unit root in \(a_t\), we need to normalize some of the variables. We denote steady state values by removing the time subindex.
From the intertemporal condition we obtain

\[ R = \frac{\Pi}{\beta} \]

From the optimality condition for investment, we can see that

\[ \Lambda A = \Phi A \]

where \( \Lambda A \) is the multiplier multiplied by \( A \), which is stationary, and similarly for \( \Phi A \). In addition, the steady state value of the investment adjustment cost is

\[ \chi(0) = 0 \]

Also, in steady state

\[ U = 1 \]

so

\[ C = 1 \]

and from the Euler condition for capital we get

\[ R^k/P = \frac{1}{\beta} - (1 - \delta) \]

where \( R^k/P \) is the steady state real rental rate. From optimality of prices

\[ P_i/P = (1 + \mu_p) MC/P \]

where \( MC/P \) are real marginal costs, and

\[ P_i/P = 1 \]

Thus

\[ \frac{W/\AP}{R^k/P} = \left[ \frac{\alpha^\alpha (1 - \alpha)^{1-\alpha}}{1 + \mu_p} \frac{1}{(R^k/P)^\alpha} \right]^{1/(1-\alpha)} \]

where \( W/\AP \) are normalized real wages. From optimal factor combination, we get

\[ \frac{K/AN}{R^k/P} = \frac{\alpha W/\AP}{1 - \alpha} \]

where \( K/AN \) is the normalized capital-to-labor ratio. From the production
function
\[
\frac{Y}{AN} = \left( \frac{K}{AN} \right)^\alpha
\]
where \(Y/AN\) is the normalized output-to-labor ratio. From resource constraints, we have
\[
I/AN = \delta K/AN
\]
where \(I/AN\) is the normalized investment-to-labor ratio. Since
\[
K/AN = \bar{K}/AN
\]
then
\[
\frac{C}{A} = \frac{Y}{A} - \frac{I}{A} - \frac{K}{A}
\]

### 3.3 SOE-RBC Model: Optimality Conditions

FOCs are obtained with respect to \(C_t, N_t, I_t, K_t, B_t, \Lambda_t, \Phi_t\), respectively:
\[
\Lambda_t = \frac{1}{C_t}
\]
\[
(1 - \alpha)\Lambda_t = N_t^{1+\varphi} \frac{1}{Y_t}
\]
\[
\Lambda_t + \Phi_t = 0
\]
\[
\Phi_t \left[1 + \nu \left( \frac{K_t}{K_{t-1}} - 1 \right) \right] + \alpha \beta E \left[ \Lambda_{t+1} \left( \frac{Y_{t+1}}{K_t} \right) \right] - \beta E \left[ \Phi_{t+1} \left( (1 - \delta) - \frac{\nu}{2} \left( 1 - \left( \frac{K_{t+1}}{K_t} \right)^2 \right) \right) \right] = 0
\]
\[
Q_t \Lambda_t = \beta E[\Lambda_{t+1}]
\]
\[
K_{t-1}^\alpha (A_t N_t)^{1-\alpha} - C_t - I_t + Q_t B_t - B_{t-1} = 0
\]
\[
K_t - (1 - \delta) K_{t-1} - I_t + \frac{\nu}{2} \left( \left( \frac{K_t}{K_{t-1}} \right)^2 - 1 \right) K_{t-1} = 0
\]

### 3.4 SOE-RBC Model: Steady State Relations

As in the NK-BW model, we look for a steady state of stationary variables. We denote steady state values by removing the time subindex.

From the intertemporal condition, obtain the condition
\[
R = \frac{1}{\beta}
\]
The steady state ratio of capital to output can be obtained from the FOCs with respect to $K_t$ and $I_t$:

$$
\frac{K}{Y} = \frac{\alpha}{1/\beta - (1 - \delta)}
$$

From the capital accumulation equation we have

$$
\frac{I}{Y} = \delta \frac{K}{Y} = \frac{\alpha \delta}{1/\beta - (1 - \delta)}
$$

The budget constraint gives

$$
1 - C \frac{Y}{Y} - I \frac{Y}{Y} = (1 - \beta) \frac{B}{Y}
$$

The resource constraint gives

$$
\frac{C}{Y} + \frac{I}{Y} + \frac{NX}{Y} = 1
$$

From these steady state relations one can conclude that the steady state level of current account surplus is given by

$$
\frac{NX}{Y} = (1 - \beta) \frac{B}{Y}
$$

Thus, in this model, the steady state level of normalized debt $B/Y$ is determined exogenously. For comparability of the model with the closed economy model above, we assume $B/Y = 0$. 
4 Results

We focus on the behavior of consumption in both the log-linearized NK-BW model and the log-linearized SOE-RBC model (the log-linearization is presented in the Appendix.) We consider two alternative parametrizations of both models. Parametrization I is one in which the response of consumption is flat (limit parametrization). Parametrization II is the typical one used in the literature (standard parametrization). We obtain results for both parametrizations.

4.1 Limit parametrization

Parametrization I is shown in Table 1. This parametrization defines a parameter region, for both models, in which the response of consumption is flat: it jumps to the long-run level of TFP after a permanent shock, computed as the cumulated sum of the growth rates implied by the shock, i.e.

\[ a_\infty = \frac{\sigma_x}{1 - \rho_x} \]

and does not move after a temporary shock (because the long-run level of TFP does not change after a temporary shock.)

The logic of this parametrization follows two theoretical results previously obtained. We discuss the parametrization of the NK-BW model first, and then we discuss the parametrization of the SOE-RBC model. We discuss the intuition behind these choices after showing the results below. Blanchard, L’Huillier, and Lorenzoni (2013) theoretically show (Online Appendix Section 6.4.2) that a baseline NK model (without capital and no bells and whistles) converges to a simple permanent income model with a fixed real interest rate and in which consumption is equal to expectations of the long-run level of labor productivity. We do not attempt to prove an equivalent theoretical result in the case of the NK-BW model, but instead numerically obtain that the result generalizes to this model as well. Following the conditions of the theorem proved in Blanchard et al. (2013), we set in Parametrization I the Calvo parameter very close to 1. We also set habit formation to 0 and we impose that the interest rate rule does not react to output (both restrictions follow from the baseline NK model).

Regarding the SOE-RBC model, Cao and L’Huillier (2014b) theoretically obtained the same result in the case of a small open economy model without
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
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<td>$h$</td>
<td>Consumption habit 0</td>
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<tr>
<td>$\alpha$</td>
<td>Capital share 0.17</td>
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<tr>
<td>$\varphi$</td>
<td>Inv. Frisch elasticity 3.79</td>
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<td>$\xi$</td>
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<tr>
<td>$\chi$</td>
<td>Investment adjustment costs 2.85</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Discount rate 0.9987</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Calvo prices $\approx 1$</td>
</tr>
<tr>
<td>$\theta_w$</td>
<td>Calvo wages 0.70</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Price markup 0.23</td>
</tr>
<tr>
<td>$\mu_p$</td>
<td>Wage markup 0.15</td>
</tr>
<tr>
<td>$\gamma_\pi$</td>
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</tr>
<tr>
<td>$\gamma_y$</td>
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</tr>
<tr>
<td>$\phi_{dy}$</td>
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</tr>
<tr>
<td>$\iota$</td>
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</tr>
<tr>
<td>$t_w$</td>
<td>Wage indexation 0.11</td>
</tr>
<tr>
<td>$\rho_r$</td>
<td>Persistence nominal interest rate 0.82</td>
</tr>
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**SOE-RBC Model**

<table>
<thead>
<tr>
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<th>Value</th>
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<tbody>
<tr>
<td>$\beta$</td>
<td>Discount rate $\approx 1$</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Elasticity of the interest rate $1 \times 10^{-12}$</td>
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<tr>
<td>$B/Y$</td>
<td>Steady state level of normalized debt 0</td>
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**Shock Processes**

<table>
<thead>
<tr>
<th>Technology</th>
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<tbody>
<tr>
<td>$\rho_x$</td>
<td>Persistence temporary shock 0.20</td>
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<tr>
<td>$\sigma_x$</td>
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</tr>
<tr>
<td>$\sigma_z$</td>
<td>Standard dev. temporary shock 1.00</td>
</tr>
</tbody>
</table>

**Notes:** Under this parametrization both models deliver flat responses of consumption. $\beta$ for SOE-RBC model is set to 0.99999, and $\theta$ to 0.99999999.
capital and fixed labor supply\textsuperscript{4}. We have a theoretical result of the model including capital (Cao and L’Huillier 2014a), but the existing proof requires fixed labor and uses the log-linearization adopted by Aguiar and Gopinath (2007).\textsuperscript{5} We do not attempt to to prove an equivalent in the case of both elastic labor supply and capital, but focus instead in numerical simulations. Following the conditions of Proposition 1 in (Cao and L’Huillier 2014b), we set the discount rate $\beta$ close to 1, the elasticity of the interest rate $\psi$ close to 0, and the ratio $\beta/(1 - \psi)$ close to 0. (Specifically, we set $\beta = 0.99999$ and $\psi = 1 \times 10^{-12}$.)

The remaining structural parameters of the models are the estimates by Justiniano et al. (2010). (They take a period to represent a quarter.) The standard deviation of the shocks is normalized to 1, and their persistence is set to 0.20.

Figure 1 shows the resulting behavior of TFP and consumption in both models. We plot the IRFs of TFP and consumption following a permanent and a temporary technology shock. In both cases, the response of consumption is flat. The responses of consumption are indistinguishable from each other, both models delivering similar responses. In the case of a permanent shock, consumption jumps to the long-run level of TFP. In the case of a temporary shock, consumption does not move.

The intuition for these results is straightforward and is given by the random-walk permanent-income consumption model. In both parameter regions identified above the real interest is (almost) constant, and therefore consumption is determined by permanent income at all periods. In the NK-BW model, when prices are very sticky inflation does not move, and therefore the nominal interest rate set by the nominal interest rate rule does not move either (so long the rate does not react to output.) As a result, the real rate is constant.\textsuperscript{6} In the SOE-RBC model, when the elasticity of the interest premium is small, the domestic real interest rate is constant and equal to the foreign rate.

This intuition highlights the role of the consumption-Euler equation in delivering a flat consumption response. In particular, this also means that it is needed to have separable preferences in order for the marginal utility for labor not to enter the consumption-Euler equation. For non-separable preferences,

\textsuperscript{4}Cao and L’Huillier (2013) contains a result in the case of elastic labor supply. This draft is available upon request.

\textsuperscript{5}This log-linearization is presented in the note by Aguiar and Gopinath (NA).

\textsuperscript{6}We obtain the same results by setting the reaction of the interest rate to inflation $\gamma_{\pi}$ close to 1 instead.
changes in the marginal utility of labor would break the random-walk property of consumption and would generate a non-flat response. This fact is also apparent after inspection of the proof in Cao and L’Huillier (2014a).

That both parametrizations achieve an (almost) constant real interest rate is fairly straightforward. What is a bit more subtle is the role of the assumption that $\beta$ is close to 1 in the case of the open economy model. Note that in the steady state the world interest rate $R^*$ is $1/\beta$. So, when $\beta$ is close to 1, the world interest rate is close to 1. In this case, temporary deviations of domestic demand from output lead to an accumulation of assets (or of debt). For the sake of the argument, consider a transition path in which the country has accumulated some debt. When the interest rate is 1, the domestic economy can roll-over the debt at infinity, and thus consumption can be freely set equal to the long-run level of output, determined on the steady state by the long-run level of TFP. We unfortunately have no intuition for the requirement that $\beta/(1-\psi)$ is close to 0.

For completeness, Figure 2 plots the IRFs of the investment, output, the labor input, capital, and inflation in the NK-BW model. In the case of a permanent shock, investment, output and the labor input comove positively. The responses of investment, output, and labor are hump-shaped, and capital builds up slowly to reach a new steady state. In the case of a temporary shock, investment, capital and output barely change. Labor shifts down strongly on
impact to compensate the productivity increase. Finally, inflation does not
move due to the high amount of price stickiness.

Figure 2: Impulse Responses: NK-BW model

<table>
<thead>
<tr>
<th>Investment</th>
<th>Output</th>
<th>Labor Input</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="#" alt="Graph" /></td>
<td><img src="#" alt="Graph" /></td>
<td><img src="#" alt="Graph" /></td>
</tr>
</tbody>
</table>

Notes: This simulation is obtained with parametrization I.

Figure 3 plots the IRFs of investment, output, the labor input, capital and net exports in the SOE-RBC model. In the case of a permanent shock, investment and output comove positively. Capital accumulates, to reach a new steady state in the long run. Net exports fall, and the labor input falls as well. In the case of a temporary shock, investment and capital barely move. Output increase temporarily, and this causes the domestic economy to save. The labor input increases.

Another way to look into our results is to consider the variance decomposition in order to gauge which, among the permanent and the temporary shocks, accounts for a higher proportion of consumption volatility. Figure 4 shows the variance decomposition of the two models at different horizons. In both models, (almost) all consumption volatility is accounted by the permanent shock.

4.2 Standard parametrization

Parametrization II is shown in Table 2. Here, we closely follow the literature. All structural parameters of the NK-BW model are from the benchmark estimation in Justiniano et al. (2010). Parameters specific to the SOE-RBC are set as follows. The coefficient on interest rate premium, $\psi$, is set to 0.0010, which
Figure 3: Impulse Responses: SOE-RBC model

Notes: This simulation is obtained with parametrization I.

Figure 4: Variance Decomposition

Notes: Percentage of forecast error explained by the permanent shock. Percentage of forecast error explained by temporary shock is just one minus the variance share depicted here in the figure. The result is obtained with parametrization I.
is the number used in the literature (Schmitt-Grohe and Uribe 2003; Aguiar and Gopinath 2007). The steady-state level of debt-to-output ratio $B/Y$ is set to 0.1 following Aguiar and Gopinath (2007). The standard deviation of the shocks is normalized to 1, and their persistence is set to 0.20.

Parametrization II (the one used in the literature) is close to parametrization I (the one that delivers exact flat responses of consumption). In the case of the NK-BW model, most of the bells and whistles introduced by Christiano et al. (2005) are real rigidities that, coupled with some nominal rigidity, form a powerful force to obtain strong effects of nominal spending on output and labor. At the same time, they allow consumption to be disconnected from actual productivity by almost shutting down the general equilibrium effect on the real interest rate. Thus, it is not crucial for our purposes that the Calvo parameter $\theta$ tends to one: So long as it is fairly high, real rigidities will help get an important effect of permanent shocks on consumption.

In the case of the SOE-RBC model, the usual parametrization naturally picks a high discount rate. More importantly, general equilibrium effects on consumption are muted by setting a small $\psi$ (of around 0.001). As discussed in the literature, $\psi > 0$ achieves stationarity of the debt holdings, however, our analysis shows that setting it to a small value also disconnects consumption from the rest of the model, pretty much as in a NK economy with sticky prices.7

Figure 5 presents the IRFs of productivity and consumption (in both models) under parametrization II (upper and middle rows). The responses of consumption are no longer flat, however, both models deliver very similar responses. The non-flatness of the consumption responses is mostly due to the presence of habit formation, which implies a slow upward adjustment of consumption after a permanent shock. The bottom row of figures plots the responses under parametrization II, but without habit formation, to show that the responses of consumption are still fairly flat.

Figure 6 plots the IRFs of investment, output, the labor input, capital, and inflation in the NK-BW model. The shape responses of investment, output and capital are similar to the ones obtained under parametrization I. The labor input now declines after a permanent shock. This is due to a smaller upward reaction of output under the same increase in TFP, itself mainly due to a smaller increase of consumption on impact. Inflation decreases after both shocks.

Figure 7 shows the IRFs of investment, output, the labor input, capital and

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7 See Garcia-Cicco, Pancrazi, and Uribe (2010) for a related feature of the model with small $\psi$ in terms of the autocorrelation function of net exports.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>Discount rate</td>
<td>0.9987</td>
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<tr>
<td>Consumption habit</td>
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<tr>
<td>Capital share</td>
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<tr>
<td>Inv. Frisch elasticity</td>
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<tr>
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<td>Investment adjustment costs</td>
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<td>Taylor rule inflation</td>
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<td>Persistence nominal interest rate</td>
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<tr>
<td>Standard dev. permanent shock</td>
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</tr>
<tr>
<td>Standard dev. temporary shock</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Figure 5: Impulse Responses

**Notes:** This simulation is obtained with parametrization II.

---

Figure 6: Impulse Responses: NK-BW model

**Notes:** This simulation is obtained with parametrization II.
net exports in the SOE-RBC model. The responses of all four first variables is very similar. The main difference comes from the reaction of net exports, which here merely turn negative after a permanent shock. The reason, again, is the slow reaction of consumption on impact. (Comparing to results under parametrization I, there is no great difference in the reaction of the labor input in this model due that in open economy net exports absorb aggregate demand shifts, not labor as in closed economy NK models.)

Even though in the presence of habit formation the responses of both models are no longer flat under parametrization II, most of the action in the reaction of consumption is still dominated by permanent shocks. Indeed, under habit formation, the impact reaction of consumption after a temporary shock is also muted. To see this, it is useful to focus on the variance decomposition of consumption into both shocks, shown in Figure 8. The variance decomposition shows that most of the impact effect on consumption is due to permanent shocks (> 99.5%), and a fortiori this remains true at longer horizons (because the effect of permanent shocks can only grow over time.)
5 Conclusions

This paper has numerically identified a parameter region for both a modern New Keynesian model (Christiano, Eichenbaum, and Evans 2005) and the small open economy RBC model (Schmitt-Grohe and Uribe 2003; Aguiar and Gopinath 2007) in which the random-walk permanent income property of consumption holds. We demonstrate this by studying the behavior of consumption conditional on both permanent and temporary shocks to TFP. At the limit, the real interest rate is (almost) fixed. The parameter region identified is relevant because it is quite close to standard parametrizations of the NK-BW and SOR-RBC models in the literature.

In terms of choices, we decided to execute these exercises using the same preferences in both models. This choice was motivated by clarity and focus on general equilibrium effects of the economy. However, this is not such a strong restriction because we could have defined the above limit parameter region by setting $\sigma = 1$ in the preferences

$$u_t = \frac{C_t^\gamma (1 - L_t)^{1-\gamma} [1-\gamma]^{1-\sigma}}{1 - \sigma}$$ \hspace{1cm} (4)

which are those used by Aguiar and Gopinath (2007) and are also considered by Schmitt-Grohe and Uribe (2003) (p. 117). When $\sigma = 1$, (4) become log-log and then separability implies that we would obtain the same results, as discussed in the body.

We believe our results shed some light on some findings of the literature and improve our understanding of the role of some of the ingredients employed
by researchers. First, we have established that temporary TFP shocks are not useful to move consumption. Many DSGE models feature only temporary (stationary) TFP shocks, thus there is need to use other shocks in order to explain consumption volatility. Some popular alternatives are preference shocks (in the form of shocks to the discount factor), or noise shocks (Lorenzoni 2009). Second, permanent TFP shocks are clearly more able to move consumption, especially on impact. It is possible that this result is useful in order to get a volatile and countercyclical current account movements in an open economy RBC model, as studied by Aguiar and Gopinath (2007). However, in this paper we have not explored the exact parametrization employed by these authors. Exploring this link more carefully seems like a fruitful research avenue.

References


A  Log-linear Approximation

A.1 NK-BW Model

First we define log-deviations for variables in the approximation. Here we also normalize variables in the model to ensure their stationarity. Specifically, we define

\[ c_t \equiv \log(C_t/A_t) - \log(C/A) \]

We define in a similar way \( y_t, k_t, \bar{k}_t, \) and \( i_t. \) \( N_t \) and \( U_t \) are already stationary, therefore we define

\[ n_t \equiv \log(N_t) - \log(N) \]
\[ u_t \equiv \log(U_t) - \log(U) \]

For nominal variables, we also need to consider non-stationarity in the price level and therefore define

\[ w_t \equiv \log(((W_t/P_t)/A_t) - \log((W/P)/A) \]
\[ r^k_t \equiv \log(R^k_t/P_t) - \log\left(R^k/P\right) \]
\[ r_t \equiv \log(R_t) \]
\[ \pi_t \equiv \log(P_t/P_{t-1}) - \Pi \]

Finally, for Lagrange multipliers, we define

\[ \lambda_t \equiv \log(\Lambda_t A_t) - \log(\Lambda A) \]
\[ \phi_t \equiv \log(\Phi_t A_t/P_t) - \log(\Phi A/P) \]

Households. The marginal utility of consumption is

\[ \Lambda_t = \frac{1}{C_t - h C_{t-1}} - \beta h \frac{1}{C_{t+1} - h C_t} \]

multiplying both sides by \( A_t \)

\[ \Lambda_t A_t = \frac{1}{C_t/A_t - h (C_{t-1}/A_{t-1}) (A_{t-1}/A_t)} - \beta h \frac{1}{(C_{t+1}/A_{t+1}) (C_{t+1}/A_{t}) - h (C_t/A_t)} \]
which with log-linearization yields

$$\lambda_t = \frac{h\beta}{(1-h\beta)(1-h)} E_t c_{t+1} - \frac{1 + h^2 \beta}{(1-h\beta)(1-h)} c_t + \frac{h}{(1-h\beta)(1-h)} c_{t-1} +$$

$$+ \frac{h\beta}{(1-h\beta)(1-h)} E_t \Delta a_{t+1} - \frac{h}{(1-h\beta)(1-h)} \Delta a_t$$

(5)

The Euler equation is

$$\Lambda_t = \beta R_t E_t \left[ \Lambda_{t+1} \frac{P_t}{P_{t+1}} \right]$$

Multiplying both sides of this Euler equation by $A_t$ gives

$$\Lambda_t A_t = \beta R_t E_t \left[ \Lambda_{t+1} A_{t+1} \frac{A_t}{A_{t+1} P_{t+1}} \right]$$

and approximating yields

$$\lambda_t = r_t + E_t \left[ \lambda_{t+1} - \Delta a_{t+1} - \pi_{t+1} \right]$$

(6)

The optimality condition for capacity utilization is

$$R^k_t / P_t = U_t^k$$

In logs,

$$r^k_t = \xi u_t$$

(7)

The provision of the capital services is

$$K_t = U_t K_{t-1}$$

dividing both sides by $A_t$

$$\frac{K_t}{A_t} = \frac{U_t}{A_t} \frac{K_{t-1}}{A_{t-1}} \frac{A_t}{A_t}$$

Approximating it leads to

$$k_t = u_t + \bar{k}_{t-1} - \Delta a_t$$

(8)
The capital accumulation equation is

$$\bar{K}_t = (1 - \delta) \bar{K}_{t-1} + \left[ 1 - \frac{\chi}{2} \left( \frac{I_t}{I_{t-1}} - 1 \right) \right] I_t$$

dividing both sides by $A_t$

$$\frac{\bar{K}_t}{A_t} = (1 - \delta) \frac{\bar{K}_{t-1}}{A_{t-1}} \left( \frac{A_{t-1}}{A_t} \right) + \left[ 1 - \frac{\chi}{2} \left( \frac{I_t}{I_{t-1}} - 1 \right) \right] \frac{I_t}{A_t}$$

Approximating this leads to

$$\frac{\bar{K}}{A} (1 + \bar{k}_t) = (1 - \delta) \frac{\bar{K}}{A} (1 + \bar{k}_{t+1} - \Delta a_t) + \left[ 1 - \frac{\chi}{2} \left( \frac{I_t}{I_{t-1}} \right)^2 \right] \frac{I}{A} (1 + i_t)$$

which summarizes to

$$\frac{\bar{K}}{A} (1 + \bar{k}_t) = (1 - \delta) \frac{\bar{K}}{A} (1 + \bar{k}_{t+1} - \Delta a_t) + \frac{I}{A} (1 + i_t)$$

Then, the log-linear approximation is

$$\bar{k}_t = (1 - \delta) (\bar{k}_{t+1} - \Delta a_t) + \delta i_t$$ (9)

The optimality condition for capital is

$$\Phi_t - \beta \Lambda_{t+1} C(U_{t+1}) - \beta (1 - \delta) \Phi_{t+1} \bar{K}_t$$

which is log-linearly approximated to

$$\phi_t = (1 - \delta) \beta \mathbb{E}_t [\phi_{t+1} - \Delta a_{t+1}] + [1 - (1 - \delta) \beta] \mathbb{E}_t [\lambda_{t+1} - \Delta a_{t+1} + r_{t+1}^k]$$ (10)

Similarly, for the optimality condition for investment

$$-\Lambda_t + \Phi_t \left[ \chi \left( \frac{I_t}{I_{t-1}} - 1 \right) \right] + \beta \Phi_{t+1} \left[ 1 - \frac{\chi}{2} \left( \frac{I_{t+1}}{I_t} - 1 \right)^2 + \frac{\chi}{2} \left( \frac{I_{t+1}}{I_t} \right)^2 \left( \frac{I_{t+1}}{I_t} - 1 \right) \right] = 0$$

The log-linear approximation obtained is

$$\lambda_t = \phi_t - \chi (i_t - i_{t-1} + \Delta a_t) + \beta \mathbb{E}_t [i_{t+1} - i_t + \Delta a_{t+1}]$$ (11)
**Final good producers.** Total output

\[ Y_t = K_t^\alpha \left( A_t N_t \right)^{1-\alpha} \]

dividing both sides by \( A_t \)

\[ \frac{Y_t}{A_t} = \frac{K_t^\alpha}{A_t^\alpha} N_t^{1-\alpha} \]

Taking logs,

\[ y_t = \alpha k_t + (1 - \alpha) n_t \quad (12) \]

**Intermediate goods producers.** The optimal factor proportions between capital and labor is

\[ \frac{K_t}{N_t} = \frac{\alpha}{1 - \alpha} \frac{W_t}{R_k^t} \]

rearranging and dividing both sides by \( A_t \),

\[ (1 - \alpha) \frac{K_t}{A_t} R_k^t = \frac{\alpha}{A_t} W_t N_t \]

Log-linearizing it leads to

\[ k_t - n_t = w_t - r^k \quad (13) \]

The marginal cost equation is

\[ MC_t = \frac{1}{\alpha^\alpha (1 - \alpha)^{1-\alpha}} \left( R_k^t \right)^\alpha \left( \frac{W_t}{A_t} \right)^{1-\alpha} \]

dividing both sides by \( P_t \)

\[ \frac{MC_t}{P_t} = \frac{1}{\alpha^\alpha (1 - \alpha)^{1-\alpha}} \left( \frac{R_k^t}{P_t} \right)^\alpha \left( \frac{W_t}{P_t A_t} \right)^{1-\alpha} \]

Log-linearizing, it leads to

\[ mc_t = \alpha r^k + (1 - \alpha) w_t \quad (14) \]

Finally, from the optimality conditions for price setters, we have

\[ \pi_t = \frac{\ell}{1 + \ell \beta} \pi_{t-1} + \frac{\beta}{1 + \ell \beta} E_t \pi_{t+1} + \kappa mc_t \quad (15) \]

where \( \kappa = (1 - \theta \beta) (1 - \theta) / \theta (1 + \ell \beta) \).
Labor market. Similar to the optimality conditions for price setters, aggregating individual optimality conditions for the wage setter lead to

\[ w_t = \frac{1}{1 + \beta} w_{t-1} + \frac{\beta}{1 + \beta} E_t w_{t+1} + \frac{\nu w}{1 + \beta} \pi_{t-1} - \frac{1 + \nu w}{1 + \beta} \pi_t + \frac{\beta}{1 + \beta} E_t \pi_{t+1} \]  

(16)

\[ + \frac{\nu w}{1 + \beta} \Delta a_{t-1} - \frac{1 + \nu w}{1 + \beta} \Delta a_t + \frac{\beta}{1 + \beta} E_t \Delta a_{t+1} - \kappa_w m_{ct}^w \]

where

\[ k_w = \frac{(1 - \theta_w/\beta) (1 - \theta_w)}{[\theta (1 + \beta) (1 + \varphi (1 + 1/\mu_w))]} \]

and

\[ m_{ct}^w = w_t - \varphi n_t + \lambda_t \]  

(17)

Monetary policy. The interest rate rule is

\[ \frac{R_t}{R} = \left( \frac{R_{t-1}}{R} \right)^{\rho_r} \left[ \left( \frac{\Pi_t}{\Pi} \right) \left( \frac{Y_t/A_t}{Y/A} \right)^{\gamma_y} \right] \left( \frac{Y_t/A_t}{Y_{t-1}/A_{t-1}} \right)^{\phi_{dy}} \]

we obtain

\[ r_t = \rho_r r_{t-1} + (1 - \rho_r) \left( \gamma_y \pi_t + \gamma_y y_t \right) + \phi_{dy} \left( y_t - y_{t-1} \right) \]  

(18)

Market clearing Market clearing in goods market implies

\[ C(1) = R^k / P \]

and

\[ C_t + I_t + C(U_t) K_{t-1} = Y_t \]

Dividing both sides by \( A_t \) and approximating

\[ \frac{C}{A} c_t + \frac{I}{A} i_t + \frac{R^k K}{PA} u_t = \frac{Y}{A} y_t \]

which leads to

\[ y_t = \frac{C}{Y} c_t + \frac{I}{Y} i_t + \frac{R^k K}{PY} u_t \]  

(19)
A.1.1 Summary

- There are 18 variables in the model. The endogenous variables are

$$\lambda_t, \phi_t, y_t, c_t, i_t, \bar{k}_t, k_t, n_t, u_t, r_t, r^k_t, w_t, \pi_t, mc_t, mc^w_t$$

and the exogenous variables are

$$a_t, x_t, z_t$$

- Equations (1)-(3) and (5)-(19) constitute the log-linearized model.

A.2 SOE-RBC Model

In order to log-linearize the model, we define the following log-deviations:

$$c_t \equiv \log(C_t/A_t) - \log(C/A)$$
$$y_t \equiv \log(Y_t/A_t) - \log(Y/A)$$
$$k_t \equiv \log(K_t/A_t) - \log(K/A)$$
$$i_t \equiv \log(I_t/A_t) - \log(I/A)$$
$$n_t \equiv \log(N_t) - \log(N)$$
$$\lambda_t \equiv \log(\Lambda_t A_t) - \log(\Lambda A)$$
$$\phi_t \equiv \log(\Phi_t A_t) - \log(\Phi A)$$

the log of the interest rate

$$r_t \equiv \log(R_t)$$

and the following absolute deviations:

$$b_t \equiv \frac{B_t}{Y_t} - \frac{B}{Y}$$
$$nx_t \equiv \frac{NX_t}{Y_t} - \frac{NX}{Y}$$

In addition to the ten endogenous variables defined above, we also have additional exogenous variables $$x_t, z_t, a_t$$ summarizing the productivity process. Log-linearization of the equilibrium conditions proceeds as follows. The marginal
utility of consumption is
\[ \Lambda_t = \frac{1}{C_t} \]
multiplying both sides by \( A_t \)
\[ \lambda_t = \frac{A_t}{C_t} \]
where \( \lambda_t = \Lambda_t A_t \). In logs the condition is
\[ \lambda_t = -c_t \]  
(20)

The production function is
\[ Y_t = K_{t-1}^\alpha (A_t N_t)^{1-\alpha} \]
dividing both sides \( A_t \)
\[ \frac{Y_t}{A_t} = \left( \frac{K_{t-1}}{A_{t-1}} \right)^\alpha \left( \frac{A_{t-1}}{A_t} \right)^\alpha N_t^{1-\alpha} \]
The log-linearized production function is
\[ y_t = \alpha k_t + (1 - \alpha)n_t - \alpha \Delta a_t \]  
(21)
The resource constraint is
\[ C_t + I_t + N X_t = Y_t \]
multiplying both sides by \( Y_t \)
\[ \frac{C_t A_t}{A_t Y_t} + \frac{I_t A_t}{A_t Y_t} + \frac{N X_t}{Y_t} = 1 \]
Since in steady state
\[ \frac{C A}{A Y} + \frac{I A}{A Y} + \frac{N X}{Y} = 1 \]
approximating,
\[ \frac{C A}{A Y}(1 + c_t - y_t) + \frac{I A}{A Y}(1 + i_t - y_t) + nx_t + \frac{N X}{Y} = 1 \]
which yields
\[ \frac{C}{Y}(c_t - y_t) + \frac{I}{Y}(i_t - y_t) + nx_t = 0 \]  
(22)
The price of debt equation is

\[ R_t = R^* + \psi \left( e^{\frac{b_t}{Y_t}} - 1 \right) \]

We have steady state relation

\[ R = R^* \]

so approximating the equation yields

\[ r_t = \psi b_t \] (23)

The optimality condition for labor is

\[ (1 - \alpha) \Lambda_t = N_t^{1+\varphi} \frac{1}{Y_t} \]

multiplying both sides by \( A_t \)

\[ (1 - \alpha) \lambda_t = N_t^{1+\varphi} \frac{A_t}{Y_t} \]

In logs

\[ \lambda_t = (1 + \varphi) n_t - y_t \] (24)

The budget constraint

\[ C_t + I_t + B_{t-1} = Y_t + Q_t B_t \]

dividing both sides by \( Y_t \),

\[ \frac{C_t}{A_t Y_t} + \frac{I_t}{A_t Y_t} + \frac{B_{t-1}}{A_t Y_{t-1}} \frac{A_t}{Y_t} \frac{A_{t-1}}{A_t} = 1 + \frac{Q_t B_t}{Y_t} \]

which leads in steady state

\[ \frac{C}{A Y} + \frac{I}{A Y} + \frac{B}{Y} = 1 + \frac{Q B}{Y} \]

Approximating

\[ \frac{C}{A Y} \left( 1 + c_t - y_t \right) + \frac{I}{A Y} \left( 1 + i_t - y_t \right) + (b_{t-1} + b) \left( 1 - \Delta y_t - \Delta a_t \right) = 1 + \frac{1}{R} (1 - r_t) (b_t + b) \]
Since $1/R = \beta$, we have that

\[
\frac{C}{Y}(c_t - y_t) + \frac{I}{Y}(i_t - y_t) = \beta b_t - b_{t-1} + b(\Delta a_t + y_t - y_{t-1} - \beta r_t)
\] (25)

The optimality condition for debt is

\[
Q_t \Lambda_t = \beta \mathbb{E}_t [\Lambda_{t+1}]
\]

multiplying both sides by $A_t$,

\[
Q_t \lambda_t = \beta \mathbb{E}_t \left[ \lambda_{t+1} \left( \frac{A_t}{A_{t+1}} \right) \right]
\]

which in steady state

\[
1 = \beta R
\]

Then, approximating

\[
\lambda_t = \mathbb{E}_t [\lambda_{t+1} - \Delta a_{t+1}] + r_t
\] (26)

The optimality condition for investment is

\[
\Lambda_t = -\Phi_t
\]

multiplying both sides by $A_t$

\[
\lambda_t = -\phi_t
\] (27)

The capital accumulation equation is

\[
K_t - (1 - \delta)K_{t-1} - I_t + \nu / 2 \left( \left( \frac{K_t}{K_{t-1}} \right)^2 - 1 \right) K_{t-1} = 0
\]

multiplying both sides by $A_t$,

\[
\frac{K_t}{A_t} - (1 - \delta) \frac{K_{t-1} A_{t-1}}{A_{t-1} A_t} - I_t A_t + \nu / 2 \left( \left( \frac{K_t}{A_t} \frac{A_{t-1}}{K_{t-1} A_{t-1}} \right)^2 - 1 \right) \frac{K_{t-1} A_{t-1}}{A_{t-1} A_t} = 0
\]

which in steady state

\[
\delta \frac{K}{A} - \frac{I}{A} = 0
\]

Then, approximating

\[
(1 + k_t) - (1 - \delta) (1 + k_{t-1} - \Delta a_t) - \frac{I}{K} (1 + i_t) + \nu / 2 (1 + k_{t-1} - \Delta a_t) ((1 + k_t - k_{t-1} + \Delta a_t)^2 - 1) = 0
\]
which yields
\[ k_t = (1 - \delta) [k_{t-1} - \Delta a_t] + \delta i_t \] (28)

Finally, the optimality condition for capital is
\[
\Phi_t \left[ 1 + \nu \left( \frac{K_t}{K_{t-1}} - 1 \right) \right] + \alpha \beta \mathbb{E}_t \left[ A_{t+1} \left( \frac{Y_{t+1}}{K_{t+1}} \right) \right] - \beta \mathbb{E}_t \left[ \Phi_{t+1} \left( 1 - \delta - \nu \left( 1 - \left( \frac{K_{t+1}}{K_t} \right)^2 \right) \right) \right] = 0
\]

multiplying both sides by \( A_t \)
\[
\phi_t \left[ 1 + \nu \left( \frac{K_t}{K_{t-1}} - 1 \right) \right] + \alpha \beta \mathbb{E}_t \left[ \lambda_{t+1} \frac{A_t}{A_{t+1}} \left( \frac{Y_{t+1}}{K_{t+1}} \right) \right] - \beta \mathbb{E}_t \left[ \phi_{t+1} \frac{A_t}{A_{t+1}} \left( 1 - \delta - \nu \left( 1 - \left( \frac{K_{t+1}}{K_t} \right)^2 \right) \right) \right] = 0
\]

which in steady state is
\[
\phi + \alpha \beta \mathbb{E}_t \left[ \lambda \frac{Y}{K} \right] - \beta \mathbb{E}_t [\phi (1 - \delta)] = 0
\]

Approximating, we end up with
\[
\phi (1 + \phi_t) [1 + \nu (1 + k_t) (1 - k_{t-1}) (1 + \Delta a_t) - \nu] + \alpha \beta \lambda (1 + \lambda_{t+1}) (1 - \Delta a_{t+1}) (1 + y_{t+1}) (1 - k_t) (1 + \Delta a_{t+1}) \frac{Y}{K} + \beta \phi (1 + \phi_{t+1}) (1 - \Delta a_{t+1}) [-(1 - \delta)] - \nu (k_{t+1} - k_t + \Delta a_{t+1}) = 0
\]

which yields after collecting terms
\[
-(1 + \phi_t + \nu k_t - \nu k_{t-1} + \nu \Delta a_t) + \alpha \beta \frac{Y}{K} (1 - k_t + \mathbb{E}_t [y_{t+1} + \lambda_{t+1}]) + (29) + \beta (1 - \delta) (1 + \mathbb{E}_t [\phi_{t+1} - \Delta a_{t+1}]) - \beta (\nu k_t - \mathbb{E}_t [\nu k_{t+1} + \nu \Delta a_{t+1}]) = 0
\]

A.2.1 Summary

- There are 13 variables in the model. The endogenous variables are

\[ \lambda_t, \phi_t, y_t, c_t, i_t, k_t, n_t, n x_t, b_t, r_t \]

and the exogenous variables are

\[ a_t, x_t, z_t \]
• Equations (1)-(3) and (20)-(29) constitute the log-linearized model.