Transferring Ownership of Public Housing to Existing Tenants: A Market Design Approach*

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Abstract

This paper explores a housing market with an existing tenant in each house and where the existing tenants initially rent their houses. The idea is to identify equilibrium prices for the housing market given the prerequisite that a tenant can buy any house on the housing market, including the one that he currently is possessing, or continue renting the house he currently is occupying. The main contribution is the identification of an individually rational, equilibrium selecting, and group non-manipulable price mechanism in a restricted preference domain that contains almost all preference profiles. In this restricted domain, the identified mechanism is the minimum price equilibrium selecting mechanism that transfers the maximum number of ownerships to the existing tenants. We also relate the theoretical model and the main findings to the U.K. Housing Act 1980 whose main objective is to transfer ownerships of houses to existing tenants.

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Keywords: Existing tenants; equilibrium; minimum equilibrium prices; maximum trade; group non-manipulability.

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1 Introduction

Matching theory has provided fundamental tools for solving a variety of house allocation problems. Examples include procedures for allocating unoccupied houses among a set of potential tenants (Svensson, 1994), methods for reallocating houses among a group of existing tenants (Shapley and Scarf, 1974), mechanisms for determining rents on competitive housing markets (Shapley and Shubik, 1971), and rules for setting rents on housing markets with legislated rent control (Andersson and Svensson, 2014). This literature is appealing from a theoretical perspective since the investigated allocation mechanisms typically satisfy a number of desirable axioms, including, e.g., individual rationality, non-manipulability, and various efficiency notions.

The housing market considered in this paper contains a finite number of houses with an existing tenant in each house. The idea is to identify equilibrium prices for this market, given the prerequisite that a tenant can buy any house on the market, including the one that he currently is occupying, or continue renting his house. In the model, a fixed lower bound for the equilibrium prices is defined (i.e., reservation prices for the owner), and in case an existing tenant buys the particular house he currently is occupying, the tenant pays only the reservation price. The reservation price can be interpreted as a personalized or discounted price for the existing tenant as the price of that specific house for all other tenants is given by the equilibrium price which is endogenously determined by the preferences of all agents. This also means that if an existing tenant decides to “Keep the House”, the tenant can either buy it at the reservation price or continue renting it. An assignment of agents to houses and a price vector constitute an equilibrium if each agent weakly prefers his assigned consumption bundle to keeping the house and to all other houses at the given prices, and if the assignment guarantees that the maximum number of agents buy a house in case there are several assignments that are compatible with the specific equilibrium price vector. Note that the first part of the equilibrium concept can be seen as a market equilibrium condition as all agents are assigned their most preferred bundle from their consumption set at the given prices.

To solve the above described house allocation problem, it is natural to search for a price mechanism that is individually rational, equilibrium selecting, and non-manipulable. This type of mechanism guarantees that no tenant can lose from participating, that no further rationing of the houses is needed, and that the reported information is reliable. Given the interest in these three specific properties, the perhaps most natural allocation mechanism is based on a “minimum equilibrium price vector” as this type of mechanism previously has been demonstrated to satisfy these specific properties in a variety of different contexts, including, e.g., single-item auction environments (Vickrey, 1961), assignment markets (Demange and Gale, 1985; Leonard, 1983), and housing markets with rent control (Andersson and Svensson, 2014).1 Another natural price mechanism for the above described housing market is an individually rational, equilibrium selecting, and trade maximizing mechanism.

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1In some cases, this is also the only mechanism that is Pareto efficient and strategy-proof. See, e.g., Svensson (2009) or Morimoto and Serizawa (2015).
The latter axiom guarantees that the maximum number of houses are transferred to the tenants in equilibrium. This is often a specific policy goal for this type of housing market as will be explained later. Here, we note that an equilibrium selecting minimum price mechanism guarantees that the number of agents who buy a house is maximal for a given minimum equilibrium price vector, whereas a trade maximizing mechanism guarantees that the number of agents who buy a house is maximal among all equilibrium price vectors.

Even if it is natural to believe that the above two mechanisms always recommend the same selection, as lower prices intuitively should increase trade, it turns out that this is generally not the case. In fact, it is not even clear what any of the above two mechanisms recommend due to the non-uniqueness of a minimum equilibrium price vector and the non-uniqueness of an allocation that maximizes trade on the full preference domain. The non-uniqueness property of the two mechanisms is somewhat unexpected, given what we know from previous literature (e.g., Demange and Gale, 1985; Leonard, 1983; Shapley and Shubik, 1971), and it is a direct consequence of the fact that agents can block trade of a house via their outside option to keep their house (see Section 4 for further discussions).

This multiplicity has the consequence that any allocation mechanism based on minimum equilibrium prices is manipulable on the full preference domain. Intuitively, this can be understood by imagining a situation when two minimum price equilibria exist for given preferences and when one agent strictly prefers one of them over the other. In this situation, it may be possible to manipulate the mechanism by misrepresenting preferences in such a way that only the strictly preferred equilibrium remains. Similarly, the multiplicity of an allocation that maximizes trade has the consequence that any allocation mechanism based on maximum trade is manipulable on the full preference domain. However, by considering a preference domain that contains almost all preference profiles, it turns out that a minimum equilibrium price vector is unique for each preference profile in the restricted domain, and that it is possible to base an individually rational, equilibrium selecting, and group non-manipulable allocation mechanism on this unique price vector. In addition, this mechanism turns out to be a maximum trade mechanism in the restricted domain, and it, therefore, guarantees that the maximum number of houses are transferred from the owner to the existing tenants.

1.1 Related Literature

To the best of our knowledge, this paper is the first to provide an individually rational, equilibrium selecting, trade maximizing, and non-manipulable allocation mechanism for a housing market with existing tenants where monetary transfers are allowed. The main results presented here are non-trivial extensions of similar and previously known results because most of the previous literature either considers the case with no initial endowments and where monetary transfers are allowed, or initial endowments but where monetary transfers are not allowed. Furthermore, we are also not aware of any matching model that includes the possibility of monetary transfers.

An individually rational, strategy-proof, and Pareto efficient mechanism for a housing market with both existing tenants and new applicants but without the possibility to transfer money has been considered by Abdulkadiroğlu and Sönmez (1999).
that can deal with the type of outside option considered in this paper when monetary transfers are allowed. In the following, we explain in more detail how our theoretical findings contribute to the related literature.

The idea to use a minimum equilibrium price vector as a key ingredient in an individually rational, equilibrium selecting, and non-manipulable allocation mechanism was first advocated by Vickrey (1961) in his single-unit sealed-bid second-price auction. This principle was later generalized by, e.g., Demange and Gale (1985) to the case when multiple heterogeneous houses are sold.\footnote{See also Crawford and Knoer (1981), Leonard (1983), Demange et al. (1986), Sun and Yang (2003), Mishra and Parkes (2009) and Morimoto and Serizawa (2015).} The significant difference between this paper and Demange and Gale (1985) is that the model in the latter paper cannot handle the case with existing tenants, and, consequently, not the case when tenants can block trade of a house through the “Keep the House” option. This outside option makes our model less “well-behaved” compared to the one in Demange and Gale (1985), e.g., the set of equilibrium price vectors generally does not have a lattice structure. In the special case when this outside option is not available, the minimum price mechanism investigated in this paper recommends the same outcome as the mechanism proposed by Demange and Gale (1985). Even though the mechanism considered here and the one in Demange and Gale (1985) are defined for different types of housing markets, they share the properties of group non-manipulability and that agents always are assigned their most preferred consumption bundle from their consumption sets at the prices determined by the mechanism.

A model with existing tenants and prices is studied by Miyagawa (2001). In his setting, each tenant owns precisely one house and is a seller as well as a buyer, so money and houses are reallocated through a price mechanism exactly as in this paper. Unlike this paper, and the ones cited in the previous paragraph, Miyagawa (2001) requires the mechanism to be “non-bossy” (Satterthwaite and Sonnenschein, 1981) which roughly means that no tenant can change the assignment for some other tenant without changing the assignment for himself. The assumption of non-bossiness has dramatic consequences on any non-manipulable allocation mechanism in this setting (see also Schummer, 2000). Namely, any individually rational, non-bossy, and non-manipulable allocation mechanism is a fixed price mechanism, and, therefore, agents need not be assigned their most preferred consumption bundle from their consumption sets. Because we do not require non-bossiness, our allocation mechanism is not a fixed price mechanism and it always selects an equilibrium outcome.

Individually rational and non-manipulable mechanisms for housing markets with existing tenants have been considered previously in the literature when monetary transfers are not allowed. Most notably Gale’s Top-Trading Cycles Mechanism (Shapley and Scarf, 1974), has been further investigated by, e.g., Abdulkadiroğlu and Sönmez (1999), Ma (1994), Postlewaite and Roth (1977), Roth (1982) and Miyagawa (2002). This mechanism always selects an outcome in the core. However, the allocation mechanisms in these papers cannot generally guarantee, in similarity with Miyagawa (2001), that agents are assigned their most preferred consumption bundle from their consumption sets. Neither can they handle the case with monetary transfers.
Finally, a recent and intermediate proposal is due to Andersson and Svensson (2014) where a set of houses with no existing tenants are allocated among a set of potential tenants using a price mechanism where prices are bounded from below and from above by price ceilings imposed by the government or the local administration. In this model, they define an individually rational, equilibrium selecting and group non-manipulable allocation mechanism that, in its two limiting cases, selects the same outcomes as the mechanism in Demange and Gale (1985) and the Deferred Acceptance Algorithm (Gale and Shapley, 1962). Both in Andersson and Svensson (2014) and in this paper, agents face exogenous constraints. More precisely, in Andersson and Svensson (2014) the price ceilings are common exogenous constraints whereas the “Keep the House” options in this paper are individual exogenous constraints. The exogenous constraint in Andersson and Svensson (2014) excludes all price equilibria for some preference profiles and, consequently, a rationing mechanism must be included in their equilibrium concept. Such a rationing mechanism need not be included in the equilibrium concept considered here. Due to these differences, the proof ideas of Andersson and Svensson (2014) cannot be applied here (details are further discussed in Remark 2 and in the main text).

1.2 Application: The U.K. Housing Act 1980

Public housing is a common form of housing tenure in which the property is owned by a local or central government authority. This type of tenure has traditionally referred to a situation where the central authority lets the right to occupy the units to tenants. In the last 30–35 years, however, many European countries have experienced important changes in their policies. In the United Kingdom, for example, the “Right to Buy” was implemented in the U.K. Housing Act 1980. As its name indicates, the legislation gave existing and eligible tenants the right to buy the houses they were living in.\textsuperscript{4} Here, an eligible tenant is defined as a tenant who has been living in the house for at least three years before applying for the “Right to Buy” (before May 2015, the requirement was five years).\textsuperscript{5} A tenant could, however, choose to remain in the house and pay the regulated rent. The U.K. Secretary of State for the Environment in 1979, Michael Heseltine, stated that the main motivation behind the Act was:

“... to give people what they wanted, and to reverse the trend of ever increasing dominance of the state over the life of the individual.”\textsuperscript{6}

In other words, the main objective of the Act was to transfer the ownership of the houses from the local or central authority to the existing tenants. To cope with this objective,

\textsuperscript{4}In the United Kingdom, local authorities have always had legal possibilities to sell public houses to tenants, but until the early 1970s such sales were extremely rare.

\textsuperscript{5}All rules and details in the U.K. Housing Act 1980 referred to in this section are taken from “Your right to buy your home: A guide for tenants of councils, new towns and registered social landlords including housing associations” (Department for Communities and Local Government, May 2015).

the Act also specified that tenants could buy the houses at prices significantly below the market price. In the latest update of the Act from May 2015, this “discount” may be as high as the lesser of £77,900 (£103,900 in London) or 60 percent of the market value. The market value of the house is estimated by the landlord, possibly together with a District Valuer (who will conduct an independent valuation), and should reflect the true market value of the house. As can be expected, the result of the Act was that the proportion of public housing in United Kingdom fell from 31 percent in 1979 to 17 percent in 2010. In fact, more than 33,000 households have taken up their “Right to Buy” since April 2012. Similar legislations, with similar effects, have been passed in several European countries, including Germany, Ireland, and Sweden, among others.

The significant difference between the U.K. Housing Act 1980, and its European equivalents, and the model considered in this paper is that we extend the situation from the case where only an existing tenant can buy the house that he currently is occupying to a situation where houses can be reallocated among all existing tenants in a pre-specified area. Note that it is impossible to immediately make this type of reallocation within the framework of the Act and at the same time transfer the ownership of the houses from the landlord to the existing tenants at the discounted prices. There are, at least, two reasons for this. First, “mutual exchange” is allowed in the U.K., but if a mutual exchange is conducted, the tenants must live in their new houses for at least three years before being eligible to apply for the “Right to Buy”. Second, it is not possible for two (or more) tenants to use their “Right to Buy” and then immediately reallocate the houses by selling them to each other since resale is not allowed within five years after purchasing the houses unless the discount is paid back to the landlord (in fact, if a tenant resells within 10 years after buying, the tenant has to offer first the house either to the former landlord or to another social landlord in the area).

A relevant question is of course why the extension of the current U.K. system, proposed in this paper, should be of any interest to a policy maker. As we will argue, there are, at least, three good reasons for this. First, as tenants in our framework always can choose to continue renting the house they live in or to buy it at the reservation price (one can think about the reservation price in our model as the discounted market price), exactly as in the prevailing U.K. system, but have the opportunity to buy some other house, all tenants are weakly better off in the considered model compared to the current U.K. system. Second, because all sold houses in the current U.K. system are sold at the reservation prices, but all sold houses in the considered model are sold at prices weakly higher than the reservation

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7 “Housing Europe Review 2012 – The nuts and bolts of European social housing systems”, European Federation of Public, Cooperative and Social Housing (Brussels).

8 Mutual exchange refers to a situation where two (or more) tenants in the public housing sector “swap” their houses when renting from a public authority. Typical requirements for a mutual exchange to take place are that none of the tenants included in the swap owes rent, is in the process of being evicted, and is moving to a home that the landlord believes is too big or small for their circumstances. In England, mutual exchange is often systematically organized on websites, e.g., www.houseexchange.org.uk and www.homeswapper.co.uk. In October, 2015, the latter website claimed on their Twitter account (@HomeSwapperteam) that they have over 1.7 million visits per month and that they currently get more than 5,000 new members per week.
prices, the public authority generates a weakly higher revenue in the considered model compared to the current U.K. system. Third, the considered model captures the idea that housing needs may change over time, e.g., a family has more children or some children move out. In such cases, a tenant may want to change house and use the “Right to Buy”. These tenants are given permission to participate in our housing market, but they are not allowed to immediately participate in the prevailing U.K. system as explained in the above.

1.3 Outline of the Paper

The remaining part of this paper is organized as follows. Section 2 introduces the formal model and some of basic definitions that will be used throughout the paper. The allocation mechanisms are introduced in Section 3 where also the main existence and non-manipulability results of the paper are stated. Section 4 contains some concluding remarks. All proofs are relegated to the Appendix.

2 The Model and Basic Definitions

The agents and the houses are gathered in the finite sets $A = \{1, \ldots, n\}$ and $H = \{1, \ldots, n\}$, respectively, where $n = |A|$ is a natural number. Note that the number of agents and houses coincide as we assume that there is an existing tenant in each house. Agent $a$ is the existing tenant of house $h$ if $h = a$.

Each house $h \in H$ has a price $p_h \in \mathbb{R}_+$. These prices are gathered in the price vector $p \in \mathbb{R}_+^n$ which is bounded from below by the reservation prices $p \in \mathbb{R}_+^n$ of the owner. The reservation price of house $h$ is $p_h$. The reservation prices are arbitrary but fixed and define a feasible set of prices $\Omega$ according to:

$$\Omega = \{p \in \mathbb{R}_+^n : p \geq p\}.$$ 

Agent $a \in A$ can continue renting house $h = a$ at the given fixed rent (“Right to Stay and Rent”) or buy the house he currently is living in at the owner’s reservation price $p_a$ (“Right to Buy His Current House”). Agent $a$ can also buy house $h \neq a$ at price $p_h$. For notational simplicity, and without loss of generality, the fixed rents will not be introduced in the formal framework. Formally, each agent $a \in A$ consumes exactly one (consumption) bundle, $x_a$, in his consumption set $X_a = (H \times \mathbb{R}_+) \cup \{a\}$ where:

$$x_a = \begin{cases} 
  a & \text{if agent } a \text{ continues to rent house } h = a, \\
  (a, p_a) & \text{if agent } a \text{ buys house } h = a \text{ at price } p_a, \\
  (h, p_h) & \text{if agent } a \text{ buys house } h \text{ at price } p_h.
\end{cases}$$

Note that each agent $a \in A$ has two outside options, i.e., the “Right to Stay and Rent”, $x_a = a$, and the “Right to Buy His Current House” (for the reservation price), $x_a = (a, p_a)$.

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9One can think of the fixed reservation prices as exogenously given and specified in the law as explained in the Introduction.
For convenience, we often denote the outside option of agent $a$ simply by house $a$, i.e., $x_a = a$ will mean that either agent $a$ continues renting or buys the house he currently is living in at its reservation price (whichever is better). We adopt the convention to say “agent $a$ keeps his house” whenever the agent exercises one of his outside options. The agent’s choice between these two options does not affect other agents. For technical reasons, an agent $a$ can also buy his own house at the price $p_a$, but that will not be the choice of a utility maximizing agent if $p_a > p_a^*$. For simplicity, a bundle of type $(h, p_h)$ will often be written as $(h, p)$, i.e., $(h, p) \equiv (h, p_h)$.  It is then understood that $(h, p)$ means house $h \in H$ with price $p_h$ at the price vector $p$.

Each agent $a \in A$ has preferences over bundles. These preferences are denoted by $R_a$ and are represented by a complete preorder on $X_a$. The strict and indifference relations are denoted by $P_a$ and $I_a$, respectively. Preferences are assumed to be continuous$^{10}$, strictly monotonic, and boundedly desirable. Strict monotonicity means that agents strictly prefer a lower price to a higher price on any given house, i.e., $(h, p) \prec (h, p')$ if $p < p'$ for any agent $a \in A$ and any house $h \in H$. Bounded desirability means that if the price of a house is “sufficiently high”, the agents will strictly prefer to keep the house they currently are living in rather than buying some other house, i.e., $aP_a(h, p_h)$ for each agent $a \in A$ and for each house $h \in H$ for $p_h$ “sufficiently high”. All preference relations $R_a$ satisfying the above properties for agent $a \in A$ are gathered in the set $\mathcal{R}_a$. A (preference) profile is a list $R = (R_1, \ldots, R_n)$ of the agents’ preferences. This list belongs to the set $\mathcal{R} = \mathcal{R}_1 \times \cdots \times \mathcal{R}_n$. We also adopt the notational convention of writing a profile $R \in \mathcal{R}$ as $R = (R_C, R_{\neg C})$ for some nonempty subset $C \subseteq A$.

A state is a triple $(\mu, \nu, p)$, where $\mu : A \to H$ is a mapping assigning agents to houses, $\nu : A \to \{0, 1\}$ is an assignment indicating if an agent $a \in A$ is renting, $\nu_a = 0$, or buying, $\nu_a = 1$, and $p \in \Omega$ is a feasible price vector. If agent $a \in A$ is assigned house $\mu_a \in H$ and $\nu_a = 1$, the agent pays:

$$p_{\mu_a} = \begin{cases} p_{\mu_a} & \text{if } \mu_a \neq a, \\ p_a & \text{if } \mu_a = a. \end{cases}$$

The assignment function $\mu$ is a bijection with the restriction $\mu_a = a$ if $\nu_a = 0$. This means that an agent cannot rent a house that currently he is not living in. Agent $a$ is also the only agent that always can buy house $h = a$ at the reservation price. We will use the simplified notation $x = (\mu, \nu, p)$ for a state, where $x \in \times_{a \in A} X_a$ and $x_a = (\mu_a, p)$ if $\nu_a = 1$ and $\mu_a \neq a$, and $x_a = a$ if $\mu_a = a$ and $\nu_a \in \{0, 1\}$. Here, it is understood that $\nu_a = 0$ only if $aP_a(a, p_a)$.

The cardinality $|\nu|$ of the assignment $\nu$ indicates the number of agents who are buying a house at state $(\mu, \nu, p)$, and it is defined by $|\nu| = \sum_{a \in A} \nu_a$.

**Definition 1.** For a given profile $R \in \mathcal{R}$, a price vector $p \in \Omega$ is an equilibrium price vector if there is a state $x = (\mu, \nu, p)$ such that the following holds for all agents $a \in A$: (i) $x_aR_a a$, and (ii) $x_aR_a (h, p)$ for all $h \in H$. If, in addition, the cardinality of the assignment $\nu$ is maximal among all states with price vector $p$ satisfying (i) and (ii), the state $x$ is an equilibrium state.

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$^{10}$Continuity means that the weak upper and lower contour sets of $R_a$ are closed in $X_a$. 

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The first condition of the definition states that each agent weakly prefers his assigned bundle to the house that he currently is renting (individual rationality) and the second one states that each agent weakly prefers his assigned bundle to all other houses at the given prices. This can be seen as an equilibrium condition as all agents are assigned their most preferred bundle from their consumption sets at the given prices. The last condition essentially states that the number of agents who are buying a house is maximal at a given equilibrium price vector. This condition reflects the fact that the owner of the houses prefers to sell the houses rather than keeping existing tenants (recall from the Introduction that this is the main goal of the U.K. Housing Act 1980 and similar legislations). Furthermore, note that any unsold house $h$ must continue to be rented by its existing tenant $a = h$, i.e., no other agent $a' \neq h$ can rent $h$. Due to this, the price of house $h$ can be above its reservation price $p_h$ (and the same is true if agent $a = h$ buys his house at its reservation price because other agents can buy house $h$ only at price $p_h$). In this respect, Definition 1 differs from some previously adopted equilibrium notions for models with indivisible items and prices (e.g., Andersson and Svensson, 2014; Demange and Gale, 1985) where prices of unsold items always equal the reservation price. The reason for deviating in this sense is that because of the outside option of keeping the house, it is impossible to guarantee that all agents are assigned their most preferred bundle from their consumption sets, at given prices, unless prices of unsold houses are allowed to exceed the reservation prices.

For a given profile $R \in \mathcal{R}$, the set of equilibrium price vectors is denoted by $\Pi_R$, and the set of equilibrium states is denoted by $E_R$. Hence:

$$\Pi_R = \{p \in \mathbb{R}^n : (\mu, \nu, p) \in \mathcal{E}_R \text{ for some assignments } \mu \text{ and } \nu\}.$$  

An equilibrium price vector $p' \in \Pi_R$ is a minimum equilibrium price vector, at a given profile $R \in \mathcal{R}$, if $p \leq p'$ and $p \in \Pi_R$ imply $p = p'$. A state $(\mu', \nu', p')$ is a minimum price equilibrium, at a given profile $R \in \mathcal{R}$, if $(\mu', \nu', p') \in \mathcal{E}_R$ and $p'$ is a minimum equilibrium price vector. A state $(\mu, \nu, p)$ is a maximum trade equilibrium, at a given profile $R \in \mathcal{R}$, if $(\mu, \nu, p) \in \mathcal{E}_R$ and $|\nu| \geq |\nu'|$ for all $(\mu', \nu', p') \in \mathcal{E}_R$. Here, we note that the requirement on an equilibrium state is that the number of agents that buy a house is maximal for a given equilibrium price vector, whereas the requirement on a maximum trade equilibrium is that the number of agents that buy a house is maximal among all equilibrium price vectors.

**Definition 2.** For a given profile $R \in \mathcal{R}$, a state $x = (\mu, \nu, p)$ is weakly efficient if there is no other state $x' = (\mu', \nu', p')$ such that $x'_a p_a x_a$ for all $a \in A$.

Hence, an equilibrium state $x$ is weakly efficient if there does not exist any other state with a feasible price vector which all agents strictly prefer to the equilibrium state $x$.

### 3 Manipulability and Non-Manipulability Results

As already explained in the Introduction, it is natural to let a minimum equilibrium price vector be a key ingredient in an allocation mechanism for the considered house allocation...
problem as this type of mechanism previously has been demonstrated to be individually rational, equilibrium selecting, and non-manipulable in various economic environments. Another natural price mechanism is based on individual rationality, equilibrium selection, and maximal trade. Here, the latter axiom guarantees that the mechanism maximizes the number of traded houses, i.e., that it achieves the objectives of the U.K. Housing Act 1980 and similar legislations. These two mechanisms are formally defined next.

Let \( E \) denote the set of all states, or equivalently \( E = \bigcup_{R \in R} E_R \). A mechanism is a function \( f : R \to E \) where, for each profile \( R \in R \), the mechanism \( f \) selects an equilibrium state \((\mu, \nu, p) \in E_R\). A mechanism is called a minimum price mechanism if it, for each profile \( R \in R \), selects an equilibrium state \((\mu, \nu, p) \in E_R \) where \( p \) is a minimum equilibrium price vector. A mechanism is called a maximum trade mechanism if it, for each profile \( R \in R \), selects a state \((\mu, \nu, p) \in E_R \) which is a maximum trade equilibrium.

To be certain that the above two mechanisms are well-defined, we need to establish that the equilibrium set \( E_R \) is nonempty for all profiles in \( R \) because in this case, \( \Pi_R \) will also be nonempty for all profiles in \( R \) by definition. The existence of a minimum equilibrium price vector then follows directly as the set of equilibrium prices is bounded from below by \( p \) and closed since preferences are continuous. The existence of a maximum trade equilibrium follows by the non-emptiness of the equilibrium set. These insights are reported in the following proposition together with the findings that any minimum price equilibrium is weakly efficient and if there is a unique minimum equilibrium price vector, at a given profile, then all agents weakly prefer any minimum price equilibrium to any other equilibrium state.

**Proposition 1.** Let \( R \in R \) be a profile and \( p \) be a vector of reservation prices. Then:

(i) The set of equilibria \( E_R \) is nonempty.

(ii) Any minimum price equilibrium is weakly efficient.

(iii) If there is a unique minimum equilibrium price vector in \( \Pi_R \), then all agents weakly prefer any minimum price equilibrium in \( E_R \) to any other equilibrium state in \( E_R \).

We will next, in a series of examples and propositions, investigate the similarities and differences between the two mechanisms defined above. The first example demonstrates that the selection of a minimum price mechanism and the selection of a maximum trade mechanism need not be identical in the full preference domain \( R \). This result is a bit counterintuitive as one would suspect that these mechanisms always recommend the same selection as lower prices intuitively should increase trade.

**Example 1.** Let \( A = \{1, 2, 3\} \) and \( H = \{1, 2, 3\} \) be the sets of agents and houses, respectively, where agent \( a \in A \) is the existing tenant of house \( h = a \). Let \( p = (0, 0, 0) \). For each agent \( a \in A \), preferences over bundles \((h, p)\) are represented by a quasi-linear utility function \( u_{ah}(p) = v_{ah} - p_h \) where the values \( v_{ah} \) are represented by real numbers. Let the
utility for agent $a \in A$ of renting house $h = a$ formally be represented by $v_{ah}$, and:

$$V = (v_{ah})_{a \in A, h \in \{0\} \cup H} = \begin{pmatrix} -2 & -2 & 3 & 2 \\ -2 & 1 & 1 & -2 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$ 

In this case, $p = (0, 0, 0)$ is the unique minimum equilibrium price vector as $x = (\mu, \nu, p)$ is an equilibrium state for $\mu = (2, 1, 3)$ and $\nu = (1, 1, 0)$. Note next that $x$ is the only possible selection of a minimum price mechanism. Note also that agent 3 continues to rent the house that he currently is living in as this is strictly preferred to buying it at the reservation price. The state $x' = (\mu', \nu', p')$ is a possible selection of a maximum trade mechanism for $\mu' = (3, 2, 1), \nu' = (1, 1, 1)$, and $p' = (0, 1, 0)$. Note that at state $x'$, agent 2 buys house 2 for its reservation price (and not for price $p'_2 = 1$). Somewhat surprisingly, agent 3 continues renting in the minimum price equilibrium but buys house 1 in the maximum trade equilibrium. Hence, $|\nu'| > |\nu|$ which demonstrates that a minimum price mechanism and a maximum trade mechanism make distinct selections for $R$. Furthermore, all agents weakly prefer the minimum price equilibrium $x$ to the maximum trade equilibrium $x'$ as the utilities are given by $(3, 1, 0)$ and $(2, 1, 0)$, respectively. □

The following example demonstrates that neither a minimum price mechanism nor a maximum trade mechanism need to make a unique price selection on the full preference domain $\mathcal{R}$.

**Example 2.** Let $A = \{1, 2, 3, 4\}$ and $H = \{1, 2, 3, 4\}$ be the sets of agents and houses, respectively. Let $p = (0, 0, 0, 0)$. As in Example 1, it is assumed that preferences are represented by quasi-linear utility functions where:

$$V = (v_{ah})_{a \in A, h \in \{0\} \cup H} = \begin{pmatrix} -1 & 0 & -2 & 0 & -2 \\ -1 & -2 & 0 & 0 & -2 \\ -1 & 2 & -2 & 0 & 1 \\ -1 & -2 & 2 & -2 & 1 \end{pmatrix}.$$ 

Note that each agent prefers buying his house to renting and $\nu = (1, 1, 1, 1)$ in any equilibrium state. In this case, both $p' = (1, 0, 0, 0)$ and $p'' = (0, 1, 0, 0)$ are minimum equilibrium price vectors (details are provided in the proof of Proposition 2). This follows since both $x' = (\mu', \nu', p')$ and $x'' = (\mu'', \nu'', p'')$ are equilibrium states for $\mu' = (1, 3, 4, 2), \nu' = (1, 1, 1, 1), \mu'' = (3, 2, 1, 4)$, and $\nu'' = (1, 1, 1, 1)$. Note that at state $x'$, agent 1 buys his house at its reservation price 0, at state $x''$ agent 2 buys his house at its reservation price 0, and that trade is maximized at states $x'$ and $x''$ since $|\nu'| = |\nu''| = |A|$. Note that agent 3's utility in $x'$ is 1 whereas agent 3's utility in $x''$ is 2. Similarly, agent 4's utility in $x'$ is 2 whereas agent 4's utility in $x''$ is 1. In other words, agents 3 and 4 have opposed preferences over $x'$ and $x''$. □

**Remark 1.** We also remark that for any profile $R$, there always exists a unique maximum price vector in $\Pi_R$ in the following sense: let $p^* \in \Omega$ be such that for all $a \in A$, (i) for any
house \( h \neq a, aR_a(h, p) \) and (ii) there exists \( a' \neq a \) such that \( a'I_{a'}(a, p^*) \). Then each agent keeps his house. The price vector \( p^* \) is unique in the sense that lowering the price of a house \( h \) results in agent \( a' \) willing to buy house \( h \) instead of keeping his house. Of course, because existing tenants buy their houses at the reservation prices or continue renting, any \( p \geq p^* \) belongs to \( \Pi_R \) as well. Choosing \( p^* \) results in the same outcome as the current separate take-it-or-leave-it offers to the existing tenants in the U.K. housing market. □

The multiplicity of a minimum equilibrium price vector and a maximum trade equilibrium is a direct consequence of the fact that agents can block the trade of a house through their outside option. This has the severe consequence that any minimum price mechanism and any maximum trade mechanism is manipulable on the full preference domain, at least if there are more than three houses on the housing market (see Proposition 2). The intuition behind this result is that if there are, say, two possible minimum price equilibria, at a given profile, and if one agent strictly prefers one of them over the other, then the agent may be able to manipulate the outcome of the mechanism by misrepresenting preferences in such a way that only the strictly preferred equilibria remains after the misrepresentation (this is also the main idea in the proof of Proposition 2). The following notion of group manipulability and (group) non-manipulability is employed.

**Definition 3.** A mechanism \( f \) is manipulable at a profile \( R \in \mathcal{R} \) by a nonempty group of agents \( C \subseteq A \) if there is a profile \( R' = (R'_C, R_{-C}) \in \mathcal{R} \), and two states \( f(R) = x = (\mu, \nu, p) \) and \( f(R'_C, R_{-C}) = x' = (\mu', \nu', p') \) such that \( x'_a P_a x_a \) for all \( a \in C \).

If the mechanism \( f \) is not manipulable by any group \( C \subseteq A \) at profile \( R \), then \( f \) is group non-manipulable at \( R \). Given \( \mathcal{R}^* \subseteq \mathcal{R} \), the mechanism \( f \) is group non-manipulable on the domain \( \mathcal{R}^* \) if for any profile \( R \in \mathcal{R}^* \), \( f \) is group non-manipulable at \( R \).

If the mechanism \( f \) is not manipulable by any group of size one (or any agent) at profile \( R \), then \( f \) is non-manipulable at \( R \). Given \( \mathcal{R}^* \subseteq \mathcal{R} \), the mechanism \( f \) is non-manipulable on the domain \( \mathcal{R}^* \) if for any profile \( R \in \mathcal{R}^* \), \( f \) is non-manipulable at \( R \).

**Proposition 2.** Let \( |A| > 3 \).

(a) Any minimum price mechanism \( f \) is manipulable on the domain \( \mathcal{R} \).

(b) Any maximum trade mechanism \( f \) is manipulable on the domain \( \mathcal{R} \).

**Remark 2.** If agents do not have any outside options (i.e., if all agents always face the same price vector and no agent can continue renting the house that he currently is living in), then there exists a unique minimum equilibrium price vector for each \( R \in \mathcal{R} \) as demonstrated by Demange and Gale (1985). In Example 2 this unique price vector is given by \( p = (1, 1, 1, 0) \) for assignment \( \mu = (3, 2, 1, 4) \). In addition, Demange and Gale (1985) showed that any minimum price mechanism is non-manipulable at \( R \). When there are only lower price constraints, the models of Andersson and Svensson (2014) and Demange

\[11\] Andersson et al. (2014) show that a minimum price mechanism is non-manipulable on the domain \( \mathcal{R} \) when \( |A| \leq 3 \).
and Gale (1985) coincide. However, as we show in the Appendix, in our setting with outside options, any minimum price mechanism is manipulable at profile \( R \). This is due to the facts that (i) kept houses can carry a price above the reservation price and (ii) in any equilibrium, the cardinality of \( \nu \) needs to be maximized. Andersson and Svensson (2014) and our paper have different exogenous constraints: in Andersson and Svensson (2014), agents face the common exogenous constraint of upper price limits on houses (and without rationing, a price equilibrium vector does not necessarily exist), whereas, in this paper, agents have individual exogenous constraints (i.e., their individual outside options) without having upper price limits on houses (and by Proposition 1, an equilibrium price vector always exists).

The implication from Proposition 2 is that if one searches for non-manipulable mechanisms on the full preference domain, one cannot search in the class of minimum price mechanisms or the class of maximum trade mechanisms, at least, if one is interested in housing markets containing more than three houses. Of course, there are non-manipulable mechanisms also on the full preference domain, e.g., a mechanism which, for any profile, sets prices “sufficiently high”\(^{12} \) so that all agents prefer either renting or buying the house they currently occupy. This mechanism always recommends the identical outcome as the mechanism which is currently used in the United Kingdom, and it will not be of any interest to a public authority that aims to transfer more houses to tenants compared to the existing U.K. system.

In the following, we will demonstrate that by excluding some profiles from the domain \( \mathcal{R} \) and instead consider the reduced preference domain \( \tilde{\mathcal{R}} \subset \mathcal{R} \) where no two outside options, or no two houses, are “connected by indifference” at any price vector \( p \in \Omega \), a minimum equilibrium price vector is unique for all profiles in the reduced domain. This unique price vector is then demonstrated to be the key ingredient in an individually rational, equilibrium selecting, and group non-manipulable mechanism, which, in addition, maximizes trade.

**Definition 4.** For a given profile \( R \in \mathcal{R} \), two houses, \( h \) and \( h' \), in \( H \) are connected by indifference if there is a price vector \( p \in \Omega \), a sequence of distinct agents \((a_1, \ldots, a_q)\), and a sequence of distinct houses \((h_1, \ldots, h_{q+1})\) for \( q \geq 2 \) such that:

(i) \( h = h_1 = a_1 \), and \( h' = h_{q+1} = a_q \),

(ii) \( a_1I_{a_1}(h_2, p) \), and \( a_qI_{a_q}(h_q, p) \),

(iii) \((h_j, p)I_{a_j}(h_{j+1}, p)\) for \( 2 \leq j \leq q - 1 \) if \( q > 2 \).

The subset of \( \mathcal{R} \) where no two houses are connected by indifference is denoted by \( \tilde{\mathcal{R}} \).

In Definition 4, house \( a_1 \) stands for agent \( a_1 \)'s outside option of keeping his house \( a_1 \) (and similarly, for house \( a_q \) and agent \( a_q \)). Note also that houses \( h = 1 \) and \( h' = 3 \) are connected.

---

\(^{12}\)Formally, for any \( R \in \mathcal{R} \), choose \( p \in \mathbb{R}_+ \) such that \( aP_a(h, p) \) for all \( a \in A \) and all \( h \in H \) (and such prices exist because preferences are boundedly desirable).
by indifference at prices \( p' = (1, 0, 0, 0) \) in Example 2. To see this, consider agents 1 and 2 and the sequence of houses \((1, 3, 2)\). In this case, \( q = 2, a_1 = 1, a_2 = a_q = 2, 1I_1(3, 0), \) and \( 2I_2(3, 0) \) so all conditions of Definition 4 are satisfied (note that the last condition in the definition is irrelevant because \( q = 2 \)), i.e., houses \( h = 1 \) and \( h' = 3 \) are connected by indifference at prices \( p' = (1, 0, 0, 0) \). This is also the reason for the existence of two minimum equilibrium price vectors in the example. If, for example, \( v_{13} \) is replaced by some arbitrary real number strictly smaller than zero, then no two houses would be connected by indifference at the preferences represented by \( V \), and \( p' \) would be the unique minimum equilibrium price vector at the given preferences.

We remark that the above type of domain restriction contains almost all preference profiles\(^{13}\) and excludes “fewer” profiles than the restriction in Andersson and Svensson (2014).\(^{14}\) In fact, one can argue that the restriction of profiles to \( \mathcal{R} \subset \mathcal{R} \) is an assumption of the same character as assuming strict preferences in the absence of monetary transfers, as, e.g., Gale and Shapley (1962), Shapley and Scarf (1974), Roth (1982) and Ma (1994) among others, as also this assumption is mild if preferences are chosen randomly since the probability of an indifference then is zero.

The next example demonstrates how to design reservation prices such that no two houses are connected by indifference given that preferences are represented by quasi-linear utility functions where the valuations are rational numbers.

**Example 3.** Let \( \mathbb{Q} \) denote the set of all rational numbers. For any agent \( a \in A \), let \( \mathcal{R}_a^q \) consist of all quasi-linear utility functions where \( v_{ah} \in \mathbb{Q} \) for all \( h \in H \cup \{ 0 \} \). Now choose reservation prices \( p \) such that \( p_h \in \mathbb{R}_+ \setminus \mathbb{Q} \) for all \( h \in H \), and \( p_h - p_{h'} \in \mathbb{R} \setminus \mathbb{Q} \) for any distinct houses \( h, h' \in H \). Then it is easy to verify that \( \mathcal{R}_1^q \times \cdots \times \mathcal{R}_n^q \subset \mathcal{R} \). This follows because if, for some \( p \in \Omega \) and two sequences of distinct agents \((a_1, \ldots, a_q)\) and distinct houses \((h_1, \ldots, h_{q+1})\), we have:

\[
\begin{align*}
(i) \quad & h_1 = a_1 \quad \text{and} \quad h_{q+1} = a_q, \\
(ii) \quad & v_{a_1 h_1} - p_{h_1} = v_{a_1 h_2} - p_{h_2} \quad \text{and} \quad v_{a_q h_q} - p_{h_q} = v_{a_q h_{q+1}} - p_{h_{q+1}},
\end{align*}
\]

\(^{13}\)Informally, this can be illustrated in the following way. Let \( R \in \mathcal{R} \) be any profile and \( h' \) and \( h'' \) be any two houses in \( H \). Further, let \((a_j)_{j=1}^t \) and \((h_j)_{j=1}^{t+1} \) be any two sequences of distinct agents and houses, respectively, such that \( h' = h_1 = a_1 \) and \( h'' = h_{t+1} = a_t \). Then, because of monotonicity and continuity of preferences, there is a unique sequence \((p_{h_j})_{j=2}^t \) of prices such that \( a_1 I_{a_1}(h_2, p_{h_2}) \) and \((h_j, p_{h_j})I_{a}(h_{j+1}, p_{h_{j+1}}) \) for all \( 2 \leq j < t \). But for houses \( h_1 \) and \( h_{t+1} \) to be connected by indifference, it is required that \((h_1, p_{h_1})I_{a_1}a_t \). This will occur with probability zero in most cases, e.g., if preferences are quasi-linear and represented by utility functions \( v_{ah} \in \mathbb{R} \) for \( a \in A \) and \( h \in H \cup \{ 0 \} \), and the various utilities are randomly drawn from a bounded interval on \( \mathbb{R} \). In this meaning a limitation to profiles in \( \mathcal{R} \) excludes very few profiles in \( \mathcal{R} \).

\(^{14}\)In Andersson and Svensson (2014), prices are bounded from below by \( p \) and from above by \( \overline{p} \). Then for any price \( p \leq p \leq \overline{p} \), houses \( h \) and \( h' \) are connected by indifference if there exists a sequence of agents \((a_1, \ldots, a_q)\) and a sequence of distinct houses \((h_1, \ldots, h_{q+1})\) for \( q \geq 2 \) such that \((i) \ h = h_1 \) and \( p_{h_1} = \overline{p}_{h_1} \), and \( h' = h_{q+1} \) and \( p_{h_{q+1}} \in \{p_{h_1}, \overline{p}_{h_{q+1}}\} \), \((ii) \ (h_j, p)I_{a}(h_{j+1}, p) \) for all \( 1 \leq j < q \). Note that there agents \( a_1 \) and \( a_{q+1} \) can be arbitrarily chosen whereas in Definition 4 we must have \( a_1 = h_1 \) and \( a_q = h_{q+1} \) and there is only one possible choice of \( a_1 \) and \( a_q \).
(iii) \( v_{a_j h_j} - p_{h_j} = v_{a_j h_{j+1}} - p_{h_{j+1}} \) for \( 2 \leq j \leq q - 1 \),

then summing all left-hand sides and all right-hand sides yields:

\[
\sum_{j=1}^{q} (v_{a_j h_j} - v_{a_j h_{j+1}}) = p_{h_1} - p_{h_{q+1}},
\]

which is a contradiction because the left-hand side belongs to \( Q \) and the right-hand side belongs to \( \mathbb{R} \setminus Q \). \( \square \)

Note that all our results apply to any subdomain \( R^* \subseteq \tilde{R} \), i.e., for applying our results one may construct/design such subdomains and define the mechanisms on this restricted subdomain.

The next observation is that under our domain restriction, one can merge the two outside options and at equilibrium states maximize the number of agents who do not keep their house instead of maximizing \( |\nu| \).\(^\text{16}\) For any assignment \( \mu \), let \( |\mu| = |\{a \in A : \mu_a \neq a\}| \) denote the number of agents who do not keep their house. For convenience, \( |\mu| \) is called the cardinality of \( \mu \). The following lemma shows that, for profiles in \( \tilde{R} \), it suffices to maximize the cardinality \( |\mu| \) instead of the cardinality \( |\nu| \).

**Lemma 1.** Let \( R \in \tilde{R} \) and \( x = (\mu, \nu, p) \) be a state such that for all \( a \in A \): (i) \( x_a R_a a \) and (ii) \( x_a R_a (h, p) \) for all \( h \in H \). If the cardinality \( |\mu| \) is maximal among all states satisfying (i) and (ii) with price vector \( p \), then \( x \) is an equilibrium state, i.e., \( |\nu| \) is maximal among all states satisfying (i) and (ii) with price vector \( p \).

Lemma 1 has the consequence that any agent could have a set of outside options which can be merged into one outside option if no two houses are connected by indifference (see the Appendix).

**Remark 3.** Lemma 1 is not true on the full domain \( R \). To see this, consider Example 1 and let \( p' = (0, 1, 0) \) and \( x'' = (\mu'', \nu'', p') \) with \( \mu'' = (2, 1, 3) \) and \( \nu'' = (1, 1, 0) \). In this case, \( |\mu''| = 2 \) and \( |\nu''| = 2 \). However, for \( x' = (\mu', \nu', p') \) with \( \mu' = (3, 2, 1) \) and \( \nu' = (1, 1, 1) \) we have \( |\mu'| = 2 \) and \( |\nu'| = 3 \), i.e., according to Definition 1, state \( x'' \) is not an equilibrium state even though the cardinality of \( \mu'' \) is maximized (because for price vector \( p' \), this cardinality cannot exceed two). \( \square \)

The first main result of the paper demonstrates that the set of equilibrium prices \( \Pi_R \) has a unique minimum equilibrium price vector at any profile \( R \in \tilde{R} \). The proof proceeds by showing that from any two equilibrium states, a new equilibrium state can be constructed by taking the minimum of the two equilibrium price vectors. This result also explains why there are two minimum equilibrium price vectors in Example 2 (and why the minimum of

\(^{15}\)In some sense, this assumption is related to the “genericity assumption” of Acemoglu et al. (2008) whereby the powers of two coalitions are never identical: here the sum of evaluations is never identical to the difference of any two reservation prices.

\(^{16}\)We are grateful to one of the referees for pointing this out.
the two minimum equilibrium price vectors is not an equilibrium price vector), because, as explained above, houses \( h = 1 \) and \( h' = 3 \) are connected by indifference at prices \( p' = (1,0,0,0) \).

**Theorem 1.** There is a unique minimum equilibrium price vector \( p^* \in \Pi_R \) for each profile \( R \in \tilde{R} \).

From Theorem 1, it is clear that the non-uniqueness problem, previously illustrated in Example 2, disappears on the restricted domain \( \tilde{R} \). What may not be so obvious, in the light of Example 1, is that a minimum price mechanism always is a maximum trade mechanism on the restricted domain \( \tilde{R} \).

**Proposition 3.** Let \( f \) be a minimum price mechanism. Then \( f \) is a maximum trade mechanism on the domain \( \tilde{R} \).

The second main result of the paper demonstrates that a minimum price mechanism, defined on the domain \( \tilde{R} \subset R \), is group non-manipulable for all profiles in \( \tilde{R} \). Note also that this mechanism always selects an individually rational equilibrium outcome which maximizes trade at each profile in \( \tilde{R} \) by the definition of the mechanism and Proposition 3.

**Theorem 2.** Let \( f \) be a minimum price mechanism. Then \( f \) is group non-manipulable on the domain \( \tilde{R} \).

Note that in Theorem 2, the minimum price mechanism is defined on the full domain of preference profiles and for any profile in \( \tilde{R} \), any possible deviation of a coalition is allowed, i.e., the resulting profile after deviation does not need to belong to the restricted domain \( \tilde{R} \).

## 4 Concluding Discussion

We considered a housing market with monetary transfers where tenants have the outside option to “Keep the House” reflecting two choices for existing tenants: continue renting the house or to buy it at the reservation price. It is natural to think about how this type of outside option influences our theoretical findings. If this outside option would not be available to the tenants at all, then our model reduces to the classical assignment game and the non-manipulability result in Theorem 2 can be found in Demange and Gale (1985, Theorem 2). Then Theorem 2 is valid on the full preference domain as a minimal equilibrium price vector is unique on the full preference domain in this special case of the model (the first sketch of this proof for general preferences can be found in Crawford and Knoer, 1981). A second modification of the outside option is that each tenant continues renting or can buy his house at the market price (and the owner’s reservation price is not available). Then all agents face the same buying prices independently of which house
is bought. More formally, each agent $a \in A$ consumes exactly one bundle, $x_a$, in his consumption set $X_a = (H \times \mathbb{R}_+) \cup \{a\}$ where:

$$x_a = \begin{cases} 
    a & \text{if agent } a \text{ continues renting house } h = a, \\
    (h, p_h) & \text{if agent } a \text{ buys house } h \text{ at price } p_h.
\end{cases} \tag{1}$$

With this modification, all results presented here continue to hold, as shown in Andersson et al. (2013) which is an early version of this paper. It is the assumption that an agent always can continue renting the house that he currently is occupying which makes the results related to uniqueness, maximal trade and preference domains to deviate from previous findings in the literature.

There are a number of interesting questions for future research, and we believe that two of them stand out more than the others. First, what is the maximal domain under which a minimum price mechanism is group non-manipulable? Second, is the minimum price mechanism the only individually rational, equilibrium selecting, trade maximizing, and group non-manipulable mechanism on the domain $\mathcal{R}$? Currently, we do not have answers to these two questions, but it would be interesting to know them, especially since the results of this paper have some potentially relevant policy implications.

We conclude with a few remarks related to the U.K. Housing Act 1980. We first remark that in the special case of the formal model where $A$ consists of one agent, the situation in the U.K. Housing Act 1980 and its European equivalents are fully reflected in our theoretical framework. That is, the single tenant is given a take-it-or-leave-it offer either to buy the house at a fixed and discounted price, or to continue renting the house. If the agent would report his preferences to the local or central administration, the minimum price mechanism would recommend the real-world outcome. Hence, the model considered in this paper can be seen as a representation of the public housing market in U.K. today where each tenant is regarded as a separate housing market. The compromise proposed here is to merge some of these separate housing markets into a new and local housing market, e.g., all houses in a specific neighborhood, and to allocate the houses based on the market prices for the local market (note that the local market prices are not the same as the market prices for which the discounted prices are calculated as the latter prices are based on a market containing all potential buyers in the U.K. whereas the former is based on potential buyers on the local market). As a tenant always has the option to buy the particular house that he is living in at the fixed reservation price or to continue renting it at the regulated rent, in the formal framework, it is clear that all agents weakly prefer the outcome of the investigated mechanism to the current U.K. system. In fact, from a revenue point of view, also the public authority weakly prefers the outcome of the minimum price mechanism to the prevailing U.K. system as the revenue from sales always is weakly higher in the suggested mechanism.\(^{18}\)

\(^{17}\)In Andersson et al. (2013), all results presented in this paper except Example 1 were derived for consumption bundles of the type described in condition (1). One can carefully design an example in the same fashion as Example 1 (and its conclusion remains true). Such an example is available from the authors upon request. The 2013 version is available on the homepage of the first author.

\(^{18}\)Informally, this follows from the fact that a tenant who buys a house in the current U.K. system...
Appendix: Proofs

The Appendix contains the proofs of all results. It also contains some additional lemmas, definitions, and concepts. All results are proved in the same order as they are presented in the main text of the paper.

The following lemma which is due to Alkan et al. (1991, Perturbation Lemma) will be useful in some of the subsequent proofs.

**Perturbation Lemma.** Let $R \in \mathcal{R}$ and $x = (\mu, \nu, p)$ be an equilibrium state. If $p > p$, then for each $0 < \epsilon \leq \min_{h \in H}(p_h - p_a)$ there exists another equilibrium state $\tilde{x} = (\tilde{\mu}, \tilde{\nu}, \tilde{p})$ such that $\tilde{p} \leq p$, $\tilde{p} \neq p$, and $p_h - \tilde{p}_h < \epsilon$ for all $h \in H$.

**Proposition 1.** Let $R \in \mathcal{R}$ be a profile and $p$ be a vector of reservation prices.

(i) The set of equilibria $\mathcal{E}_R$ is nonempty.

(ii) Any minimum price equilibrium is weakly efficient.

(iii) If there is a unique minimum equilibrium price vector in $\Pi_R$, then all agents weakly prefer any minimum price equilibrium in $\mathcal{E}_R$ to any other equilibrium state in $\mathcal{E}_R$.

**Proof.** (i) Since each house is boundedly desirable for each agent $a \in A$ at each profile $R \in \mathcal{R}$, by assumption, there is a price vector $\overline{p} > p$ such that $aP_a(h, \overline{p})$ for all $a \in A$ and all $h \in H$. Then, $x = (\mu, \nu, \overline{p})$ constitutes an equilibrium state if $\mu_a = a$ for all $a \in A$, $\nu_a = 1$ if $(a, p_a)R_a a$, and $\nu_a = 0$ if $aP_a(a, p_a)$. Hence, $x \in \mathcal{E}_R$, and, consequently, $\mathcal{E}_R \neq \emptyset$ for each profile $R \in \mathcal{R}$.

(ii) Let $R \in \mathcal{R}$ and $x = (\mu, \nu, p)$ be an equilibrium state such that $p$ is a minimum price vector. If $x$ is not weakly efficient, then there exists another state $\hat{x} = (\hat{\mu}, \hat{\nu}, \hat{p})$ with a feasible price vector $\hat{p} \geq p$ such that $\hat{x}_aP_a\hat{x}_a$ for all $a \in A$. Because $x$ is an equilibrium state, it must hold that $x_aR_a a$ for all $a \in A$. Thus, $\hat{x}_aP_a x_a$ implies that $\hat{x}_aP_a a$ and $\hat{\mu}_a \neq a$. Again, because $x$ is an equilibrium state and $\hat{\mu}_a \neq a$, it follows that $x_aR_a(\hat{\mu}_a, p)$. Thus, $\hat{x}_aP_a x_a$ implies that $\hat{x}_aP_a(\mu_a, p)$ and $\hat{p}_a < p_a$. Because $\hat{\mu}$ is a bijection and $\hat{x}$ is a state, it follows that $p_h - \hat{p}_h < p_h$ for all $h \in H$. Let now $\epsilon = \min_{h \in H}(p_h - \hat{p}_h)$. Note next that $\epsilon > 0$ and that any agent $a$’s preference satisfies weak monotonicity: for all $p', p'' \in \mathbb{R}_+^n$ and all $h \in H$, $(h, p')R_a(h, p'')$ if $p'_h < p''_h$ (where $x'_aI_a x''_a$ for any two states $x'$ and $x''$ for which $x'_a = a = x''_a$). Since Alkan et al. (1991) only require the weak monotonicity property, by the Perturbation Lemma, there exists another equilibrium state $\hat{x} = (\hat{\mu}, \hat{\nu}, \hat{p})$ such that $\hat{p} \leq p$, $\hat{p} \neq p$, and $p_h - \hat{p}_h < \epsilon$ for all $h \in H$. Thus, by our choice of $\epsilon$, we have $\hat{p} \geq p$ and $\hat{p}$ is a feasible equilibrium price vector. Since $p \geq \hat{p}$ and $p \neq \hat{p}$, this is a contradiction to the assumption that $p$ is a minimum equilibrium price vector.

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also buys a house in our framework. Consequently, if a house is sold in the current U.K. system, then it is sold in our framework, but a house which is not sold in the current U.K. system may be sold in our framework (since an agent is allowed to buy a house different from the one that he currently lives in). The conclusion then follows since the prices chosen by the minimum price mechanism are weakly higher than the reservation prices.
Example 2, it is clear that either $\hat{p}$ is replaced by vector. Suppose that $\hat{\mu}$ is a minimum equilibrium price vector by assumption. Analogous arguments denote the profile of quasi-linear preferences where the entry $v_{ah}$ $\hat{\mu}$ is a minimum equilibrium price vector at profile $x$. Consequently, $\hat{\mu}_1 = 3, \hat{p}_3 = 0$ by individual rationality for agent 1, and it then follows that $\hat{\mu}_2 = 2$ by individual rationality for agent 2. But then it must be the case that $\hat{\mu}_4 = 4$ and $\hat{p}_2 \geq 1$ because otherwise agent 4 will envy agent 2 at state $\hat{x}$. Hence, $\hat{p} \geq p''$. But then $\hat{p} = p''$, by definition, as $\hat{p}$ is a minimum equilibrium price vector by assumption. Analogous arguments can be used to show that $\hat{p} = p'$ if $\hat{p}_2 < 1$.

Let now $f$ be minimum price mechanism on domain $\mathcal{R}$. Then $f$ chooses either $p'$ or $p''$. If $f$ chooses $p'$, then $\mu' = (1, 3, 4, 2)$ and agent 3’s utility is equal to $v_{34} - p'_4 = 1$. Let $R'$ denote the profile of quasi-linear preferences where the entry $v_{32}$ in the matrix $V$ from Example 2 is replaced by $v_{32}' = 2$. Obviously, $x' \notin \mathcal{E}_{R'}$ because $(2, p'_2)P_3x_3'$. On the other hand, it is easy to check that $x'' \in \mathcal{E}_{R'}$. Also, $p''$ is the unique minimum equilibrium price vector at profile $R'$. To see this, suppose that $\hat{x} = (\hat{\mu}, \hat{v}, \hat{p}) \in \mathcal{E}_{R'}$ and that $\hat{p} \neq p''$ is a minimum equilibrium price vector at profile $R'$. Then $\hat{p}_2 < 1$, which implies that $\hat{\mu}_4 = 2$ and $\hat{p}_3 = 1$. But then individual rationality cannot be satisfied for both agents 1 and 2 at state $x''$. Thus, $p''$ must be chosen by $f$ for profile $R'$. Then, by individual rationality for agents 1 and 2, it follows that agent 3 must receive house 1. Because $R' = (R'_3, R_{-3})$ and agent 3’s utility from $(1, p'')$ under $R_3$ is equal to $v_{31} - p''_1 = 2 > 1$, agent 3 can profitably manipulate $f$ at $R$.

**Proposition 2.** Let $|A| > 3$.

(a) Any minimum price mechanism $f$ is manipulable on the domain $\mathcal{R}$.

(b) Any maximum trade mechanism $f$ is manipulable on the domain $\mathcal{R}$.

**Proof.** In showing (a), we identify a profile $R \in \mathcal{R}$ where some agent can manipulate the outcome of an arbitrary minimum price mechanism $f$ when $|A| > 3$. It is sufficient to demonstrate the result for $|A| = 4$. As in Example 2, let preferences be represented by quasi-linear utility functions where:

$$
V = (v_{ah})_{a \in A, h \in \{0\} \cup H} = \begin{pmatrix}
-1 & 0 & -2 & 0 & -2 \\
-1 & -2 & 0 & 0 & -2 \\
-1 & 2 & -2 & 0 & 1 \\
-1 & -2 & 2 & -2 & 1
\end{pmatrix}.
$$

Let the profile $R \in \mathcal{R}$ denote the preferences that are represented by the above values.

Consider now the state $\hat{x} = (\hat{\mu}, \hat{v}, \hat{p}) \in \mathcal{E}_{R}$ and suppose that $\hat{p}$ is a minimum equilibrium price vector. We first demonstrate that $\hat{p} = p' = (1, 0, 0, 0)$ or $\hat{p} = p'' = (0, 1, 0, 0)$. From Example 2, it is clear that either $\hat{p}_1 < 1$ or $\hat{p}_2 < 1$ as $\hat{p}$ is a minimum equilibrium price vector. Suppose that $\hat{p}_1 < 1$. Then $\hat{\mu}_3 = 1$ as $\hat{x}$ is an equilibrium state. Consequently, $\hat{\mu}_1 = 3, \hat{p}_3 = 0$ by individual rationality for agent 1, and it then follows that $\hat{\mu}_2 = 2$ by individual rationality for agent 2. But then it must be the case that $\hat{\mu}_4 = 4$ and $\hat{p}_2 \geq 1$ because otherwise agent 4 will envy agent 2 at state $\hat{x}$. Hence, $\hat{p} \geq p''$. But then $\hat{p} = p''$, by definition, as $\hat{p}$ is a minimum equilibrium price vector by assumption. Analogous arguments can be used to show that $\hat{p} = p'$ if $\hat{p}_2 < 1$. 

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If $f$ chooses $p''$, it can be shown, by identical arguments as in the above, that agent 4 can manipulate the mechanism by replacing the entry $v_{41}$ in the matrix $V$ from Example 2 by $v_{41}' = 2$.

In showing (b), we identify a profile $R \in R$ where some agent can manipulate the outcome of an arbitrary maximum trade mechanism $f$ when $|A| > 3$. It is sufficient to demonstrate the result for $|A| = 4$. We slightly modify Example 2 and let preferences be represented by quasi-linear utility functions where:

$$V = (v_{ah})_{a \in A, h \in \{0\} \cup H} = \begin{pmatrix} -1 & 0 & -2 & 0 & -2 \\ -1 & -2 & 0 & 0 & -2 \\ 0 & 2 & -2 & -1 & 1 \\ -1 & -2 & 2 & -2 & 1 \end{pmatrix}.$$ 

Let the profile $R \in R$ denote the preferences that are represented by the above values.

Note that $x = (\mu, \nu, p)$ is an equilibrium state for $\mu = (1, 3, 4, 2)$, $\nu = (1, 1, 1, 1)$, and $p = (1, 0, 0, 0)$. Note, in particular, that $|\nu'| = 4$. This also means that any selection $x' = (\mu', \nu', p')$ of a maximum trade mechanism $f$ at profile $R$ must have the property that $|\nu'| = 4$. But then we must have $\mu'_3 \neq 3$ and $\mu'_3 \in \{1, 4\}$. We consider two cases.

**Case 1:** $\mu'_3 = 4$.

Then $p'_3 \leq 1$ because $v_{30} = 0$ and $v_{34} = 1$. Hence, by $0 \leq p'_4 \leq 1$, the maximal utility of agent 3 is equal to 1. Consider next the profile $\bar{R} \in R$ where all agents except agent 3 have the same values as in profile $R$, and:

$$(\bar{v}_{ah})_{h \in \{0\} \cup H} = (0, 0, -2, -2, -2),$$

i.e., agent 3 misrepresents his values. Let $\bar{x} = (\bar{\mu}, \bar{\nu}, \bar{p})$ be a maximum trade equilibrium for $\bar{R}$. First, note that if $\bar{v}_3 = 1$, then $\bar{\mu}_3 = 1$ and $\bar{p}_1 = 0$ (by individual rationality of $\bar{x}$) which implies $\bar{x}_3 P_3 x'_3$. It suffices to show that $\bar{x}_3 = (1, \bar{p})$. Indeed, let $\bar{\mu} = (3, 2, 1, 4)$, $\bar{\nu} = (1, 1, 1, 1)$ and $\bar{p} = (0, 2, 0, 2)$. It is straightforward to check that $\bar{x}$ is an equilibrium state. Thus, $|\bar{\nu}| = 4$ and in any maximum trade equilibrium agent 3 buys house 1.

**Case 2:** $\mu'_3 = 1$.

Then $\{\mu'_1, \mu'_2\} = \{2, 3\}$ and $\mu'_4 = 4$ (and $x'_4 = 4$) which implies that agent 4 gets utility 1. Consider next the profile $\bar{R} \in R$ where all agents except agent 4 have the same values as in profile $R$, and:

$$(\bar{v}_{ah})_{h \in \{0\} \cup H} = (0, -2, 0, -2, -2),$$

i.e., agent 4 misrepresents his values. Let $\bar{x} = (\bar{\mu}, \bar{\nu}, \bar{p})$ be a maximum trade equilibrium for $\bar{R}$. First, note that if $\bar{v}_4 = 1$, then $\bar{\mu}_4 = 2$ and $\bar{p}_2 = 0$ (by individual rationality of $\bar{x}$) which implies $\bar{x}_4 P_4 x'_4$. It suffices to show that $\bar{x}_4 = (2, \bar{p})$. Second, let $\bar{\mu} = (1, 3, 4, 2)$, $\bar{\nu} = (1, 1, 1, 1)$ and $\bar{p} = (2, 0, 0, 0)$. It is straightforward to check that $\bar{x}$ is an equilibrium state. Thus, $|\bar{\nu}| = 4$ and in any maximum trade equilibrium agent 4 buys house 2.

To prove the subsequent results, some consequences of the domain restriction (stated as lemmas) are derived for which some additional concepts are needed.

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Let \( q > 1 \) and \( a_j \in A \) for \( 1 \leq j \leq q \). Given two assignments \( \mu \) and \( \mu' \), a trading cycle from \( \mu \) to \( \mu' \) is a sequence \( G = (a_1, \ldots, a_q) \) of distinct agents such that \( \mu'_{a_j} = \mu_{a_{j+1}} \) for \( 1 \leq j < q \) and \( \mu'_{a_q} = \mu_{a_1} \). For simplicity, we will use the same notation for the sequence \( G \) and the corresponding set \( G = \{a_1, \ldots, a_q\} \). The following lemma shows that the complete trade from \( \mu \) to \( \mu' \) can be decomposed uniquely into a number of trading cycles.

**Lemma 2.** Let \( a \in A \) be such that \( \mu_a \neq \mu'_a \). Then there exists a unique trading cycle \( G \) from \( \mu \) to \( \mu' \) such that \( a \in G \).

**Proof.** Let \( a_1 \in A \) be such that \( \mu_{a_1} \neq \mu'_{a_1} \). Because \( \mu : A \to H \) and \( \mu' : A \to H \) are bijections, there exists \( a_2 \in A \setminus \{a_1\} \) such that \( \mu_{a_2} = \mu'_{a_1} \). By the same argument, \( \mu'_{a_2} \neq \mu_{a_2} \). If \( \mu'_{a_2} = \mu_{a_1} \), then \( q = 2 \) and \( G = (a_1, a_2) \) is a trading cycle from \( \mu \) to \( \mu' \). Otherwise \( \mu'_{a_2} \neq \mu_{a_1} \) and \( \mu'_{a_2} \neq \mu_{a_2} \), and there exists \( a_3 \in A \setminus \{a_1, a_2\} \) such that \( \mu_{a_3} = \mu'_{a_2} \). Again \( \mu'_{a_3} \neq \mu_{a_3} \), \( \mu'_{a_2} \) (because \( \mu' \) is a bijection and \( \mu'_{a_1} = \mu_{a_2} \)). If \( \mu'_{a_3} = \mu_{a_1} \), then \( q = 3 \) and \( G = (a_1, a_2, a_3) \) is a trading cycle from \( \mu \) to \( \mu' \). Otherwise, by induction, finiteness of \( A \) and \( H \) and \( |A| = |H| \), there exists \( 3 < q \leq n \) such that \( \mu'_{a_1} = \mu_{a_2}, \ldots, \mu'_{a_{q-1}} = \mu_{a_q} \) and \( \mu'_{a_q} = \mu_{a_1} \), which means that \( G = (a_1, \ldots, a_q) \) is a trading cycle from \( \mu \) to \( \mu' \). \( \square \)

Recall that for any assignment \( \mu \), \( |\mu| = |\{a \in A : \mu_a \neq a\}| \). The next lemma shows that for profiles in \( \mathcal{R} \), it suffices maximizing \( |\mu| \) instead of maximizing \( |\nu| \). In the proof, it will be useful to note that for any two equilibrium states \( x = (\mu, \nu, p) \) and \( x' = (\mu', \nu', p') \) with \( p = p' \), it holds that:

\[
x_a I_a x'_a \quad \text{for all } a \in A.
\]

**Lemma 1.** Let \( R \in \mathcal{R} \) and \( x = (\mu, \nu, p) \) be a state such that for all \( a \in A \): (i) \( x_a R_a a \) and (ii) \( x_a R_a (h, p) \) for all \( h \in H \). If the cardinality \( |\mu| \) is maximal among all states satisfying (i) and (ii) with price vector \( p \), then \( x \) is an equilibrium state, i.e., \( |\nu| \) is maximal among all states satisfying (i) and (ii) with price vector \( p \).

**Proof.** To obtain a contradiction, suppose that \( |\nu| \) is not maximal among all states satisfying (i) and (ii) with price vector \( p \). Then there exists an equilibrium state \( x' = (\mu', \nu', p') \) with \( p' = p \) and \( |\nu'| > |\nu| \). Then for some \( a \in A \), \( \nu'_a = 1 \) and \( \nu_a = 0 \). Thus, \( \mu_a = a \) and \( x_a = a \). If \( \mu'_a = a \), then \( (a, p)I_a a \) which contradicts our assumption \( \nu_a = 0 \) only if \( aP_a(a, p_a) \). Thus, \( \mu'_a \neq a \) and \( a \) belongs to some trading cycle \( G = (a_1, \ldots, a_q) \) from \( \mu \) to \( \mu' \) where \( a_1 = a, \mu'_{a_j} = \mu_{a_{j+1}} \) for all \( 1 \leq j < q \) and \( \mu'_{a_q} = \mu_a = a \). Because (i) and (ii) hold for \( x \), (2) holds for \( x \) and \( x' \), in particular \( aI_a (\mu'_a, p) \). If for all \( 1 \leq j \leq q, \mu'_a \neq a_j \), then \( (\mu, \nu, p) \) could not have maximized \( |\mu| \) (simply define \( x'' = (\mu'', \nu'', p'') \) by \( p'' = p, x''_{a''} = x_{a''} \) for all \( a'' \in A \setminus G \), and \( x_{a''} = x'_{a''} \) for all \( a'' \in G \); then \( |\mu''| > |\mu| \)), which is a contradiction. Thus, choose \( 1 < l \leq q \) minimal such that \( \mu'_{a_l} = a_l \). Note that \( \mu_{a_l} \neq \mu_{a_l} \). Because (2) holds for \( x \) and \( x' \), we have \( a_1 I_{a_l} (\mu'_{a_l}, p), (\mu_{a_l}, p)I_{a_l} (\mu_{a_{l+1}}, p) \) for all \( 1 < j < l \), and \( (\mu_{a_l}, p)I_{a_l} a_l \). Thus, houses \( a_1 \) and \( a_l \) are connected by indifference at profile \( R' \) which contradicts that the profile \( R' \) belongs to \( \mathcal{R} \). \( \square \)

Condition (2) together with Lemma 1 have the important consequence that, without loss of generality, for profiles \( R \in \mathcal{R} \) below we focus on equilibrium states \( x = (\mu, \nu, p) \) where
the cardinality $|\mu|$ is maximized for price vector $p$. This allows to ignore the assignment $\nu$ and redefine (equilibrium) states. Throughout below we will use the following simplified notions. A state is a tuple $x = (\mu, p)$ where $\mu : A \to H$ is an assignment and $p \in \Omega$ is a feasible price vector. Then $x_a = a$ if $\mu_a = a$ and $x_a = (\mu_a, p)$ if $\mu_a \neq a$.

**Definition 1’.** For a given profile $R \in \tilde{R}$, a price vector $p \in \Omega$ is an **equilibrium price vector** if there is a state $x = (\mu, p)$ such that the following holds for all agents $a \in A$: (i) $x_aR_a a$, and (ii) $x_aR_a (h, p)$ for all $h \in H$. If, in addition, the cardinality of $\mu$ is maximal among all states with price vector $p$ satisfying (i) and (ii), the state $x$ is an **equilibrium state**.

Let $R, R' \in \tilde{R}$ be two profiles such that $R' = (R'_C, R_{-C})$ for some $C \subseteq A$ (where possibly $C = \emptyset$ and $R = R'$), and consider two equilibrium states $x = (\mu, p) \in E_R$ and $x' = (\mu', p') \in E_{R'}$. Let $H_0, H_1, H_2, H_3$ be defined as:

$$
H_0 = \{ a \in A : \mu'_a = \mu_a = a \}, \\
H_1 = \{ h \in H : p'_h < p_h \} \setminus H_0, \\
H_2 = \{ h \in H : p'_h = p_h \} \setminus H_0, \\
H_3 = \{ h \in H : p'_h > p_h \} \setminus H_0.
$$

**Lemma 3.** Let $R$ and $R' = (R'_C, R_{-C})$ be two profiles in $\tilde{R}$, and $x = (\mu, p) \in E_R$ and $x' = (\mu', p') \in E_{R'}$ be two equilibrium states. Let $G = (a_1, \ldots, a_q)$ be a trading cycle from $\mu$ to $\mu'$, and $(\mu_a, \ldots, \mu_a)$ the corresponding set of houses.

(i) If $a_k \in G$ and $x'_{a_k} P_a x_{a_k}$, then $\mu'_{a_k} \in H_1$. \\
(ii) If $\mu_a \in H_1, \mu'_a \in H_2 \cup H_3$ and $a \notin C$, then $x_a = a, \mu'_a \in H_2$, and $x_a I_a x'_a$.

**Proof.** Part (i) is proved by contradiction. Assume that $a_k \in G$ and $x'_{a_k} P_a x_{a_k}$, but that $\mu'_{a_k} \notin H_1$. Note first that $\mu'_{a_k} \neq a_k$. To see this, suppose that $\mu'_{a_k} = a_k$ or, equivalently, that $x'_{a_k} = a_k$. Because $x$ is an equilibrium state, it must be the case that $x_aR_a a_k$. But then $x_aR_{a_k} x'_{a_k}$, which contradicts the assumption that $x'_{a_k} P_a x_{a_k}$. Hence, $\mu'_{a_k} \neq a_k$. Note next that $x_aR_{a_k} (\mu'_{a_k}, p)$ and $(\mu'_{a_k}, p)R_{a_k} (\mu'_{a_k}, p')$ because $x$ is an equilibrium state and $\mu'_{a_k} \notin H_1$, respectively. But then $x_aR_{a_k} (\mu'_{a_k}, p')$. Now, $x'_{a_k} = (\mu'_{a_k}, p')$, because $\mu'_{a_k} \neq a_k$.

Consequently, $x_aR_{a_k} x'_{a_k}$ which, again, contradicts the assumption that $x'_{a_k} P_a x_{a_k}$. Hence, $\mu''_{a_k} \in H_1$.

To prove part (ii), note that because $\mu_a \in H_1$ and $\mu'_a \in H_2 \cup H_3$, by assumption, it follows that $\mu_a \neq \mu'_a$. Since $a \notin C$ we have $R'_a = R_a$. Thus, $x_aR_{a}(\mu'_a, p)R_{a}x'_a$ as $\mu'_a \in H_2 \cup H_3$ and $x'_a = (\mu'_a, p')$. Suppose that $\mu'_a \in H_3$. If $x_a = a$, then $aR_{a}(\mu'_a, p)P_{a}x'_a$, which contradicts that $x'$ is individually rational. If $x_a \neq a$, then $(\mu_a, p')P_{a}(\mu_a, p)R_{a}x'_a$, which contradicts to $x' \in E_{R'}$. Hence, $\mu'_a \notin H_3$, i.e., $\mu'_a \in H_2$. It now follows directly that if $x_a = a$ then $x_a I_a x'_a$, and if $x_a \neq a$ then $(\mu_a, p')P_{a}x'_a$, which is a contradiction to $x' \in E_{R'}$. Hence, $x_a = a, x'_a \in H_2$, and $x_a I_a x'_a$. 

\[\square\]
Lemma 4. Let $R$ and $R' = (R'_C, R_{-C})$ be two profiles in $\tilde{R}$, and $x = (\mu, p) \in \mathcal{E}_R$ and $x' = (\mu', p') \in \mathcal{E}_{R'}$ be two equilibrium states where $x'_a P_a x_a$ for all $a \in C$. Let also $G = (a_1, \ldots, a_q)$ be a trading cycle from $\mu$ to $\mu'$, and $(\mu_{a_1}, \ldots, \mu_{a_q})$ the corresponding set of houses. Then $\mu_{a_k} \in H_1$ for some $a_k \in G$ implies that $\mu_{a_j} \notin H_3$ for all $a_j \in G$.

Proof. For notational simplicity, let $h_j = \mu_{a_j}$ for all $1 \leq j \leq q$. To obtain a contradiction, suppose that $h_k \in H_1$ but $h_l \in H_3$, where, without loss of generality, $k < l$ and $h_j \in H_2$ for all $k < j < l$. A first observation is that $a_j \notin C$ for $k < j < l$. This follows directly from Lemma 3(i) as $\mu'_{a_j} \in H_2 \cup H_3$ for all $k \leq j < l$ by construction. Hence, $R'_{a_j} = R_{a_j}$ for all $k \leq j < l$. We will consider the cases when $k + 1 = l$ and $k + 1 < l$. By Lemma 3(ii), $x_{a_k} = a_k = h_k$ and $h_{k+1} \in H_2$, implying that $k + 1 = l$ is impossible.

A second observation is that $x'_{a_k} I_{a_j} x_{a_j}$ for $k \leq j < l$. For $j = k$ this follows from Lemma 3(ii). For all $k < j < l - 1$, it follows from the facts that $h_j, h_{j+1} \in H_2$, $R'_{a_j} = R_{a_j}$, and that $x$ and $x'$ are equilibrium states. For $j = l - 1$, we have $a_{l-1} \notin C$, $\mu_{a_{l-1}} \in H_2$ and $\mu'_{a_{l-1}} \in H_3$. Note that now we can apply Lemma 3(iii) by interchanging the roles of $x$ and $x'$ and obtain $x'_{a_{l-1}} = a_{l-1}$ and $(\mu'_{a_{l-1}} = h_{l-1}, p') I'_{a_{l-1}} a_{l-1}$. Now, because $k + 1 < l$ and both $x_{a_k} = a_k$ and $x'_{a_{l-1}} = a_{l-1}$, there exist $l'$ and $l''$ where $k \leq l' < l'' < l - 1$ such that for all $l'' < j < l''$ we have $x_j \neq a_j \neq x'_j$ and both $x_{a_{l'}} = a_{l'}$ and $x'_{a_{l''}} = a_{l''}$. But then:

(a) $x'_{a_{l'}} = (h_{l'+1}, p), a_{l'} I_{a_{l'}} (h_{l'+1}, p)$ and $R'_{a_{l'}} = R_{a_{l'}}$,
(b) $(h_j, p) I_{a_j} (h_{j+1}, p)$ and $R'_{a_j} = R_{a_j}$ for all $l' < j < l''$,
(c) $(h_{l''}, p) I'_{a_{l''}} a_{l''}$ (where $R'_{a_{l''}} = R_{a_{l''}}$ if $l'' < l$).

Now by (a)-(c), houses $a_{l'}$ and $a_{l''}$ are connected by indifference at profile $R'$ which contradicts that the profile $R'$ belongs to $\tilde{R}$. \hfill $\Box$

An immediate consequence of Lemma 4 is that the set of trading cycles from $\mu$ to $\mu'$ can be partitioned into two disjoint groups as explained in the following definition.

Definition 5. Let $R$ and $R' = (R'_C, R_{-C})$ be two profiles in $\tilde{R}$, and $x = (\mu, p) \in \mathcal{E}_R$ and $x' = (\mu', p') \in \mathcal{E}_{R'}$ be two equilibrium states where $x'_a P_a x_a$ for all $a \in C$. Let $A^+ \subseteq A$ be such that $a \in A^+$ precisely when there is a trading cycle $G$ from $\mu$ to $\mu'$ such that $a \in G$ and $\mu_{a'} \in H_1$ for some $a' \in G$. Let $A^- = A \setminus A^+$.

The notations $A^+$ and $A^-$ are chosen because all agents in $A^+$ are weakly better off at the equilibrium state $x'$ than at the equilibrium state $x$, while no agent in $A^-$ is strictly better off at the equilibrium state $x'$ than at the equilibrium state $x$. This is demonstrated in the next lemma.

Lemma 5. Let $R$ and $R' = (R'_C, R_{-C})$ be two profiles in $\tilde{R}$, and $x = (\mu, p) \in \mathcal{E}_R$ and $x' = (\mu', p') \in \mathcal{E}_{R'}$ be two equilibrium states where $x'_a P_a x_a$ for all $a \in C$. Let $G \subseteq A$ be a trading cycle from $\mu$ to $\mu'$. If $x'_a P_a x_a$ for some agent $a \in G$, then $G \subseteq A^+$ and $x'_a R_{a} x_{a}$ for all $\hat{a} \in G$. 

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Proof. Note first that Lemma 3(i) implies that \( \mu'_a \in H_1 \) since \( x'_a P_a x_a \). Hence, \( G \subseteq A^+ \) and \( \mu_{\hat{a}} \in H_1 \cup H_2 \) for all \( \hat{a} \in G \) by Lemma 4 and by the fact that \( G \) is a trading cycle from \( \mu \) to \( \mu' \). Note next that either \( x'_a P_a x_a \) or \( x_a R_{\hat{a}} x'_a \) for all \( \hat{a} \in G \). In the latter case, \( \hat{a} \notin C \) and \( R'_a = R_{\hat{a}} \). But then from \( \mu_{\hat{a}} \in H_1 \cup H_2 \) we obtain \( x'_a R_{\hat{a}} (\mu_{\hat{a}}, p') R_{\hat{a}} (\mu_{\hat{a}}, p) \) because \( x' \) is an equilibrium state. If \( x_{\hat{a}} = (\mu_{\hat{a}}, p) \), then \( x'_a R_{\hat{a}} x_{\hat{a}} \). Otherwise, \( x_{\hat{a}} = \hat{a} \) from the fact that \( x' \) is an equilibrium state, \( x'_a R_{\hat{a}} x_{\hat{a}} \). Hence, \( x'_a R_{\hat{a}} x_{\hat{a}} \) for all \( \hat{a} \in G \).

\[ \text{Theorem 1.} \] There is a unique minimum equilibrium price vector \( p^* \in \Pi_R \) for each profile \( R \in \hat{R} \).

Proof. Let \( R \in \hat{R} \), and let \( x' = (\mu', p') \in \mathcal{E}_R \) and \( x'' = (\mu'', p'') \in \mathcal{E}_R \) be two equilibrium states. We will demonstrate that \( x = (\mu, p) \in \mathcal{E}_R \) for some assignment \( \mu \) if \( p_h = \min \{ p'_h, p''_h \} \) for each \( h \in H \). The proof of the theorem then follows directly since the set \( \Pi_R \) is closed and bounded from below.

Consider the equilibrium states \( x' = (\mu', p') \) and \( x'' = (\mu'', p'') \), and the trading cycles from \( \mu' \) to \( \mu'' \). Let the sets \( A^+ \) and \( A^- \) be defined as in Definition 5 with the restriction \( R = R' \), and define \( p_h = \min \{ p'_h, p''_h \} \) for each \( h \in H \). Let:

\[
\mu_a = \begin{cases} 
\mu''_a & \text{if } a \in A^+, \\
\mu'_a & \text{if } a \in A^-.
\end{cases}
\]

Note that the assignment \( \mu \) defined in this way becomes bijective because no agent \( a \in A^+ \) belongs to a trading cycle containing an agent in \( A^- \) by Lemma 4. Let now \( x_a = x''_a \) for all \( a \in A^+ \), and \( x_a = x'_a \) for all \( a \in A^- \). To prove the theorem, we need to demonstrate that \( x = (\mu, p) \) is an equilibrium state. Now, the following is true for any \( a \in A^+ \) (recall that \( R = R' \)):

(a) \( x''_a R_a a \) because of individual rationality,

(b) \( x''_a R_a (h, p'') \) for all \( h \in H \) as \( x'' \) is an equilibrium state,

(c) \( x''_a R_a x'_a \) by Lemma 5 since \( a \in A^+ \),

(d) \( x'_a R_a (h, p') \) for all \( h \in H \) as \( x' \) is an equilibrium state.

From (a)-(d) in the above and the construction that \( x_a = x''_a \) for all \( a \in A^+ \), it now follows that \( x_a R_a a \) and \( x_a R_a (h, p) \) for all \( a \in A^+ \) and all \( h \in H \). Symmetric arguments now give that \( x_a R_a a \) and \( x_a R_a (h, p) \) for all \( a \in A^- \) and all \( h \in H \). Hence, \( x = (\mu, p) \) is an equilibrium state if \( |\mu| \) is maximal. If this condition is not satisfied, it only remains to change the assignment \( \mu \) so that the number of agents who do not keep their house becomes maximal for price vector \( p \). Hence, \( x = (\mu, p) \) is an equilibrium state.

\[ \text{Proposition 3.} \] Let \( f \) be a minimum price mechanism. Then \( f \) is a maximum trade mechanism on the domain \( \hat{R} \).
Lemma 6 is the key in the proof of Theorem 2.

Proof. Let $R \in \hat{R}$ and $f$ be a minimum price mechanism. Let the state $x' = (\mu', p') \in \mathcal{E}_R$ be selected by the minimum price mechanism, i.e., $p'$ is the unique minimum equilibrium price vector in $\Pi_R$ by Theorem 1. Let $x = (\mu, p) \in \mathcal{E}_R$ be such that $p' \neq p$ and $p'_h \leq p_h$ for all $h \in H$. We show that for all $a \in A$, if $\mu'_a = a$, then $\mu_a = a$. It then follows that $f$ is a maximum trade mechanism because (i) by Lemma 1 for price vector $p'$ it suffices to maximize $|\mu|$ and (ii) for any price vector $p \in \Pi_R \setminus \{p'\}$ and any equilibrium state $(\mu, p)$, any agent who keeps his house under $(\mu', p')$, also keeps his house under $(\mu, p)$ and each of these agents exercises the same outside option for both $(\mu', p')$ and $(\mu, p)$.

Suppose by contradiction, that there exists $a_l \in A$ such that $\mu'_a = a_l$ and $\mu_a \neq a_l$. Recall next that the complete trade from $\mu$ to $\mu'$ can be decomposed uniquely into a number of trading cycles. This means that there must be a trading cycle $G = (a_1, \ldots, a_q)$ from $\mu$ to $\mu'$ with $a_l \in G$. Note also that $\mu_a \in H_1 \cup H_2$ for all $a_j \in G$ as $p'_h \leq p_h$ for all $h \in H$.

Let now agent $a_l \in G$ be chosen as above, and let $a_j \in G$ for all $1 \leq j \leq q$. We will next demonstrate that $\mu_{a_l} \in H_2$ and $a_l I_{a_l}(\mu_{a_l}, p)$. As $\mu_{a_l} \in H_1 \cup H_2$, it suffices to show that $\mu_{a_l} \notin H_1$ to prove the first part of the statement. To obtain a contradiction, suppose that $\mu_{a_l} \in H_1$. Because $\mu_{a_l} \in H_1$, $\mu_a \neq a_l$, and $(\mu, p) \in \mathcal{E}_R$, it follows that:

$$(\mu_{a_l}, p') P_{a_l}(\mu_{a_l}, p) R_{a_l} a_l.$$ But then state $(\mu', p')$ cannot belong to $\mathcal{E}_R$ since $x'_{a_l} = a_l$. Hence, $\mu_{a_l} \in H_2$. But if $\mu_{a_l} \in H_2$ and $x'_{a_l} = a_l$, it is immediately clear that $x'_{a_l} I_{a_l} x_{a_l}$ as both $x$ and $x'$ belong to $\mathcal{E}_R$. Note that the latter condition may also be written as $a_l I_{a_l}(\mu_{a_l}, p)$ since $x'_{a_l} = a_l$.

Let again agent $a_l \in G$ be defined as in the above. Given the above findings, we next remark that either (i) $\mu_{a_l} \in H_2$ for all $k < j \leq l$ and $\mu_{a_k} \in H_1$ for some $a_k \in G$, or (ii) $\mu_{a_j} \in H_2$ for all $a_j \in G$. We will demonstrate that both these cases lead to the desired contradiction.

Case (i). Suppose first that $x_{a_k} \neq a_k$. Because $(\mu, p) \in \mathcal{E}_R$, $\mu_{a_k} \in H_1$, and $\mu'_{a_k} = \mu_{a_{k+1}} \in H_2$ it follows that:

$$(\mu_{a_{k+1}}, p') P_{a_{k+1}}(\mu_{a_{k+1}}, p) R_{a_{k+1}} x'_{a_{k+1}},$$ which contradicts that $x' = (\mu', p') \in \mathcal{E}_R$. Hence, $x_{a_k} \neq a_k$ cannot be the case. Suppose instead that $x_{a_k} = a_k$, and note that $x_{a_l} I_{a_l} x'_{a_l}$ for all $k \leq j \leq l - 1$. This follows as both $x$ and $x'$ belong to $\mathcal{E}_R$ and $\mu_{a_j} \in H_2$ for all $k < j \leq l$. But then houses $a_k$ and $a_l$ are connected by indifference at prices $p$ as $a_k I_{a_k}(\mu_{a_{k+1}}, p)$, $a_l I_{a_l}(\mu_{a_{k+1}}, p)$ for $k < j < l$, and $a_l I_{a_l}(\mu_{a_l}, p)$. Hence, $x_{a_k} = a_k$ cannot be the case.

Case (ii). If $\mu_{a_j} \in H_2$ for all $a_j \in G$, it follows, by the same arguments as in Case (i), that $x_{a_l} I_{a_l} x'_{a_l}$ for all $a_j \in G$. Now, if $\mu_{a_{l'}} = a_{l'}$ for some $a_{l'} \in G$ and $\mu_{a_l} \neq a_{l'}$ for all $l' < j \leq l$, houses $a_{l'}$ and $a_l$ are connected by indifference at prices $p$ as $a_{l'} I_{a_{l'}}(\mu_{a_{l'+1}}, p)$, $a_{l'} I_{a_{l'}}(\mu_{a_{l'+1}}, p)$ for $l' < j < l$, and $a_l I_{a_l}(\mu_{a_l}, p)$. On the other hand, if $\mu_{a_{l'}} \neq a_{l'}$ for all $a_{l'} \in G$, then the cardinality of $\mu'$ can be increased since $\mu'_{a_l} = a_l$, which contradicts that $x' \in \mathcal{E}_R$ and $|\mu'|$ is maximal for price vector $p'$.

\[\square\]

Lemma 6 is the key in the proof of Theorem 2.
Lemma 6. Let \( R, R' \in \tilde{R} \) be two profiles such that \( R' = (R_C, R_{-C}) \) for some \( C \subset A \). Let also \( x = (\mu, p) \in E_R \) and \( x' = (\mu', p') \in E_{R'} \) be two equilibrium states such that \( x'_a P_a x_a \) for all \( a \in C \). If \( H_1 \neq \emptyset \), then there is a subset \( S \subseteq H_1 \) such that \( A_S = \emptyset \) where:

\[
A_S = \{ a \in A : \mu_a \notin S \text{ and } x_a I_a(h, p) \text{ for some } h \in S \setminus \{a\} \}.
\]

Proof. Without loss of generality, we may assume that \( C = \{ a \in A : x'_a P_a x_a \} \) as \( R'_a = R_a \) is an allowed report for all agents \( a \in C \). To obtain a contradiction, suppose that \( A_S \neq \emptyset \) for each \( S \subseteq H_1 \). Then \( A_{H_1} \neq \emptyset \) and there is an agent \( a_0 \in A \) with \( \mu_{a_0} \notin H_1, x_{a_0} I_{a_0}(h, p) \) for some \( h \in H_1 \), and \( h \neq a_0 \). Now, \( x_{a_0} R_{a_0} a_0 \) by individual rationality and \( (h, p')P_{a_0}(h, p) \) as preferences are strictly monotonic, \( h \neq a_0 \) and \( p'_h < p_h \). Hence, \( (h, p')P_{a_0}(h, p)I_{a_0} x_{a_0} R_{a_0} a_0 \) and, obviously, \( a_0 \in C \) and \( \mu_{a_0}' \in H_1 \).

Note next that agent \( a_0 \) belongs to some trading cycle \( G = (a_t, a_{t-1}, \ldots, a_1, a_0) \) from \( \mu \) to \( \mu' \) (throughout the proof of this lemma we switch indexing agents in a trading cycle from increasing to decreasing). Since \( \mu_{a_0}' \in H_1 \), it follows from Lemma 4 that \( \mu_{a_0} \in H_1 \cup H_2 \) for all \( 0 < j < l \), and \( \mu_{a_0} \in H_2 \) because \( \mu_{a_0} \notin H_1 \). Let \( 0 < j < l \) be minimal such that \( \mu_{a_0} \in H_1 \), and note that such an index exists because \( \mu_{a_0}' = \mu_{a_0} \in H_1 \). But then \( \mu_{a_0}' \in H_2 \) and, consequently, \( a_t \notin C \) by Lemma 3(i). Because \( \mu_{a_t} \in H_1 \) and \( \mu_{a_t}' \in H_2 \), Lemma 3(ii) yields \( x_{a_t} = a_t \) and \( a_t I_{a_t}(\mu_{a_t-1}, p) \). Furthermore, for \( 0 < j < l \), we have \( \mu_{a_j}' \in H_2, a_j \notin C, x_{a_j} I_{a_j} x_{a_j}, \) and \( x_{a_j} \neq a_j \) because \( R \) belongs to \( \tilde{R} \).

It is next established that \( x_{a_j} P_{a_j}(a_t, p) \) for all \( 0 < j < l \). To obtain a contradiction, suppose that \( x_{a_0} I_{a_0}(a_t, p) \). Then it is possible to define an assignment \( \mu'' \) such that \( \mu'' = \mu_a \) for \( a \in A \setminus \{a_0, a_1, \ldots, a_l\} \), \( \mu_{a_j}'' = \mu_{a_j-1} \) for \( 0 < j < l \), and \( \mu_{a_0}'' = a_t \). Let \( x'' = (\mu'', p) \). Note that \( x'' \) is a utility maximizing choice for agent \( a \in \{a_1, \ldots, a_l\} \) since for \( 0 < j < l \), \( x_{a_j} R_{a_j} a_j, x_{a_j} R_{a_j}(h, p) \) for all \( h \in H, x_{a_j} I_{a_j}(\mu_{a_j-1}, p) \) and \( x_{a_j}' \neq a_j \). Because \( x_{a_0} I_{a_0}(a_t, p) \), then the tuple \( (\mu'', p) \) satisfies the requirements of Definition 1. However, by comparing \( x'' = (\mu'', p) \) and \( x = (\mu, p) \), we see that \( |\mu''| > |\mu|, \) i.e., \( \mu_{a_t} = \mu_t \) while \( \mu_{a_j}' \neq \mu_t \) (and \( \mu_{a_j}'' = \mu_{a_j}' \neq a_j \) for \( 0 < j < l \)). This contradicts that \( x \) is an equilibrium state where the cardinality of \( \mu \) is maximized. Similar arguments can be used to derive a contradiction if \( x_{a_j} I_{a_j}(a_t, p) \) for some \( 0 < j < l \). Hence, \( x_{a_j} P_{a_j}(a_t, p) \) for all \( 0 \leq j < l \) by the above conclusion. Consider next the following two cases:

(I) Suppose that \( \mu_{a_0} \notin H_1 \). Then, again, \( a_0 \) belongs to some trading cycle \( \hat{G} = (\hat{a}_r, \hat{a}_{r-1}, \ldots, \hat{a}_1, a_0) \) from \( \mu \) to \( \mu' \). Again \( \hat{a}_0 \in C \) and \( \mu_{a_0} \in H_2 \), and there exists \( 0 < k \leq r \) such that \( \mu_{\hat{a}_k} \in H_1 \) and both \( \mu_{\hat{a}_j} \in H_2 \) and \( x_{\hat{a}_j}' \neq \hat{a}_j \) for all \( 0 < j < k \).

Again \( x_{\hat{a}_k} = \hat{a}_k \) and \( \hat{a}_k I_{\hat{a}_k}(\mu_{\hat{a}_k}', p) \). Because \( G \) and \( \hat{G} \) are trading cycles, we have \( \hat{a}_j \notin \{a_0, a_1, \ldots, a_l\} \) for all \( 0 \leq j \leq k \). Now if (a) \( \hat{a}_k = \mu_{a_0}' \) (this is possible if \( G = \hat{G} \)) or (b) for some \( 0 \leq j < k, x_{\hat{a}_j} I_{\hat{a}_j}(\hat{a}_k, p) \) or (c) for some \( 0 \leq j < l \) and \( 0 < j' < k \), we have \( x_{\hat{a}_j} I_{\hat{a}_j}(\mu_{\hat{a}_j}', p) \), then, in all cases (a), (b) and (c), we can construct similarly as above from \( x \) an equilibrium state \( x'' \) where the cardinality of \( \mu'' \)
is larger compared to the cardinality of \( \mu \), which is a contradiction. Thus, suppose that (a), (b) and (c) are not true, in particular, \( x_a, P_{\hat{\mu}}(\hat{\alpha}, p) \) for all \( 0 \leq j \leq l \), and \( x_{\hat{\alpha}}, P_{\hat{\mu}}(\hat{\alpha}, p) \) for all \( 0 \leq j < k \). By \( \hat{\alpha} \in H_1 \) and the above assumption that \( A_S \neq \emptyset \) for each \( S \subseteq H_1 \), it then follows that \( A_{\{\hat{\alpha}\}} \neq \emptyset \). Let \( a_0' \in A_{\{\hat{\alpha}\}} \). Now we have \( a_0' \notin \{\hat{\alpha}, \hat{\alpha}1, \ldots, \hat{\alpha}k\} \cup \{a_0, a_1, \ldots, a_l\} \).

(II) Suppose that \( \mu_{a_0} \in H_1 \). Now, if \( x_{a_j}, I_{\alpha_j}(\mu_{a_0}, p) \) for some \( 0 \leq j \leq l \), it is possible to construct, in a similar fashion as in the above, an equilibrium state \( x'' \) where the cardinality of \( \mu'' \) is larger compared to the cardinality of \( \mu \), which is a contradiction. Thus, \( x_{a_j}, P_{\alpha_j}(\mu_{a_0}, p) \) for all \( 0 \leq j \leq l \). By \( \mu_{a_0} \in H_1 \) and the above assumption that \( A_S \neq \emptyset \) for each \( S \subseteq H_1 \), it then follows that \( A_{\{\mu_{a_0}\}} \neq \emptyset \). Let \( a_0' \in A_{\{\mu_{a_0}\}} \). Since \( x_{a_j}, P_{\alpha_j}(\mu_{a_0}, p) \) for all \( 0 \leq j \leq l \), we have \( a_0' \notin \{\hat{\alpha}0, a_0, a_1, \ldots, a_l\} \).

We next observe that if \( \mu_{a_0} \notin H_1 \), then as in (I) we can find another sequence \( a_0', a_1', \ldots, a_o' \) where \( x_{a_o'} = a_0' \in H_1 \), \( \mu_{a_o'} \in H_2 \), \( \mu_{a_j} = a_{j-1} \), \( x_{a_j} ' \neq a_j ' \) for \( 0 < j < o \), and \( \mu_{a_o'} = a_{o-1} ' \neq a_o ' \); and otherwise as in (II) \( \mu_{a_0} \in H_1 \). Then either we can use similar arguments as above to construct another equilibrium state \( x'' \) from \( x \) where the cardinality of \( \mu'' \) is larger compared to the cardinality of \( \mu \) or otherwise we continue to construct another (disjoint) sequence as in (I) or (II) (where in (II) the sequence consists of one agent) which eventually leads to a contradiction because in any such sequence there is a new agent \( a \in A \), who does not belong to any of the previously identified sequences, with \( \mu_a \in H_1 \). The contradiction then follows as \( H_1 \) is finite.

\( \square \)

**Theorem 2.** Let \( f \) be a minimum price mechanism. Then \( f \) is group non-manipulable on the domain \( \hat{R} \).

**Proof.** To obtain a contradiction, suppose first that some nonempty group \( C \subseteq A \) can manipulate the minimum price mechanism \( f \) at a profile \( \hat{R} \in \hat{R} \) by reporting preferences \( \hat{R}' = (R'_C, R_{-C}) \in \hat{R} \). More precisely, let \( x = (\mu, p) \in E \) and \( x' = (\mu', p') \in E' \) be two equilibrium states such that \( x_a P_a x_a \) for all \( a \in C \). As in the proof of Lemma 6, we will, without loss of generality, assume that \( C = \{a \in A : x_a P_a x_a \} \). Then \( \mu_a' \in H_1 \) for all \( a \in C \) by Lemma 3(i) as \( x_a P_a x_a \) for all \( a \in C \) and \( C \neq \emptyset \). Hence, \( H_1 \neq \emptyset \) and \( p_h > p_h' \geq p_h \) for all \( h \in H_1 \).

From Lemma 6, it then follows that there exists a nonempty set \( S \subseteq H_1 \) such that \( A_S = \emptyset \). By continuity of preferences, there exists \( 0 < \epsilon < \min_{h \in S}(p_h - p'_h) \) such that for all \( a \in A \) with \( \mu_a \notin S \), \( x_a R_a (h, p_h - \epsilon) \) for all \( h \in S \). By the Perturbation Lemma, it is then possible to weakly decrease \( p_h \) for all \( h \in S \) (since \( p_h > p_h' \geq p_h \) for all \( h \in S \), and if \( h = a \in S \) and \( \mu_a \notin S \), then agent \( a \) is not affected by the decrease because \( x_a R_a a \) and \( p_a > p_a' \)) and obtain a new equilibrium state which contradicts that state \( x \) is selected by \( f \) at profile \( \hat{R} \).**19** Hence, it is impossible for an arbitrary nonempty group \( C \subseteq A \) to

\[ \text{More precisely, there exists } \hat{x} = (\hat{\mu}, \hat{p}) \text{ such that (i) } \hat{x}_a = x_a \text{ and } \hat{p}_{\mu_a} = p_{\mu_a} \text{ for all } a \in A \text{ with } \mu_a \notin S, \]

\[ \text{ (ii) } \hat{p} \leq p \text{ and } \hat{p} \neq p, \text{ and (iii) } p_h - \hat{p}_h \leq \epsilon \text{ for all } h \in S. \]

\[ \text{19} \]
manipulate the minimum price mechanism \( f \) at a profile \( R \in \bar{R} \) by reporting preferences \( R' = (R'_C, R_{-C}) \in \bar{R} \).

Given the above conclusion, it remains to prove that it is impossible for an arbitrary nonempty group \( C \subseteq A \) to manipulate the minimum price mechanism \( f \) at a profile \( R \in \bar{R} \) by reporting preferences \( R' = (R'_C, R_{-C}) \in \bar{R} \setminus \bar{R} \) because this conclusion, combined with the above conclusion, demonstrates that it is impossible for an arbitrary nonempty group \( C \subseteq A \) to manipulate the minimum price mechanism \( f \) at a profile \( R \in \bar{R} \) by reporting preferences \( R' = (R'_C, R_{-C}) \in \bar{R} \). To prove the statement, suppose that the nonempty group \( C \subseteq A \) can manipulate the minimum price mechanism \( f \) at \( R \in \bar{R} \) by reporting preferences \( R' = (R'_C, R_{-C}) \in \bar{R} \). More precisely, let \( f(R) = x = (\mu, p) \in \mathcal{E}_R \) and \( f(R') = x' = (\mu', p') \in \mathcal{E}_{R'} \) be such that \( x_a P_a x_a \) for all \( a \in C \).

Note first that because \( x_a R_a a \) for all \( a \in C \), it follows that \( \mu'_a \neq a \) for all \( a \in C \). Consider now a profile \( R'' = (R''_C, R_{-C}) \in \bar{R} \) and let \( x^* = (\mu^*, p^*) \) be a minimum price equilibrium at profile \( R'' \). By Theorem 1, \( p^* \) is the unique minimal equilibrium price vector at profile \( R'' \). To complete the proof, it suffices to specify a profile \( R'' = (R''_C, R_{-C}) \in \bar{R} \) such that (a) \( x' \in \mathcal{E}_{R''} \) and (b) \( \mu'^*_a = \mu'_a \) for all \( a \in C \). To see this, by Theorem 1, \( p^* \leq p' \) and as both \( x' \in \mathcal{E}_{R''} \) and \( \mu'^*_a = \mu'_a \) for all \( a \in C \), we have \( x'_a P_a x'_a \) for all \( a \in C \). Hence, by \( x'_a P_a x'_a \) for all \( a \in C \), we have \( x'_a P_a x'_a \) for all \( a \in C \). Consequently, the nonempty group \( C \subseteq A \) can manipulate the minimum price mechanism \( f \) at \( R \in \bar{R} \) by reporting preferences \( R'' = (R''_C, R_{-C}) \in \bar{R} \), which is a contradiction to the first part of the proof.

To complete the proof it is, as demonstrated in the above, sufficient to specify a profile \( R'' = (R''_C, R_{-C}) \in \bar{R} \) such that (a) \( x' \in \mathcal{E}_{R''} \) and (b) \( \mu'^*_a = \mu'_a \) for all \( a \in C \). Let now \( R''_a \) for each \( a \in C \) be represented by a quasi-linear utility function where:

\[(i) \quad v_{aa} = p'_a - \epsilon^a \text{ for } 0 < \epsilon^a < 1, \]

\[(ii) \quad v_{aa} = -1, \text{ and;} \]

\[(iii) \quad v_{ah} = p'_h - k^a_h \text{ for all } h \in H \setminus \{\mu'_a, a\} \text{ where } -k^a_h < -(p'_h - p^*_h) - 1. \]

Note that \( x' \in \mathcal{E}_{R''} \) and if \( (\mu^*, p^*) \) is a minimum price equilibrium such that \( p^* \leq p' \), then \( \mu'^*_a = \mu'_a \) for all \( a \in C \): for any feasible price \( p^* \leq p' \) and any \( a \in C \), the utility from consuming \( (\mu'_a, p^*) \) satisfies:

\[p'_a \mu'_a - \epsilon^a - p^*_a = (p'_a \mu'_a - p^*_a) - \epsilon^a > -1, \quad (3)\]

where the inequality follows from \( p^* \leq p' \) and \(-\epsilon^a > -1\); and the utility from consuming \( (h, p^*) \) with \( h \in H \setminus \{\mu'_a, a\} \) satisfies:

\[p'_h - k^a_h - p^*_h < p'_h - (p'_h - p^*_h) - 1 - p^*_h = (p'_h - p^*_h) - 1 < -1, \quad (4)\]

where the last inequality follows from \( p^* \geq p \). Thus, by conditions (3)–(4) and \( v_{aa} = -1 \), \( x' \in \mathcal{E}_{R''} \) implies \( x' \in \mathcal{E}_{R''} \) and for any \( (\mu^*, p^*) \in \mathcal{E}_{R''} \) such that \( p^* \leq p' \), we have \( \mu'^*_a = \mu'_a \) for all \( a \in C \).
Hence, by the conclusions from the previous paragraph, if the numbers \(e^a\) and \(k^a_h\) can be chosen such that \(R'' \in \mathcal{R}\), then \(p^* \leq p'\) (as \(p^*\) then is the unique minimal equilibrium price vector at profile \(R''\) by Theorem 1), and, consequently, (a) \(x' \in \mathcal{E}_{R''}\) and (b) \(\mu_a'' = \mu_a'\) for all \(a \in C\) as desired. The remaining part of the proof demonstrates that the numbers \(e^a\) and \(k^a_h\) indeed can be chosen such that \(R'' \in \mathcal{R}\).

Suppose first that \(|C| = 1\), say \(C = \{1\}\), but that \(R'' \notin \mathcal{R}\). Then, by Definition 4, there exist two houses, \(h\) and \(h'\) in \(H\), a price vector \(p \in \Omega\), a sequence of distinct agents \((a_1, \ldots, a_q)\) of agents and a sequence of distinct houses \((h_1, \ldots, h_{q+1})\) for \(q \geq 2\) such that \(h = h_1 = a_1, h' = h_{q+1} = a_q, a_1I''_{a_1}(h_2, p), a_qI''_{a_q}(h_q, p)\) and \((h_j, p)I''_{a_j}(h_{j+1}, p)\) for all \(2 \leq j \leq q - 1\). Because \(R \in \mathcal{R}\) and \(R'' = (R''_1, R_{-1})\), it must be the case that \(1 \in \{a_1, \ldots, a_q\}\). Suppose that \(a_1 = 1\) with \(l \in \{1, \ldots, q\}\) and \(l < q\). Since \(1 < l \leq q\), the indifferences \(a_1I''_{a_1}(h_2, p)\) and \((h_j, p)I''_{a_j}(h_{j+1}, p)\) for \(2 \leq j \leq l - 1\) determine uniquely the price \(p_h\) (and similarly, if \(l < q\), the price \(p_{h_{l+1}}\) is uniquely determined). The idea is to distort (arbitrarily small) the parameter \(k^1_{h_1}\) and such that \((h_1, p)P''_{a_1}(h_{l+1}, p)\) (if \(l < q\)) or \((h_1, p)P''_{1, 1}(l = q)\), in such a way that \(h\) and \(h'\) are no longer connected via the sequences \((a_1, \ldots, a_q)\) and \((h_1, \ldots, h_{q+1})\).

Formally, for any \(2 \leq l \leq n - 1\), let \(\vec{h} = (h_1, \ldots, h_l)\) be a sequence of \(l\) distinct houses and \(\vec{a} = (a_1, \ldots, a_{l-1})\) be a sequence \(l - 1\) distinct agents in \(A \setminus \{1\}\) such that \(h = h_1 = a_1\) and \(h_j \neq a_j \neq h_{j+1}\) for \(2 \leq j \leq l - 1\). We say that \((\vec{h}, \vec{a})\) ends in \(h_l\). Denote all these sequences \((\vec{h}, \vec{a})\) by \(S_{h_l}\). The empty sequence \(\emptyset\) corresponds to the sequence consisting of agent 1 and by convention is the only element belonging to \(S_1\). Let \(S = \cup_{h \in H}S_h\). Now if for some \(p \in \Omega\) we have \(a_1I''_{a_1}(h_2, p)\) and \((h_j, p)I''_{a_j}(h_{j+1}, p)\) for all \(2 \leq j \leq l - 1\), then the price \(h_l\) is uniquely determined and we denote it by \(p^*_{h_l}\) (and it also implicitly stands for agent 1’s consumption bundle \((h_1, p^*_{h_1})\)).

Note that the set \(S\) is finite, and we may choose an order of the houses \(H \setminus \{1\}\) starting with \(p_1'\), say \(\mu_1' = h_1, h_2, \ldots, h_{n-1}\) such that we increase \(e^1\) by an amount \(\delta(h_1) > 0\) in order to break agent 1’s indifferences in \(S\) involving \(h_1\) while not reversing any strict preferences in \(S\): choose \(\delta(h_1) > 0\) such that for \(v_{1p_1'} = p_1' - e^1 + \delta(\mu_1')\) we have for any \(h \in H \setminus \{h_1\}\) and any sequences \((\vec{h}, \vec{a})\) ending in \(h_1\) and \((\vec{h}', \vec{a}')\) ending in \(h\):

(a) (for \(h = 1\)) \(p_{h_1'} - e^1 - \tilde{p}_{h_1} < -1\) iff \(p_{h_1'} - e^1 + \delta(h_1) - \tilde{p}_{h_1} < -1\), and

(b) (for \(h \neq 1\)) \(p_{h_1'} - e^1 - \tilde{p}_{h_1} < p_{h_1} - k^1_{h_1} - \tilde{p}_{h_1'}\) iff \(p_{h_1'} - e^1 + \delta(h_1) - \tilde{p}_{h_1} < p_{h_1} - k^1_{h_1} - \tilde{p}_{h_1'}\).

Note that agent 1’s indifferences in \(S\) involving \(h_1\) are broken favor of \(h_1\) because \(\delta(h_1) > 0\): in (a), if \(p_{h_1'} - e^1 - \tilde{p}_{h_1} = -1\), then \(p_{h_1'} - e^1 + \delta(h_1) - \tilde{p}_{h_1} > -1\); and in (b), if \(p_{h_1'} - e^1 + \tilde{p}_{h_1} = p_{h_1} - k^1_{h_1} - \tilde{p}_{h_1'}\), then \(p_{h_1'} - e^1 + \delta(h_1) - \tilde{p}_{h_1} > p_{h_1} - k^1_{h_1} - \tilde{p}_{h_1'}\).

For \(2 \leq j \leq n - 1\) we increase \(k^1_{h_j}\) by an amount \(\delta(h_{j-1}) > \delta(h_j) > 0\) in order to resolve agent 1’s remaining indifferences in \(S\) involving \(h_j\) while not reversing any strict preferences in \(S\): for \(v_{1h_j} = p_{h_j} - k^1_{h_j} + \delta(h_j)\) we have for any \(h \in H \setminus \{h_1, \ldots, h_j\}\) and any sequences \((\vec{h}, \vec{a})\) ending in \(h_j\) and \((\vec{h}', \vec{a}')\) ending in \(h\):
(a) (for $h = 1$) $p_{h_{j}}^{1} - k_{h_{j}}^{1} - \tilde{p}_{h_{j}}^{\tilde{h}, \tilde{a}} < -1$ iff $p_{h_{j}}^{1} - k_{h_{j}}^{1} + \delta(h_{j}) - \tilde{p}_{h_{j}}^{\tilde{h}, \tilde{a}} < -1$, and;

(b) (for $h \neq 1$) $p_{h_{j}}^{1} - k_{h_{j}}^{1} - \tilde{p}_{h_{j}}^{\tilde{h}, \tilde{a}} < p_{h_{j}}^{1} - k_{h_{j}}^{1} - \tilde{p}_{h_{j}}^{\tilde{h}', \tilde{a}'}$ iff $p_{h_{j}}^{1} - k_{h_{j}}^{1} + \delta(h_{j}) - \tilde{p}_{h_{j}}^{\tilde{h}, \tilde{a}} < p_{h_{j}}^{1} - k_{h_{j}}^{1} - \tilde{p}_{h_{j}}^{\tilde{h}', \tilde{a}'}$.

Note that all remaining indifferences in $S$ involving $h_{j}$ are broken favor of $h_{j}$ and that $\delta(h_{j-1}) > \delta(h_{j})$ guarantees that strict preferences in $S$ involving $h_{j-1}$ and $h_{j}$ are never reversed: for any $(\tilde{h}, \tilde{a}) \in S_{h_{j-1}}$ and any $(\tilde{h}', \tilde{a}') \in S_{h_{j}}$, $p_{h_{j-1}}^{1} - k_{h_{j-1}}^{1} - \tilde{p}_{h_{j-1}}^{\tilde{h}, \tilde{a}} > p_{h_{j}}^{1} - k_{h_{j}}^{1} - p_{h_{j}}^{\tilde{h}', \tilde{a}'}$ implies $p_{h_{j-1}}^{1} - k_{h_{j-1}}^{1} + \delta(h_{j-1}) - \tilde{p}_{h_{j-1}}^{\tilde{h}, \tilde{a}} > p_{h_{j}}^{1} - k_{h_{j}}^{1} + \delta(h_{j}) - p_{h_{j}}^{\tilde{h}', \tilde{a}'}$, i.e., no new indifferences in $S$ are added when we replace $-k_{h_{j}}^{1}$ by $-k_{h_{j}}^{1} + \delta(h_{j})$.

Thus, we may without loss of generality assume that $R'' \in \tilde{R}$ when $|C| = 1$. If $|C| \geq 2$, then we may use the above argument to replace the preferences of the agents in $C$ in $R$ one at a time by quasi-linear preferences: for example, if $C = \{1, 2\}$, then first we replace $R_{1}$ by $R''_{1}$ and obtain $(R''_{1}, R_{-1}) \in \tilde{R}$ and then we obtain $(R''_{1}, R''_{2}, R_{-1,2}) \in \tilde{R}$. □

**References**


