The Role of Learning for Asset Prices, Business Cycles and Monetary Policy

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Abstract

The importance of financial frictions for the business cycle is widely recognised, but what is less recognised is that the results obtained from studying these frictions depend heavily on the underlying asset pricing theory. I examine the implications of learning-based asset pricing for business cycles with financial frictions. I construct a model in which stock market valuations affect firms’ ability to access credit, and in which investors rely on past observation to predict the future. Learning greatly improves asset price properties such as return volatility and predictability. In combination with financial frictions, a powerful feedback loop emerges between beliefs, stock prices and real activity, leading to substantial amplification of shocks. The model-implied subjective expectations are found to be consistent with patterns of forecast error predictability in survey data. A reaction of monetary policy to asset prices stabilises expectations and substantially improves welfare, which is not the case under rational expectations.

Keywords: Learning, Asset Pricing, Credit Constraints, Monetary Policy, Survey Data

JEL Classification: D83, E32, E44, E52

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1 Introduction

*I think financial factors in general, and asset prices in particular, play a more central role in explaining the dynamics of the economy than is typically reflected in macroeconomic models, even after the experience of the crisis.*

- Andrew Haldane, 30 April 2014

The above statement, made by the Chief Economist of the Bank of England during a parliamentary hearing, may provoke disbelief among macroeconomists. After all, a wealth of research in the last fifteen years has been dedicated precisely to the links between the financial sector and the real economy. Financial frictions are now seen as a central mechanism by which asset prices interact with macroeconomic dynamics.

Still, our understanding of this interaction remains incomplete, in part due to the inherent difficulty of modelling asset prices.

The typical business cycle model employs an asset pricing theory based on time-separable preferences with moderate degrees of risk aversion and rational expectations. Such an asset pricing theory is well known to be inadequate for many empirical regularities such as return volatility (Shiller, 1981) and return predictability (Fama and French, 1988). This is not problematic when asset prices are disconnected from the real economy, since asset pricing and business cycle dynamics can then be separated.\(^1\) In the presence of financial frictions however, the prices of assets used as collateral affect borrowing constraints and hence the dynamics of the economy. A failure to generate realistic endogenous asset price dynamics can then become a potentially important source of model misspecification.

This paper examines the business cycle implications of a learning-based asset pricing theory. I construct a model of firm credit frictions in which agents are unable to form rational expectations about the price of equities in the stock market, and instead have to learn from past observation to form subjective beliefs. The learning-based approach to stock pricing has been shown to perform surprisingly well in endowment economies, without the need to rely on non-separable preferences or habit (Adam, Marcet and Nicolini, 2013; Adam, Beutel and Marcet, 2014). The interpretation of price dynamics under learning is quite different from rational expectations. With learning, stock prices fluctuate not because of variations in the discounting of prices and returns, but because of variations in subjective beliefs about the prices and returns themselves. The deviation of these subjective beliefs from rational expectations is a natural measure of “price misalignments”, “over-” and “undervaluation”. These notions are often present in informal arguments about financial markets, but absent in most asset pricing theories.\(^2\)

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\(^1\)For the real business cycle model with recursive preferences, Tallarini Jr. (2000) showed that business cycle properties are driven almost entirely by the intertemporal elasticity of substitution, while asset price properties are almost entirely governed by the degree of risk aversion.

\(^2\)For example, in October 2014, the IMF warned of “highly correlated mispricing [...] across assets” in its Global Financial Stability Report (p. 6).
A second model ingredient is that firms are subject to credit constraints, the tightness of which depends on firm market value. This type of constraint emerges from a limited commitment problem in which defaulting firms can be restructured and resold (similar to Chapter 11 of the US Bankruptcy Code) as opposed to being liquidated. It provides a mechanism by which high stock market valuations translate into easier access to credit. The model has a “financial accelerator” mechanism similar to Bernanke et al. (1999), with the strength and properties of this mechanism crucially depending on the endogenous dynamics of stock prices.

The analysis of the model yields three results. First, a positive feedback loop emerges between beliefs, asset prices and the production side of the economy, which leads to considerable amplification and propagation of business cycle shocks. When investor beliefs are more optimistic, their demand for stocks increases. This increases firm valuations and relaxes credit conditions. This in turn allows firms to move closer to their profit optimum. Provided counteracting general equilibrium forces are not too strong, they will also be able to pay higher dividends to their shareholders, raising stock prices further and propagating investor optimism even more. The financial accelerator mechanism becomes much more powerful than under rational expectations. At the same time, the learning mechanism greatly improves asset price properties such as price and return volatility and predictability without the need to impose complex preferences or high degrees of risk aversion. This result suggests that the relatively weak quantitative strength of the financial accelerator effect in many existing models (Cordoba and Ripoll, 2004) is at least in part due to low endogenous asset price volatility.

Second, while agents’ subjective expectations are not rational expectations, they are consistent in a number of ways with data obtained from surveys. I document that forecast errors on several macroeconomic aggregates (from the US Survey of Professional Forecasters) as well as on stock returns (from the Duke-Fuqua CFO Survey) can be predicted by the price/dividend ratio as well as forecast revisions. This is also true in the model, despite the fact that learning only adds one free parameter to the model and agents have model-consistent expectations for all relevant prices and outcomes except for stock prices. When they are over-predicting asset prices, they also over-predict credit limits depending on those prices and therefore aggregate activity, just like in the data.

Third, I show that the model has important normative implications. A recurring question in monetary economics is whether policy should react to asset price “misalignments”. Gali (2014) writes that justifying such a reaction requires “the presumption that an increase in interest rates will reduce the size of an asset price bubble” for which “no empirical or theoretical support seems to have been provided”. This paper is a first step towards filling this gap. Indeed, I find that under learning, the welfare-maximising monetary policy within a class of interest rate rules reacts strongly to asset price growth. By raising interest rates when stock prices are rising, policy is able to curb the endogenous build-up of over-optimistic investor beliefs. Such a reaction reduces both asset price volatility and business
cycle volatility. In contrast, under rational expectations, a policy reaction to asset prices does not improve welfare, in line with earlier findings in the literature.

The remainder of this paper is structured as follows. Section 2 reviews the related literature. Section 3 provides several empirical facts relating to the macroeconomic effects and properties of stock prices as well as discrepancies of measured expectations from rational expectations. Section 4 presents a highly stylised version of the model that permits an analytic solution. It shows that credit frictions or asset price learning alone does not generate either amplification of shocks or interesting asset price dynamics, while their combination does. The full model which can be used for quantitative analysis is then presented in Section 5. Section 6 contains the quantitative results. Section 7 contains the monetary policy analysis. Section 8 concludes.

2 Related literature

This paper starts from learning-based asset pricing developed in a series of papers by Klaus Adam and Albert Marcet (Adam and Marcet, 2011; Adam et al., 2013, 2014). They show that parsimonious models of learning about stock prices succeed in explaining key aspects of observed stock price data such as the excess volatility, equity premium, and return predictability puzzles. They also show consistency with investor expectations, which are hard to reconcile with rational expectations. Recent work by Barberis et al. (forthcoming) goes in a similar direction. While these papers study endowment economies, I take their approach to an economy with production. This allows one to look at the interactions between financial markets and the real economy, as well as policy implications.

There exist other approaches to asset pricing in production economies. For models with financial frictions in particular, it is popular to simply include exogenous shocks to explain the observed fluctuations in asset prices. Iacoviello (2005) and Liu et al. (2013), for example, set up economies in which exogenous shocks to housing demand drive house prices, which in turn affect credit constraints, and study the financial accelerator mechanism. Xu et al. (2013) have a model with a credit friction similar to that in my model, in which borrowing limits also depend on stock market valuations. They prove the existence of rational liquidity bubbles and introduce a shock that governs the size of this bubble, thus enabling them to match the stock prices seen in the data. In all of these models, the simple preferences and rational expectations would not allow realistic asset price dynamics in the absence of asset price shocks, and the structural interpretation of these shocks is often not clear.\(^3\) In order to advance our understanding of the interaction between asset prices and the real economy, I believe that it is necessary to have macroeconomic models that can endogenise asset price fluctuations. This paper is a step in this direction.

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\(^3\)Even the disaster-risk model of Gourio (2012) can be interpreted as such a model in which exogenous shocks to discount factors drive asset prices, even though quantities are not affected by financial frictions but by the changes in discount factors themselves.
The macro-finance literature has two main, rational expectations-based propositions to obtain realistic asset price dynamics. The first one is due to Campbell and Cochrane (1999) and relies on a non-linear form of habit formation combined with high risk aversion. The second, so-called “long-run risk” approach due to Bansal and Yaron (2004), introduces small, predictable and observable components to long-run consumption and dividend growth combined with Epstein-Zin preferences. There are some papers that try to embed these alternative approaches in production economies: Boldrin et al. (2001) for habit formation, Tallarini Jr. (2000) and Croce (2014) for long-run risk. These papers consider real business cycle models and are mainly concerned with endogenising consumption and dividend streams in a production economy while preserving the asset price implications. To my knowledge, there are no studies which take either approach to larger business cycle models with financial frictions, possibly because they are computationally quite demanding. They also require the use of preferences which have some rather counter-intuitive properties (shown by Lettau and Uhlig (2000) for habit and Epstein et al. (2013) for long-run risk). In my view, learning-based asset pricing is a promising alternative. It is intuitively appealing to think that asset prices are to some degree driven by self-amplifying waves of over- and under-confidence, and such a view is supported by survey evidence. Importantly, it also has implications for policy, as this paper shows.

The paper also makes a contribution to the literature on adaptive learning in business cycles. A number of papers in this area have studied learning in combination with financial frictions (Caputo, Medina and Soto, 2010; Milani, 2011; Gelain, Lansing and Mendicino, 2013). The conventional approach taken in this literature consists of two steps: first, derive the linearised equilibrium conditions of the economy under rational expectations; second, replace all terms involving expectations with parametrised forecast functions, and update the parameters using recursive least squares every period. Such models certainly produce very rich dynamics, but they are problematic on several grounds. First, it is not clear that first-order conditions parametrised in this way correspond in a meaningful sense to intertemporal optimisation problems. Second, these models are often very complex and intransparent. The need to parametrise every expectation in the first-order conditions requires a large number of parameters. In all but the simplest models, it then becomes prohibitively difficult to analyse equilibrium dynamics. In this paper, I propose a more transparent and parsimonious approach. Beliefs are restricted to be model-consistent as under rational expectations (with the only exception being the beliefs about stock prices) and agents make optimal choices given this set of beliefs. Even in a medium-sized DSGE model, the introduction of learning then adds only one parameter and one state variable to the model.

Finally, the paper also relates to the debate on whether monetary policy should react to...
asset price “misalignments”. Bernanke and Gertler (2001) found in a financial frictions model with rational exogenous asset price bubbles that the answer is “no”. This view, although not unchallenged (Filardo, 2001; Cecchetti et al., 2002), forms the consensus opinion and indeed the practice of most central banks. It has also recently been reinforced by Gali (2014), who argues that since rational bubbles are predicted to grow at the rate of interest, the optimal policy to deflate a bubble might even be to lower interest rates when asset prices are rising too fast. Without incorporating bubbles, Faia and Monacelli (2007) find a similar result, and conclude that a strong exclusive anti-inflationary stance remains welfare-maximising. This paper shows that such policy recommendations depend critically on the underlying asset price theory. In a world of less than fully rational expectations, raising interest rates in an asset price boom can be effective in curbing exuberant investor expectations and mitigate a surge (and subsequent reversal) in real activity due to high asset prices and easy access to credit.

3 Empirical evidence

The purpose of this section is to document three sets of facts. First, movements in the aggregate stock market have sizeable effects on investment and credit constraints, consistent with the credit friction in my model. Second, stock price movements exhibit high volatility and return predictability. Third, measures of expectations from survey data, both for stock prices and macro variables, reveal systematic deviations from rational expectations. Some but not all of these observations have been documented previously in the literature.

3.1 Effect of the stock market on investment and credit constraints

One of the oldest documented links between financial markets and the real economy is that the stock market predicts investment (Barro, 1990). Of course, prediction does not imply causation. It is plausible that new information about improved economic fundamentals causes both stock prices and investment to rise, with stock prices responding faster. This view is taken by Beaudry and Portier (2006) who show that in an estimated vector error correction model, innovations in stock prices orthogonal to current changes in TFP predict a substantial portion of long-run TFP variation. This suggests that stock price fluctuations are in fact “news shocks” about future productivity. However, it is also conceivable that stock market movements have a direct effect on investment even when they do not reflect changing expectations about the economic fundamentals. Blanchard et al. (1993) construct a measure of expected fundamentals and find that stock prices retain their predictive power even when controlling for fundamentals. In general though, it is hard to come to any definitive conclusions about causality without spelling out a structural model.

In the model of this paper, higher stock prices affect investment because of financial frictions: Firms with higher market value have easier access to external finance and can
therefore increase investment. Is this consistent with the data? It is, at least when looking at aggregate time series. I estimate a VAR using quarterly US data. The VAR includes six variables: investment, total factor productivity, dividends, the Federal Funds rate, a corporate credit spread, and the aggregate price/dividend-ratio.\(^5\)

In order to isolate movements in stock prices that are unrelated to contemporaneous productivity, monetary, or other shocks, I examine the effects of a “stock price shock”, identified as having an immediate effect on the P/D ratio but no contemporaneous effect on any other variables.\(^6\) This shock alone accounts for more than two thirds of the forecast error variance of the P/D ratio at all horizons.

Figure 1 plots the estimated impulse response functions. The shock leads to a persistent

\(^5\)Investment is real private non-residential fixed investment. Productivity is adjusted for capacity utilisation as in Kimball et al. (2006). Dividends are four-quarter moving averages from the S&P Composite index. The corporate credit spread is Moody’s baa-aaa corporate bond spread, serving as a proxy for credit market conditions. The P/D ratio is again from the S&P composite index. The lag length is set to two as per the Bayesian Information Criterion.

\(^6\)This ordering is chosen on the premise that financial markets adjust faster to shocks than either real variables or monetary policy. As for the ordering among the financial variables, the results are robust to inverting the order of the P/D ratio and the credit spread.
rise in the P/D ratio. It also significantly increases investment and dividends while reducing credit spreads. The effect on TFP however is insignificant throughout and initially negative. This casts doubt on the view that most stock price movements are a reflection of news about future productivity. They also do not seem to reflect news about interest rates, since the response of the Federal Funds rate, too, is flat and insignificant.

3.2 Asset price “puzzles”

Asset prices in general, and stock prices in particular, are known to exhibit a number of characteristics that are difficult to reconcile with a basic consumption-based asset pricing model (by which I mean a representative investor with time-separable power utility and rational expectations). Here, I document two of them: excess volatility and return predictability. These are summarised in Table 1 for quarterly aggregate US data.

The first row shows the ratio of the standard deviation of the cyclical components of stock prices and dividends. By this measure prices are 2.63 times more volatile than dividends. The log price/dividend ratio (second row) and log stock returns (third row) are also highly volatile. Shiller (1981) showed that this amount of volatility cannot be accommodated in an asset pricing theory based on rational expectations and constant discount rates. If one starts from the premise that asset prices equal discounted cash flows, then this implies that either discount rates must vary a lot, or expectations are not rational (or both).

The fourth and fifth rows of the table document return predictability at the one- and five-year horizon, respectively. A high P/D ratio reliably predicts low future returns at these horizons, even if short-run stock returns are almost unpredictable. Cochrane (1992)

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Table 1: Stock market statistics.

<table>
<thead>
<tr>
<th>statistic</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>excess volatility</td>
<td>$\frac{\sigma(\text{hp}(\log P_t))}{\sigma(\text{hp}(\log D_t))}$</td>
</tr>
<tr>
<td>$\sigma(\log \frac{P_t}{D_t})$</td>
<td>.408 (.017)</td>
</tr>
<tr>
<td>$\sigma(\log R_{t,t+1})$</td>
<td>.335 (.022)</td>
</tr>
<tr>
<td>return predictability</td>
<td>$\rho(\frac{P_t}{D_t}, R_{t,t+4})$</td>
</tr>
<tr>
<td>$\rho(\frac{P_t}{D_t}, R_{t,t+20})$</td>
<td>-.439 (.049)</td>
</tr>
</tbody>
</table>


Another equally famous fact due to Mehra and Prescott (1985) is the size of the equity premium. Adam et al. (2013) show that learning models are able to generate sizeable equity premia, but in this paper, I only focus on volatility and return predictability.
shows that the variance of the P/D ratio can be decomposed into its covariance with future returns and future dividend growth. Since dividend growth is not very volatile, not well predicted by the P/D ratio, and the P/D ratio itself is volatile, it follows that returns must be predictable. Assuming rational expectations, Cochrane identifies the predictable component of returns with a time-varying discount rate. Again, the alternative is that expectations used to price assets are distinct from rational expectations. In the model of this paper, a high P/D ratio is not a result of low required returns, but of high expected returns - where the subjective expectation is distinct from the statistical prediction.

### 3.3 Survey data on expectations

The rational expectations hypothesis is a fundamental building block of modern macroeconomics. Sometimes, it is criticised as an unrealistic modelling device which asserts that agents are hyper-rational, endowed with infinite computing power and knowledge of the structural shocks and relationships of the economy. But in fact, it makes no such claim. In the words of Sargent (2008), it simply asserts that “outcomes do not differ systematically [...] from what people expect them to be”. Put differently, any agent’s forecast error should not be predictable by information available to the agent at the time of the forecast.

The rational expectations hypothesis is testable based on survey measures of expectations. It is almost always rejected. Here, I document some of these tests, and characterise some of the predictability patterns of forecast errors.

Expectations of returns are positively correlated with past returns and the P/D-ratio, whereas the best statistical prediction would call for a negative correlation. The difference is strong enough to be statistically rejected. This pattern is observable across many different sources of survey data (Greenwood and Shleifer, 2014). I illustrate it with data from the CFO survey by John Graham and Campbell Harvey at Duke University. The survey respondents are CFOs of major US corporations, which are likely to possess good knowledge of financial markets. Since 2000, the survey includes a question on stock market return expectations (“Over the next year, I expect the average annual S&P 500 return will be ...”). Figure 2 compares the survey expectations with realised returns. The left panel plots the mean survey response against the value of the P/D ratio in the month preceding the survey. The correlation is strongly positive: return expectations are more optimistic when stock valuations are high. However, high stock valuations actually predict low future returns, as documented above and illustrated again in the right panel of the figure. Such a pattern cannot be reconciled with rational expectations, as it implies that agents’ forecast errors are predictable by the P/D ratio, a publicly observable statistic.\(^8\)

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\(^8\)To be precise, one can test the hypothesis that the univariate regression coefficient of the P/D ratio on expected (survey) returns is the same as that on realised returns, e.g. using the SUR estimator. The hypothesis is rejected at the 0.1% level. The survey asks for a return estimate of a 12-month period starting at varying days of the month, whereas the realised return measure is taken at the beginning of the month. But the results are robust to taking realised returns at the end of the month.
Expected nominal returns are the mean response in the Graham-Harvey survey, P/D ratio and realised nominal returns data from the S&P 500. Data period 2000Q3-2012Q4. Correlation coefficient for expected returns $\rho = .54$, for realised returns $\rho = -.40$.

Of course, survey data are only an imperfect measure of expectations. When answering questions on a survey, respondents might wilfully misstate their true expectations, answer carelessly, or misunderstand the question. When being asked for a point estimate, they might report a statistic other than the mean of their belief distribution. Still, survey data are the best available test for the rational expectations hypothesis.

Tests of forecast error predictability can be applied to other variables of macroeconomic significance. Table 2 describes tests using the Federal Reserve’s Survey of Professional Forecasters (SPF) as well as the CFO survey data. Each row and column corresponds to a correlation of a mean forecast error with a variable that is observable by respondents at the time the survey is conducted. Under the null of rational expectations, the true correlation coefficients should all be zero.

The first column shows that the P/D ratio negatively predicts with forecast errors. When stock prices are high, people systematically under-predict economic outcomes. This holds in particular for stock returns, as was already shown in the scatter plot above. But it also holds true for macroeconomic aggregates, albeit at lower levels of significance. The second column shows that the change in the P/D ratio is a better predictor of forecast errors, strongly rejecting the rational expectations hypothesis, but in the opposite direction. At times when stock prices are rising, people systematically under-predict economic outcomes. Since the stock market itself positively predicts economic activity (as shown above), this suggests that agents’ expectations are too cautious and under-predict an expansion in its beginning, but then overshoot and over-predict it when it is about to end. Such a pattern emerges naturally in the model of this paper under learning.
Table 2: Forecast error predictability: Correlation coefficients.

<table>
<thead>
<tr>
<th>forecast variable</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>log PD&lt;sub&gt;t&lt;/sub&gt;</td>
<td>Δ log PD&lt;sub&gt;t&lt;/sub&gt;</td>
<td>forecast revision</td>
</tr>
<tr>
<td>( R_{stock}^{t+1} )</td>
<td>(-.44^{***})</td>
<td>.06</td>
<td>( n/a )</td>
</tr>
<tr>
<td></td>
<td>(-3.42)</td>
<td>(.41)</td>
<td></td>
</tr>
<tr>
<td>( Y_{t+3} )</td>
<td>-.21*</td>
<td>.22**</td>
<td>.29***</td>
</tr>
<tr>
<td></td>
<td>(-1.78)</td>
<td>(2.42)</td>
<td>(3.83)</td>
</tr>
<tr>
<td>( I_{t+3} )</td>
<td>-.20*</td>
<td>.25***</td>
<td>.31***</td>
</tr>
<tr>
<td></td>
<td>(-1.74)</td>
<td>(2.88)</td>
<td>(3.79)</td>
</tr>
<tr>
<td>( C_{t+3} )</td>
<td>-.19*</td>
<td>.21**</td>
<td>.23***</td>
</tr>
<tr>
<td></td>
<td>(-1.85)</td>
<td>(2.37)</td>
<td>(2.67)</td>
</tr>
<tr>
<td>( u_{t+3} )</td>
<td>.05</td>
<td>-.27***</td>
<td>.43***</td>
</tr>
<tr>
<td></td>
<td>(.12)</td>
<td>(-3.07)</td>
<td>(6.07)</td>
</tr>
</tbody>
</table>

Correlation coefficients for mean forecast errors on one-year ahead nominal stock returns (Graham-Harvey survey) and three-quarter ahead real output growth, investment growth, consumption growth and the unemployment rate (SPF). t-statistics for the null of zero correlation in parentheses. One, two, and three asterisks correspond to significance at the 10%, 5%, and 1% levels. Regressors: Column (1) is the S&P 500 P/D ratio and Column (2) is its first difference. Column (3) is the forecast revision as in Coibion and Gorodnichenko (2010). Data from Graham-Harvey covers 2000Q3-2012Q4. Data for the SPF covers 1981Q1-2012Q4.

The third column reports the results of a particular test of rational expectations devised by Coibion and Gorodnichenko (2010). Since for any variable \( x_t \), the SPF asks for forecasts at one- through four-quarter horizon, it is possible to construct a measure of agent’s revision of the change in \( x_t \) as \( \hat{E}_t [x_{t+3} - x_t] - \hat{E}_{t-1} [x_{t+3} - x_t] \). Forecast errors are positively predicted by this revision measure. Coibion and Gorodnichenko take this as evidence for sticky information models in which information sets are gradually updated over time. As I will show later, it is also consistent with the learning model developed in this paper.

4 Understanding the mechanism

In this section, I construct a simplified version of the model which illustrates the interaction between asset prices and credit frictions under learning. I impose several strong assumptions permitting a closed-form solution. Quantitative analysis will require a richer model, the development of which is relegated to the next section. The main message of this section is that financial frictions alone do not generate either sizeable amplification of business cycle shocks or asset price volatility, but in combination with learning they do.
4.1 Model setup

Time is discrete at \( t = 0, 1, 2, \ldots \). The model economy consists of a representative household and a representative firm. The representative household is risk-neutral and inelastically supplies one unit of labour. Its utility maximisation programme is as follows:

\[
\max_{(C_t, S_t, B_t)} \mathbb{E}^P \sum_{t=0}^{\infty} \beta^t C_t
\]

s.t. \( C_t + S_t P_t + B_t = w_t + S_{t-1} (P_t + D_t) + R_{t-1} B_{t-1} \)

\( S_t \in [0, \bar{S}], S_{-1}, B_{-1} \)

\( C_t \) is the amount of non-durable consumption goods purchased by the household in period \( t \). The consumption good is traded in a competitive spot market and serves as the numéraire. \( w_t \) is the real wage rate. Labour is also traded in a competitive spot market, so that the wage \( w_t \) is taken as given by the household. Moreover, the household can trade two financial assets, again in competitive spot markets: one-period bonds, denoted by \( B_t \) and paying gross real interest \( R_t \) in the next period; and stocks \( S_t \) which trade at price \( P_t \) and entitle their holder to dividend payments \( D_t \). The household cannot short-sell stocks and his maximum stock holdings are capped at some \( \bar{S} > 1 \).

The household maximises the expectation of discounted future consumption under the probability measure \( P \). This measure is the subjective belief system held by agents in the model economy at time \( t \), which will be discussed in detail further below. The first order conditions describing the household’s optimal plan under an arbitrary \( P \) are

\[
R_t = R = \beta^{-1}
\]

\[
S_t \begin{cases} 
= 0 & \text{if } P_t > \mathbb{E}_t^P [P_{t+1} + D_{t+1}] \\
\in [0, \bar{S}] & \text{if } P_t = \mathbb{E}_t^P [P_{t+1} + D_{t+1}] \\
= \bar{S} & \text{if } P_t < \mathbb{E}_t^P [P_{t+1} + D_{t+1}] 
\end{cases}
\]

Let us now turn to the firm. It engages in the production of a good which can be used both for consumption and investment. It is produced using capital \( K_{t-1} \), which the firm owns, and labour \( L_t \) according to the constant returns to scale technology

\[
Y_t = K_{t-1}^\alpha (A_t L_t)^{1-\alpha}
\]

\(9\)The constraint on \( S_t \) is necessary to guarantee existence of the learning equilibrium, although it never binds along the equilibrium path.
where $A_t$ is its productivity. Here, I only allow for permanent shocks to productivity:

$$\log A_t = \log G + \log A_{t-1} + \varepsilon_t, \varepsilon_t \sim \mathcal{N}\left(-\frac{\sigma^2}{2}, \sigma^2\right) \text{iid} \tag{4.4}$$

In particular, the expected growth rate of productivity is constant at $E_t A_{t+1}/A_t \equiv G$. The capital stock is predetermined and owned by the firm. It depreciates at the rate $\delta$ at the end of each period. The firm can also issue shares and bonds as described above. Thus, its period budget constraint reads as follows:

$$Y_t + (1 - \delta) K_{t-1} + B_t + S_t P_t = w_t L_t + K_t + S_{t-1} (P_t + D_t) + RB_{t-1} \tag{4.5}$$

Before describing the equilibrium, it is useful to introduce the marginal return on capital:

$$R^k_t = \frac{\partial Y_t}{\partial K_{t-1}} + 1 - \delta \tag{4.6}$$

### 4.2 Frictionless equilibrium

In the absence of financial constraints, the Modigliani-Miller theorem will render the composition of firm financing redundant and the model collapses to a standard stochastic growth model. In particular, the optimal choice of the capital stock (under rational expectations) equates the marginal return on capital with the inverse of the discount factor:

$$E_t R^k_{t+1} = \beta^{-1}. \tag{4.7}$$

Whatever one assumes about the financial structure of the firm, its total value will equal the size of the capital stock. The capital stock and firm value co-moves perfectly with productivity:

$$\frac{K_t}{A_t} = \tilde{K}_t = K^* \tag{4.7}$$

$$\frac{P_t + B_t}{A_t} = \tilde{K}_t = K^* \tag{4.8}$$

where $K^* = G \left(\frac{\alpha}{\beta - 1 + \delta}\right)^{1/(1-\alpha)} e^{-\alpha \sigma^2/2}$.

### 4.3 Rational expectations equilibrium with financial frictions

In introducing financial frictions, I impose constraints on both the equity and debt instruments. On the equity side, the firm is not allowed to change the quantity of shares outstanding, fixed at $S_t = 1$. Further, it is not allowed to use retained earnings to finance investment. Instead, all earnings net of interest and depreciation have to be paid out to shareholders:

$$D_t = Y_t - w_t L_t - \delta K_{t-1} - (R - 1) B_{t-1} \tag{4.9}$$
Combined with equation (4.5), this assumption implies that the firm’s capital stock must be entirely debt-financed: $K_t = B_t$ at all $t$.\textsuperscript{10} In other words, the firm’s book value of equity after dividend payouts is constrained to be zero. This does not mean, however, that the market value of equity is also zero as long as the firm’s expected dividend payouts are strictly positive.

On the debt side, the level of debt that can be acquired by the firm is limited to a fraction $\xi \in [0, 1]$ of the total market value of its assets, i.e. the sum of debt and equity:

\[
B_t \leq \xi (B_t + P_t) \\
\iff K_t \leq \frac{\xi}{1 - \xi} P_t
\]  

Equation (4.10) is a simple constraint on leverage, i.e. debt divided by value of total assets. I depart from the standard assumption that total assets enter with their liquidation value (in this case $K_t$, the book value) and instead let them enter with their market value (in this case $B_t + P_t$). This captures the idea that a firm which is more highly valued by financial markets will have easier access to credit. This could be because high market value acts as a signal to lenders for the firm’s ability to repay, or because the amount lenders can recover in the event of default depends on the price at which a firm can be resold to other financial market participants. In the full version of the model in the next section, I formally derive (4.10) from a limited commitment problem.

The firm maximises the presented discounted sum of future dividends, using the household discount factor:

\[
\max_{(K_t, L_t, D_t)_{t=0}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t D_t
\]

s.t. $D_t = Y_t - w_t L_t + (1 - \delta - R) K_{t-1}$

\[
K_t \leq \frac{\xi}{1 - \xi} P_t
\]

In particular, it makes its decisions under the same belief system $\mathcal{P}_t$ as the household.

Due to constant returns to scale in production, we can write dividends at the optimum as follows:

\[
\max_{L_t} D_t = \left( R^*_t - R \right) K_{t-1} \]  

The optimal choice of capital is to exhaust borrowing limits as long as the expected internal

\textsuperscript{10}This holds under the suitable initial condition $K_{-1} = B_{-1}$, e.g. the firm starts with zero book value of equity on its balance sheet. This assumption is relaxed in the full model.
When solving for the equilibrium, market clearing needs to be imposed. The market clearing condition for bonds is just $R = \beta^{-1}$. That for equity is $S_t = 1$, which means the Euler equation (4.2) has to hold with equality:

$$P_t = \beta E_t [P_{t+1} + D_{t+1}]$$

(4.13)

Goods market clearing requires

$$Y_t + K_t = C + (1 - \delta) K_{t-1}$$

(4.14)

and labour market clearing requires $L_t = 1$.

The equilibrium under rational expectations admits a closed-form solution. First, note that the equilibrium return on capital depends on the aggregate capital stock due to decreasing returns to scale at the aggregate level:

$$R^k_t = R^k(\tilde{K}_{t-1}, \varepsilon_t) = \alpha \left( \frac{Ge^{\varepsilon_t}}{K_{t-1}} \right)^{1-\alpha} + 1 - \delta$$

(4.15)

This dependency comes about through a general equilibrium effect: A higher level of the capital stock increases labour demand, which increases real wages and therefore lowers the return on capital. Next, one can write expected dividends as a function of the capital stock:

$$\tilde{D}(\tilde{k}_t, \tilde{K}_t) = E_t \frac{D_{t+1}}{A_t} = \left( E_t R^k(\tilde{K}_t, \varepsilon_{t+1}) - R \right) \tilde{k}_t$$

(4.16)

Here, I have made a distinction between the capital choice $\tilde{k}_t$ of the representative firm that takes future wages as given, and the aggregate capital stock $\tilde{K}_t$ which determines wages and the return on capital in general equilibrium. Of course, in equilibrium the two are equal. Finally, the effective stock price under rational expectations is simply the discounted sum of future dividends:

$$\tilde{P}_t = \frac{P_t}{A_t} = \beta E_t \sum_{s=0}^{\infty} \frac{\beta^s A_{t+s}}{A_t} \tilde{D}(\tilde{K}_{t+s}, \tilde{K}_{t+s})$$

(4.17)

First, let us consider the case $\xi = 1$. In this case, $\tilde{K}_t$ is constant across time and states in equilibrium and is at its efficient level $K^*$ such that $E_t R^k(K^*, \varepsilon_{t+1}) = R$. By consequence, expected dividends and the market value of equity are zero: $\tilde{D}(K^*, K^*) = 0$ and $P_t = 0$. Intuitively, when the firm can borrow up to the total amount of its market value, it faces
no financial friction. Book and market value of the firm coincide. The expected return on capital equals the interest rate due to risk neutrality of households, and since all capital is financed by debt and the production function has constant returns to scale, in expectation all profits are paid out as interest payments to debt holders. The residual equity claims trade at a price of zero. This result is the expectation version of the zero-profit condition under constant returns to scale.

Next, let us turn to the case in which $\xi$ is strictly smaller than one. The equilibrium effective capital stock $\tilde{K}_t$ and stock price $\tilde{P}_t$ turn out to be constant here as well. The equilibrium is characterised by two equations:

$\bar{P} = \tilde{D}(\bar{K}, \tilde{K}) + \frac{\alpha P}{R}$ (4.18)

$\bar{K} = \frac{\xi}{1 - \xi} \bar{P}$ (4.19)

The first equation pins down the stock market value of the firm as a function of its capital stock, while the second determines the capital stock that can be reached by exhausting the borrowing constraint that depends on the stock market value. In particular, the internal rate of return is always greater than the return on debt and so the borrowing constraint is always binding. The equilibrium is depicted graphically in Figure 3. Equation (4.19) is represented by a straight line of slope $(1 - \xi)/\xi$, while the market value of equity (4.18) is a hump-shaped curve. The equilibrium lies at the intersection (Point A). Firms pay positive dividends (in expectation) and the market value of equity is positive. The capital stock $\bar{K}$ remains inefficiently low: $\bar{K} < K^*$. In addition, Figure 3 plots as a dotted line firm value as a function of the current choice of capital $\tilde{k}_t$. From the firm perspective (taking factor prices as given), firm value is increasing everywhere, so it always wants to exhaust the borrowing constraint.

While the capital stock and expected output are an increasing function of maximum leverage $\xi$, expected dividends are non-monotonous and hump-shaped. Why is that so?
There are two opposing forces affecting expected dividends, as can be seen from the following decomposition:

\[
\frac{d}{d\xi} \tilde{D}(\bar{K}, \bar{K}) = \left[ \mathbb{E}_t R^k (\bar{K}, \varepsilon_{t+1}) - R + \mathbb{E}_d \frac{dR^k}{d\bar{K}} (\bar{K}, \varepsilon_{t+1}) \bar{K} \right] \frac{d\bar{K}}{d\xi} > 0 \quad (4.20)
\]

The first term in brackets captures a partial equilibrium effect, which is internalised by the firm. When a firm is financially constrained, its internal rate of return is higher than the return it has to pay to debt holders. By borrowing more, it can increase its scale of production and make more profit. The second term, however, captures a general equilibrium effect: higher investment lowers the marginal product of capital, which in practice is realised through an increase in the equilibrium wage \(w_{t+1}\). When \(\xi\) is small (financial frictions are severe) the partial equilibrium effect dominates, while for a large \(\xi\) the general equilibrium effect dominates.

Most importantly however, financial frictions do not lead to any amplification or propagation of shocks in the rational expectations equilibrium. They have a level effect on output, capital, etc., but the dynamics of the model are identical for any value of \(\xi\). This can be seen by looking at the variances of log stock price and output growth which do not depend on \(\xi\):

\[
\text{Var} [\Delta \log P_t] = \sigma^2 \quad (4.21)
\]
\[
\text{Var} [\Delta \log Y_t] = (1 - 2\alpha + 2\alpha^2) \sigma^2 \quad (4.22)
\]

Intuitively, with financial frictions, a shock to productivity raises asset prices just as much as to allow the firm to instantly adjust the capital stock proportionately.

At the same time, the model cannot replicate many of the stylised facts on stock price data. Up to a first-order approximation, the relative volatility of asset price growth with respect to dividend growth is bounded from below:

\[
\frac{\sigma (\Delta \log P_t)}{\sigma (\Delta \log D_t)} < \left(1 - 2\alpha + 2\alpha^2\right)^{1/2} \leq \sqrt{2} \quad (4.23)
\]

The asset price volatility observed in the data can therefore not be matched. The volatility of the price/dividend ratio can also not be matched: in fact, the forward P/D ratio is even constant:

\[
PD_t = \frac{P_t}{\mathbb{E}_t D_{t+1}} = \frac{1}{R - G} \quad (4.24)
\]

Furthermore, excess returns are unpredictable: \(\mathbb{E}_t [(P_{t+1} + D_{t+1})/P_t] - R = 0\). Finally, by definition of rational expectations, forecast errors are unpredictable, again at odds with the data.
4.4 Learning equilibrium

I now describe the equilibrium under learning. Qualitatively, this alteration to the expectation formation process can address many of the issues encountered in the last section: it will increase the volatility of stock prices, account for return and forecast error predictability, and most importantly, induce endogenous amplification and propagation on the production side of the model economy. The only departure from rational expectations will be that agents do not understand the pricing function that maps fundamentals into an equilibrium stock price. Instead, they form subjective beliefs about the law of motion of prices and update them using realised price observations.

More specifically, agents continue to make optimal choices for a consistent belief system governed by the measure \( \mathcal{P} \). The equilibrium therefore satisfies “internal rationality” as formalised by Adam and Marcet (2011) and used in Eusepi and Preston (2011). I impose the following restrictions to beliefs. Under \( \mathcal{P} \),

1. agents observe the exogenous productivity process \((\varepsilon_t)_{t=0}^{\infty}\) and have correct beliefs about its distribution;

2. agents believe that \( P_t \) evolves according to

\[
\log P_t - \log P_{t-1} = \mu_t + \eta_t \tag{4.25}
\]

\[
\mu_t = \mu_{t-1} + \nu_t \tag{4.26}
\]

where

\[
\begin{pmatrix} \eta_t \\ \nu_t \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} -\frac{1}{2} & \sigma_{\eta}^2 \\ \sigma_{\nu}^2 & 0 \end{pmatrix}, \begin{pmatrix} \sigma_{\eta}^2 & 0 \\ 0 & \sigma_{\nu}^2 \end{pmatrix} \right) \text{iid}, \tag{4.27}
\]

the variable \( \mu_t \) and the disturbances \( \eta_t \) and \( \nu_t \) are unobserved and the prior about \( \mu_t \) in period 0 is given by

\[
\mu_0 | \mathcal{F}_0 \sim \mathcal{N} \left( \hat{\mu}_0, \sigma_{\mu}^2 \right) \text{where } \sigma_{\mu}^2 = \frac{-\sigma_{\nu}^2 + \sqrt{\sigma_{\nu}^4 + 4\sigma_{\eta}^2\sigma_{\nu}^2}}{2}; \tag{4.28}
\]

3. agents update their beliefs about \( \mu_t \) after making their choices and observing equilibrium prices in period \( t \);

4. for any model variable \( x_t \), any future date \( t + \tau \), and any sequence of exogenous productivity shocks \( \varepsilon_t, \ldots, \varepsilon_{t+\tau} \) and stock prices \( P_t, \ldots, P_{t+\tau} \) which lies on the equilibrium path:

\[
\mathbb{E}_t^\mathcal{P} \left[ x_{t+\tau} \mid \varepsilon_t, P_t, \ldots, \varepsilon_{t+\tau}, P_{t+\tau} \right] = x_{t+\tau} \text{ a.s.}
\]

i.e. agents’ beliefs are correct *conditional* on the realisation of stock prices and fundamentals.
The first assumption implies that agents have as much information about the fundamental shocks of the economy as under rational expectations. The second assumption amounts to saying that agents believe stock prices to be a random walk. This random walk is believed to have a small, unobservable, and time-varying drift $\mu_t$. Learning about this drift is going to be the key driver of asset price dynamics. The third assumption imposes that forecasts of stock prices are updated after equilibrium prices are determined, so as to avoid possible multiple equilibria in price and forecast determination.\footnote{This “lagged belief updating” is common in the learning literature. It makes all feedback between forecasts and prices inter- rather than intramoral. For further discussion see Adam et al. (2014).}

To the best of my knowledge, the fourth and final assumption is novel to the learning literature. To make their choices, agents must form expectations about more than just the stock price and the exogenous processes of the economy. For example, investors need to form a belief about dividends, which are endogenous and depend on wages and the capital choice of the firm in equilibrium. The typical way to deal with this is to introduce additional expectation formation processes for all forecast endogenous variables. My view is that this increases complexity to the point of rendering a learning model opaque. Instead, I retain the simplicity of rational expectations as much as possible: Conditional on a realisation of stock prices and fundamentals on the equilibrium path, agents do not make systematic forecast errors. In particular, the choices they make are conditionally consistent with equilibrium outcomes. I also solve for these beliefs in much the same way as one would under rational expectations; the procedure is discussed in Section 5 and Appendix C. The only difference to rational expectations lies in the word “conditional”. Since agents’ beliefs about stock prices produce biased forecast errors, their unconditional forecasts about other endogenous variables $E^P_{t}[x_{t+1}]$ are also biased, even if the conditional forecasts $E^P_{t}[x_{t+1} | \varepsilon_{t+1}, P_{t+1}]$ are not.

Despite such relatively accurate beliefs, agents are still left with the important problem of making a good guess about the unobservable drift $\mu_t$ of stock prices. Optimal Bayesian belief updating amounts to Kalman filtering in this case, since (4.25)-(4.28) is a linear state-space system. Under $P$, agents’ beliefs about $\mu_t$ at time $t$ are normally distributed with stationary variance $\sigma_{\mu}^2$ and mean $\hat{\mu}_{t-1}$. This belief about the mean of $\mu_t$, which I will usually just call “the belief”, evolves according to the updating equation:

$$\hat{\mu}_t = \hat{\mu}_{t-1} - \frac{\sigma_{\mu}^2}{2} + g \left( \log P_t - \log P_{t-1} + \frac{\sigma_n^2 + \sigma_{\mu}^2}{2} - \hat{\mu}_{t-1} \right) \quad (4.29)$$

In this equation, $P_t$ and $P_{t-1}$ are observed, realised stock prices. These are determined in equilibrium under the actual law of motion of the economy and do not follow the perceived law of motion described by (4.25)-(4.28). The parameter $g$ is called the “learning gain”. It governs the speed with which agents move their prior in the direction of the last forecast error.\footnote{The gain is related to the variances of the disturbances by the formula $g = \frac{1}{2} \left( \frac{\sigma_n^2}{\sigma_{\mu}^2} + \sqrt{\frac{\sigma_n^2}{\sigma_{\mu}^2} + 4 \frac{\sigma_n^2}{\sigma_{\mu}^2}} \right)$.} When $g$ is high, agents are confident that observed changes in the growth rate
of asset prices are due to changes in the trend $\mu_t$ rather than the noise $\eta_t$. The gain is not decreasing in time: Agents believe that the drift in asset prices is itself time-varying, so that even after a long period of time it remains difficult to forecast it. A consequence of this is that beliefs never converge: Agents always entertain the possibility of some structural change to the law of motion of asset prices, so that even an infinite number of observations is not completely informative about the future.

It is important to keep in mind that the disturbances $\eta_t$ and $\nu_t$ are objects that exist only in the subjective belief system $\mathcal{P}$. The actual equilibrium under learning does not contain any shock process other than the productivity shock $\varepsilon_t$. Instead, the equilibrium still contains the actual market clearing conditions (4.13) and (4.14), even if they are unknown to the agents. By Walras’ law, it is sufficient to impose stock market clearing. Under $\mathcal{P}$, (4.13) reads as follows:

$$P_t = \frac{E^\mathcal{P}_t P_{t+1} + E^\mathcal{P}_t D_{t+1}}{R} = \frac{P_t \exp(\hat{\mu}_{t-1} + \frac{1}{2} \sigma^2_\mu) + A_t \tilde{D}(\hat{K}_t, \hat{K}_t)}{R} = \frac{A_t \tilde{D}(\hat{K}_t, \hat{K}_t)}{R - \exp(\hat{\mu}_{t-1} + \frac{1}{2} \sigma^2_\mu)}$$

The second line is obtained by substituting in agents’ beliefs about the evolution of the future stock price $E_t P_{t+1}$ and dividends $E_t D_{t+1}$. Under $\mathcal{P}$, agents forecast future dividends accurately conditional on their belief about stock prices. Therefore, their expectations about dividends depend on the current capital stock in the same way as under rational expectations.

In sum, the learning equilibrium is the solution to the following:

$$\hat{P}_t = \frac{\tilde{D}(\hat{K}_t, \hat{K}_t)}{R - \exp(\hat{\mu}_{t-1} + \frac{1}{2} \sigma^2_\mu)}$$

$$\hat{K}_t = \frac{\xi}{1 - \xi} \hat{P}_t$$

$$\hat{\mu}_t = \hat{\mu}_{t-1} - \frac{\sigma^2_\nu}{2} + g \left( \log \frac{\hat{P}_t}{\hat{P}_{t-1}} + \log G + \varepsilon_t - \hat{\mu}_{t-1} + \frac{\sigma^2_\eta + \sigma^2_\nu}{2} \right)$$

The first two equations are static and the third is dynamic. The third equation also depends on the productivity innovation $\varepsilon_t$, and as such $\hat{P}_t$ and $\hat{K}_t$ are not constant any more. The resulting stock price dynamics after a positive innovation at $t = 1$ are depicted in Figure 4.\textsuperscript{13} The initial shock at $t = 1$ raises stock prices proportional to productivity. In the rational expectations equilibrium, this would be all that happens. Learning investors and is strictly increasing in the signal-to-noise ratio $\sigma^2_\nu/\sigma^2_\varepsilon$.

\textsuperscript{13}The figure depicts the case in which beliefs start at their rational expectations value $\hat{\mu}_t = \log G$ and subjective uncertainty is vanishing in the sense that $\sigma^2_\eta, \sigma^2_\nu \to 0$ while $g$ remains constant.
Figure 4: Stock price dynamics under learning.

however are not sure whether the rise in $P_1$ is indicative of a transitive or permanent increase in the growth rate of stock prices. They therefore revise their beliefs upwards. In $t = 2$ then, demand for stocks is higher and stock prices need to rise further to clear the market. Beliefs continue to rise as long as observed asset price growth (dashed black line in Figure 4) is higher than the current belief $\hat{\mu}_t$ (solid red line). The differences between observed and expected price growth are the forecast errors (dotted red lines). In the figure, the increase in prices and beliefs ends at $t = 3$, when the forecast error is zero. There is no need for a further belief revision. Now, by equations (4.33)-(4.34), in the absence of subsequent shocks, $P_t$ just co-moves with beliefs $\hat{\mu}_{t-1}$, so when there is no belief revision at $t = 4$, realised asset price growth is also zero. This triggers an endogenous reversal in prices, as investors observe stalling asset prices at the peak of their optimism. They subsequently revise their beliefs $\hat{\mu}_t$ downwards, pushing the stock price down until it returns to its steady-state level.

Figure 4 can be used to illustrate the key properties of beliefs and prices under learning. First, the described dynamics do not depend on time: The system of $\tilde{K}_t$, $\tilde{P}_t$, and $\hat{\mu}_t$ is stationary. Second, asset prices are more volatile than under rational expectations. Third, (excess) returns are predictable even though agents are risk-neutral and the discount factor is constant. To see this, it is again convenient to look at the forward P/D ratio:

$$PD_t = \frac{1}{R - \exp (\hat{\mu}_{t-1} + \frac{1}{2}\sigma^2)}$$

The forward P/D ratio is directly related to the belief $\hat{\mu}_t$. A high P/D ratio is realised at the asset price peak in periods 3 and 4 of Figure 4, immediately after which stock prices start declining. Therefore, the P/D ratio negatively predicts future returns. Furthermore, forecast errors are also predictable: By Equation (4.35), forecast errors are a linear function of $\hat{\mu}_t - \hat{\mu}_{t-1}$. Since a high P/D ratio implies declining beliefs in the future, forecast errors are predictable in the same way as future returns, as in the data.
The aforementioned asset pricing implications are present even when dividends are completely exogenous, as in Adam et al. (2013). But the model considered here also contains a link between asset prices, output and dividends. The effective capital stock \( \tilde{K}_t \) is directly related to equity valuations \( \tilde{P}_t \) through Equation (4.34). Thus, the fluctuations in the stock market translate into corresponding fluctuations in investment, the capital stock and hence output. Thus, the presence of financial frictions in combination with learning leads to amplification of productivity shock, whereby under rational expectations amplification was zero.

It is also possible that this amplification mechanism is further enhanced by positive feedback from capital to expected dividends. This additional feedback, however, depends on the slope of the function \( \tilde{D} \). As demonstrated above, the expected dividend \( \tilde{D} \left( \tilde{k}_t, \tilde{K}_t \right) \) is increasing in the firm’s capital choice \( \tilde{k}_t \), but decreasing in the aggregate capital stock \( \tilde{K}_t \). In the equilibrium \( \left( \tilde{k}_t = \tilde{K}_t \right) \) it is increasing if financial frictions are sufficiently severe. This case is depicted in Panel (a) of Figure 5. When the degree of financial frictions is high, the credit constraint line is steep. Assume that the initial equilibrium in period 0 is at \( \tilde{P}_0 \) and \( \tilde{\mu}_0 \). Now consider the effect of a positive productivity shock in period 1 as before. The immediate effect will be a proportionate rise in stock prices and capital which leaves \( \tilde{P}_1 \) and \( \tilde{K}_1 \) unchanged, but raises beliefs from \( \tilde{\mu}_0 \) to \( \tilde{\mu}_1 \). This leads to higher stock prices at \( t = 2 \) and allows the firm to invest more and increase its expected profits \( \tilde{D} \left( \tilde{K}_2, \tilde{K}_2 \right) \). But this adds further to the rise in realised stock prices, further relaxing the borrowing constraint and increasing next period’s beliefs. Stock prices, beliefs, investment, and output all rise by more compared to a situation in which \( \tilde{D} \) is constant.\(^{14}\)

\(^{14}\)To my knowledge, this paper is the first to establish a positive feedback from fundamentals to beliefs under learning. Adam et al. (2012) also model economies with endogenous fundamentals. Their learning specification is similar, but the “dividend” in their asset pricing equation is simply the marginal utility of housing which is strictly decreasing in the level of the housing stock. Their model dynamics are therefore always as in case (b) described above.
However, this additional amplification channel only works when $\xi$ is sufficiently low. In Panel (b), $\xi$ is large and the firm is operating in the downward-sloping bit of the profit curve $\tilde{D}$. A relaxation of the borrowing constraint due to a rise in $\hat{\mu}$ still allows the firm to invest and produce more, but dividends fall in equilibrium. This is due to the general equilibrium forces mentioned earlier: The marginal product of capital has to fall in equilibrium, which in practice derives from an increase in real wages, effectively reducing the firm’s profits. In this situation, the endogenous response of dividends dampens rather than amplifies the dynamics of investment and asset prices.

This can also be seen algebraically. Under rational expectations, the derivative of log stock prices with respect to the productivity shock is simply $d \log P_t / d\xi_{t-s} = 1$ for all $t, s \geq 0$. With learning, the corresponding expression contains additional terms:

$$
\frac{d \log P_t}{d\xi_{t-s}} = 1 + \frac{1}{1 - \epsilon_D \left( \tilde{K}_t \right)} \frac{e^{\hat{\mu}_{t-1}}}{R - e^{\hat{\mu}_{t-1}}} \frac{d\hat{\mu}_t}{d\xi_{t-s}}
$$

(4.36)

where $\epsilon_D \left( \tilde{K}_t \right) = \frac{dD}{d\tilde{K}_t} \frac{\tilde{K}_t}{D}$ is the elasticity of expected dividends with respect to the capital stock. Learning adds a product of three terms. The last term is the effect of the productivity shock on subjective beliefs. The second term is the effect of beliefs on stock prices. Small variations in beliefs $\hat{\mu}_t$ cause large fluctuations in stock prices because the denominator $(R - e^{\hat{\mu}_{t-1}})$ is close to zero. The first term captures the general equilibrium effects mentioned earlier: when dividends rise after a relaxation of credit constraints, $\epsilon_k^d$ is positive and the term is greater than one, leading to additional amplification, and to dampening when $\epsilon_k^d$ is negative.

It can also be shown that the learning dynamics vanish as the economy approaches the unconstrained first-best:

$$
\frac{d \log P_t}{d\xi_{t-s}} \xrightarrow{\xi \to \infty} 1
$$

In other words, amplification rests on the interaction between learning and financial frictions, not on either of them separately. Intuitively, as financial frictions disappear, the economy moves into a region where the general equilibrium effects become so strong that any potential rise in beliefs or asset prices is countered by a fall in expected dividends.

In sum, the learning equilibrium can qualitatively account for a number of asset pricing facts and for the predictability of forecast errors. At the same time, the larger endogenous asset price volatility induces corresponding fluctuations in the slackness of the firm’s borrowing constraint. The presence of financial frictions thus magnifies productivity shocks, while there is no amplification under rational expectations. When financial frictions are sufficiently severe, a two-sided positive feedback loop emerges between beliefs, asset prices and firm profits, which further amplifies the dynamics. The presence of positive feedback from assets prices to profits depends on the relative strength of general equilibrium forces, which in the model in this section operate through the real wage.

I now turn to the development of a richer model that embeds the same mechanism, but
can also be taken to the data to study its quantitative implications.

5 Full model for quantitative analysis

This section embeds the mechanism discussed previously into a New-Keynesian business cycle model with a financial accelerator. Compared to the simple model in the previous section, there are a number of new elements. First, capital no longer has to be financed entirely out of debt. Instead, I allow for endogenous fluctuations in net worth. To prevent firms from saving until they become unconstrained, I impose exogenous entry and exit, as in Bernanke, Gertler and Gilchrist (1999). Second, I provide a microfoundation of the borrowing constraint by means of a limited commitment problem. Third, I add several standard business cycle frictions: nominal rigidities, which enables me to introduce monetary policy and later on analyse its effects on welfare under learning; and investment adjustment costs, which allow for a better fit of the model.

5.1 Model setup

The economy is closed and operates in discrete time. There are a number of different agents:

1. **Intermediate goods producers** (or simply *firms*) are at the heart of the model. They combine capital and and differentiated labour to produce a homogeneous intermediate good. They are financially constrained and borrow funds from households.

2. **Firm owners** only consume differentiated final goods. They trade shares in intermediate goods producers and receive dividend payments.

3. **Households** consume differentiated final goods and supply homogeneous labour to labour agencies. They lend funds to intermediate goods producers.

4. **Labour agencies** transform homogeneous household labour into differentiated labour services, which they sell to intermediate goods producers. They are owned by households.

5. **Final good producers** transform intermediate goods into differentiated final goods. They are owned by households.

6. **Capital goods producers** produce new capital goods from final consumption goods subject to an investment adjustment cost.

7. **The fiscal authority** sets certain tax rates to offset steady-state distortions from monopolistic competition.

8. **The central bank** sets nominal interest rates.
Since most elements of the model are standard, I focus on the financially constrained firms, firm owners, households and the central bank. Additional details are provided in Appendix A.

5.1.1 Households

A representative household with time-separable preferences maximises utility as follows:

\[
\max_{(C_t, L_t, B^g_t, B^j_t)} \mathbb{E}_0^T \sum_{t=0}^{\infty} \beta^t u(C_t, L_t)
\]

s.t. \( C_t = \tilde{w}_t L_t + B^g_t \cdot (1 + i_{t-1}) \frac{P_{t-1} \cdot B^g_{t-1}}{p_t} + \int_0^1 (B_{j,t} - R_{j,t-1} B_{j,t-1}) dj + \Pi_t \)

The utility function \( u \) satisfies standard concavity and Inada conditions and \( \beta \in (0, 1) \). Further, \( \tilde{w}_t \) is the real wage received by the household and \( L_t \) is the amount of labour supplied. \( B^g_t \) are real quantities of nominal one-period government bonds (in zero net supply) that pay a nominal interest rate \( i_t \) and \( p_t \) is the price level, defined below. Households also lend funds \( B^j_t \) to intermediate goods producers indexed by \( j \in [0, 1] \) at the real interest rate \( R_{j,t} \). These loans are the outcome of a contracting problem described later on. \( \Pi_t \) represents lump-sum profits and taxes. Finally, consumption \( C_t \) is itself a composite utility flow from of a variety of differentiated goods that takes the familiar CES form:

\[
C_t = \max_{C_{it}} \left( \int_0^1 (C_{it})^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}
\]

s.t. \( p_t C_t = \int_0^1 p_{it} C_{it} di \)

As usual, the price index \( p_t \) of composite consumption consistent with utility maximisation and the demand function for good \( i \) is given by

\[
p_t = \left( \int_0^1 (p_{it})^{1-\sigma} di \right)^{\frac{1}{1-\sigma}} ; \quad C_{it} = \left( \frac{p_{it}}{p_t} \right)^{-\sigma} C_t.
\] (5.1)

Consequently, the inflation rate is given by \( \pi_t = p_t/p_{t-1} \). The first order conditions of the household are also standard and given by

\[
\tilde{w}_t = -\frac{u_{CL}}{u_{C_t}} \quad \text{and} \quad 1 = \mathbb{E}_t^P \beta u_{C_{t+1}} \frac{1 + i_t}{\pi_t}.
\] (5.2, 5.3)

We can define the stochastic discount factor of the households as \( \Lambda_t = \beta u_{C_{t+1}}/u_{C_t} \).
5.1.2 Central bank

Like most of the New-Keynesian literature, the model is cashless, with the central bank affecting allocations in the presence of nominal rigidities by setting the nominal interest rate. In the baseline version of the model, I assume that the central bank conducts monetary policy through the use of a Taylor-type interest rate rule:

$$i_t = \rho_i i_{t-1} + (1 - \rho_i) \left(1/\beta + \pi_t + \phi_\pi (\pi_t - \pi_t^*)\right) \quad (5.4)$$

where $\pi_t^*$ is the central bank’s (time-varying) inflation target, $\rho_i$ is the degree of interest rate smoothing and $\phi_\pi > 1$.

5.1.3 Intermediate good producers (firms)

The production of intermediate goods is carried out by a continuum of firms, indexed $j \in [0, 1]$. Firm $j$ enters period $t$ with capital $K_{jt-1}$ and a stock of debt $B_{jt-1}$ which needs to be repaid at the gross real interest rate $R_{jt-1}$. First, capital is combined with a labour index $L_{jt}$ to produce output

$$Y_{jt} = (K_{jt-1})^\alpha (A_t L_{jt})^{1-\alpha}, \quad (5.5)$$

where $A_t$ is aggregate productivity. The labour index is a CES combination of differentiated labour services parallel to the differentiated final goods bought by the household:

$$L_{jt} = \max_{L_{jht}} \left(\int_0^1 (L_{jht})^{\frac{\sigma_{w^{-1}}}{\sigma_w}} dh\right)^{\frac{\sigma_w}{\sigma_{w^{-1}}}} \quad (5.6)$$

s.t. \( w_t p_t L_{jt} = \int_0^1 W_{jht} L_{jht} dh \) \quad (5.7)

The firm’s problem can then be treated as if the labour index was acquired in a competitive market at the real wage index $w_t$.\(^\text{15}\) Output is sold competitively to final good producers at price $q_t$. During production, the capital stock depreciates at rate $\delta$. This depreciated capital can be traded by the firm at the price $Q_t$.

At this point, the net worth of the firm is the difference between the value of its assets and its outstanding debt:

$$N_{jt} = q_t Y_{jt} - w_t L_{jt} + Q_t (1 - \delta) K_{jt-1} - R_{jt-1} B_{jt-1} \quad (5.8)$$

I assume that the firm exits with a probability $\gamma$. This probability is exogenous and independent across time and firms. As in Bernanke et al. (1999), exit prevents firms from becoming financially unconstrained. If a firm does not exit, it needs to pay out a fraction

\(^{15}\)This real wage index does not necessarily equal the wage $\tilde{w}_t$ received by households due to wage dispersion.
\( \zeta \) of its earnings as dividends, where earnings are given by
\[ E_{jt} = N_{jt} - Q_t K_{jt-1} + B_{jt-1}. \]
If it exits, it must pay out its entire net worth as dividends. It is subsequently replaced
by a new firm which receives the index \( j \). I assume that this new firm gets endowed with
a fixed number of shares, normalised to one, and is able to raise an initial amount of net
worth. This amount equals \( \omega (N_t - \zeta E_t) \) where \( \omega \in (0, 1) \) and \( N_t \) and \( E_t \) are aggregate
net worth and earnings, respectively.

The net worth of firm \( j \) after equity changes, entry and exit is given by
\[
\tilde{N}_{jt} = \begin{cases} 
N_{jt} - \zeta E_{jt} & \text{for continuing firms,} \\
\omega (N_t - \zeta E_t) & \text{for new firms.}
\end{cases}
\]
This firm then decides on the new stock of debt \( B_{jt} \) and the new capital stock \( K_{jt} \). Its
balance sheet must satisfy:
\[
Q_t K_{jt} = B^t_j + \tilde{N}_{tj} \tag{5.9}
\]
where the price of capital \( Q_t \) can vary in the presence of adjustment costs.

Firms maximise the present discounted value of their dividend payments using the discount
factor of their owners. In doing so, they face financial constraints. Before describing these
constraints though, I first turn to the description of the firms’ owners.

### 5.1.4 Firm owners

Firm owners differ from households in their capacity to own intermediate firms. The
representative firm owner is risk-neutral and discounts future income at the rate \( \tilde{\beta} = \beta G^{-\theta} < \beta \), with \( G \) being the growth rate of consumption in the non-stochastic steady
state. He can buy shares in firms indexed by \( j \in [0, 1] \). As described above, when a firm
exits it pays out its net worth \( N_{jt} \) as dividends, and is replaced by a new firm which raises
equity \( \omega (N_t - \zeta E_t) \). Let the set of exiting firms in each period \( t \) be denoted by \( \Gamma_t \subset [0, 1] \).
Then, the firm owner’s utility maximisation problem is given by:
\[
\max_{(C^f_t, S^f_t)} \mathbb{E}_0 \sum_{t=0}^{\infty} \tilde{\beta}^t C^f_t
\]
\footnote{The optimal dividend policy in this model would be to never pay dividends until exit. In this case, aggregate dividends would be proportional to aggregate net worth. This implies a dividend process that is not nearly as volatile as in the data, and thus makes it impossible to obtain good asset pricing properties even under learning. Imposing that firms need to pay out a fraction of their earnings greatly improves the quantitative fit of the model.}
\[ \text{s.t. } C^f_t + \int_0^1 S_{jt} P_{jt} dj = \int_{j \notin \Gamma_t} S_{jt-1} (P_{jt} + D_{jt}) dj \]
\[ + \int_{j \in \Gamma_t} [S_{jt-1} D_{jt} - \omega (N_t - \zeta E_t) + P_{jt}] dj \]
\[ S^f_{jt} \in [0, \bar{S}] \]

for some $\bar{S} > 1$. Here, firm owners’ consumption $C^f_t$ is the same aggregator of differentiated final goods as for households.

The first term on the right hand side of the budget constraint deals with continuing firms and is standard: Each share in firm $j$ pays dividends $D_{jt}$ and continues to trade, at price $P_{jt}$. The second term deals with firm entry and exit. If the household owns a share in the exiting firm $j$, he receives a terminal dividend. The firm is then delisted in the stock market, and so $S_{jt-1} P_{jt}$ does not appear. At the same time, a new firm $j$ appears which is able to raise a limited amount of equity $\omega (N_t - \zeta E_t)$ from the firm owner in exchange for a unit amount of shares that can be traded at price $P_{jt}$. In addition, upper and lower bounds on traded stock holdings are introduced to make firm owners’ demand for stocks finite under arbitrary beliefs, as in the stylised model of the previous section. In equilibrium, they are never binding.

The first order conditions of the firm owner are

\[ S_{jt} = \begin{cases} 
0 & \text{if } P_{jt} > \tilde{\beta}_E P_t \\
\tilde{S} & \text{if } P_{jt} < \tilde{\beta}_E P_t \\
\tilde{S} & \text{if } P_{jt} = \tilde{\beta}_E P_t
\end{cases} \]

\[ \in [0, \bar{S}] \]

\[ \text{if } P_{jt} > \tilde{\beta}_E P_t \]

\[ \text{if } P_{jt} = \tilde{\beta}_E P_t \]

\[ \text{if } P_{jt} < \tilde{\beta}_E P_t \]

5.1.5 Borrowing constraint

In choosing their debt holdings, firms are subject to a borrowing constraint. The constraint is the solution to a particular limited commitment problem in which the outside option for the lender in the event of default depends on equity valuations.

Each period, lenders (households) and borrowers (firms) meet to decide on the lending of funds. Pairings are anonymous to rule out repeated interactions. The incompleteness of contracts imposed is that repayment of loans cannot be made contingent. Only the size $B_{jt}$ and the interest rate $R_{jt}$ of the loan can be contracted in period $t$. Both the lender (a household) and the firm have to agree on a contract $(B_{jt}, R_{jt})$. Moreover, there is limited commitment in the sense that at the end of the period, but before the realisation of next period’s shocks, firm $j$ can always choose to enter a state of default. In this case, the value of the debt repayment must be renegotiated. If the negotiations are successful, then wealth is effectively shifted from creditors to debtors. The outside option of this renegotiation process is the seizure of the firm by the lender, in which case the current firm owners receive zero.
The lender, a household, does not have the ability to run the firm though. The usual assumption in the literature is that she has to liquidate the firm’s asset in this case. In this model, the lender can always liquidate as well. In this case, all debt and a fraction $1 - \xi$ of the firm’s capital is destroyed. The remaining capital can be sold in the next period, resulting in a total recovery value of $\xi Q_{t+1} K_{jt}$. On top of this, with some probability $x$ (independent across time and firms), the lender gets the opportunity to “restructure” the firm. Restructuring means that, similar to Chapter 11 bankruptcy proceedings, the firm gets partial debt relief but remains operational. I assume that the lender has to sell the firm to another firm owner, retaining a fraction $\xi$ of the initial debt. It will turn out in equilibrium that the recovery value in this case is just $\xi \left( P_{jt} + B_{jt} \right)$ and that lenders always prefer restructuring to liquidation. The debt contract then takes the form of a leverage constraint in which total firm value is a weighted average of liquidation and market value:

$$B_{jt} \leq \xi \left( x E_t^P A_{t+1} Q_{t+1} \xi K_{jt} + (1 - x) \left( P_{jt} + B_{jt} \right) \right)$$

(5.14)

5.1.6 Further model elements and market clearing

Final good producers, indexed by $i \in [0, 1]$, combine the homogeneous intermediate good into a differentiated final good using a one-for-one technology. Their revenue is subsidised by the government at the rate $\tau$.\textsuperscript{17} Per-period profits of producer $i$ are $\Pi_{Yit} = (1 + \tau) \left( p_{it}/p_t \right) Y_{it} - q_{it} Y_{it}$. They are subject to a Calvo price setting friction: Every period, each final-good producer can change his price only with probability $1 - \kappa$, independent across time and producers. Similarly, labour agencies (indexed by $h \in [0, 1]$) combine the homogeneous labour provided by households into differentiated labour goods which they sell on to intermediate good producers. Labour agencies’ revenue is subsidised at the rate $\tau_w$, the per-period profit of agency $h$ is $\Pi_{Lht} = (1 + \tau) \left( W_{ht}/p_t \right) L_{ht} - \tilde{w}_t L_{ht}$ and each agency can change its nominal wage $W_{ht}$ only with probability $1 - \kappa_w$. The government collects subsidies as lump sum taxes from households and runs a balanced budget each period. The government sets the subsidy rates such that under flexible prices, the markup over marginal cost is zero in both the labour and output markets.

Capital goods producers produce new capital goods subject to standard investment adjustment costs and have profits $\Pi_{I}$. Thus, the total amount of lump-sum payments $\Pi_t$ received by the household is the sum of the profits of all final good producers, labour agencies and capital goods producers, minus the sum of all subsidies.

\textsuperscript{17}This assumption is standard in the New-Keynesian literature. It eliminates distortions from monopolistic competition where firms price above marginal cost. The only distortion is then due to sticky prices, which simplifies the solution by perturbation methods.
Finally, the exogenous stochastic processes are productivity and the inflation target shock:

\[
\log A_t = t (1 - \rho) \log G + \rho (\log A_{t-1} + \log G) + \log \varepsilon_{At} \tag{5.15}
\]

\[
\pi^*_t = \rho \pi^*_{t-1} + \log \varepsilon_{\pi t} \tag{5.16}
\]

\[
\varepsilon_{At} \sim \mathcal{N} \left(0, \sigma^2_A\right) \tag{5.17}
\]

\[
\varepsilon_{\pi t} \sim \mathcal{N} \left(0, \sigma^2_{\pi}\right) \tag{5.18}
\]

Market clearing needs to take into account the distortions from price and wage dispersion. All market clearing conditions are listed in Appendix A.

5.2 Rational expectations equilibrium

I first describe the equilibrium under rational expectations. Equilibrium is a set of stochastic processes for prices and allocations, a set of strategies in the limited commitment game, and an expectation measure \(\mathcal{P}\) such that the following holds for all states and time periods: Markets clear; allocations solve the optimisation programmes of all agents given prices and expectations \(\mathcal{P}\); the strategies in the limited commitment game are a subgame-perfect Nash equilibrium for all lender-borrower pairs; and the measure \(\mathcal{P}\) coincides with the actual probability measure induced by the equilibrium.

Under appropriate parameter restrictions, there exists a rational expectations equilibrium characterised by the following properties (proofs and characterisation of the restrictions are relegated to Appendix B):

1. All firms choose the same capital-labour ratio \(K_{jt}/L_{jt}\). This allows one to define an aggregate production function and an internal rate of return on capital:

\[
Y_t = \alpha K_{t-1}^\alpha \left(\frac{A_t}{\bar{L}_{t-1}}\right)^{1-\alpha} \tag{5.19}
\]

\[
R^k_t = q_t \alpha \frac{Y_t}{K_t} + Q_t (1 - \delta) K_{t-1} \tag{5.20}
\]

2. The expected return on capital is higher than the internal return on debt: \(\mathbb{E}_t R^k_{t+1} > R_{jt}\).

3. At any time \(t\), the stock market valuation \(P_{jt}\) of a firm \(j\) is proportional to its net worth after entry and exit \(\mathcal{N}_{jt}\). This permits one to write an aggregate stock market index as

\[
P_t = \int_0^1 P_{jt} = \beta \mathbb{E}_t \left[ D_{t+1} + \frac{1 - \gamma}{1 - \gamma + \gamma \omega} P_{t+1} \right]. \tag{5.21}
\]

The “correction” term in the continuation value \(P_{t+1}\) can be understood as follows. The stock market index \(P_t\) is the value of all currently existing firms, not including firms that are not yet born. In \(t+1\), a fraction \(\gamma\) of firms exit and pay dividends
γN_t. The remaining firms are left with net worth \((1 - γ) N_{t+1}\) and pay dividends \((1 - γ) \zeta E_t\) and, but a mass \(γ\) of new firms also enters, each endowed with initial net worth \(ω (N_t - \zeta E_t)\).

4. Borrowers never default on the equilibrium path and borrow at the risk-free rate

\[
R_{jt} = R_t = (E_t \beta u_{Ct+1}/u_{Ct})^{-1}.
\] (5.22)

The lender only accepts debt payments up to a certain limit \(\bar{B}_{jt}\). The firm always exhausts this limit, \(B_{jt} = \bar{B}_{jt}\), which is proportional to the firm’s net worth \(\bar{N}_{jt}\). If the firm defaulted and the lender seized the firm, she would always prefer restructuring to liquidation. Intuitively, this is because capital is more valuable inside the firm than outside of it. Because acquiring capital is difficult due to financial frictions, firm owners will always pay the lender a higher price for a restructured, operational firm than for its capital stock alone.

5. As a consequence of the previous properties of the equilibrium, all firms can be aggregated. Aggregate debt, capital, and net worth are sufficient to describe the intermediate goods sector and evolve as

\[
N_t = R_{it} K_{t-1} - R_{t-1} B_{t-1}
\] (5.23)

\[
Q_t K_t = (1 - γ + γω) ((1 - \zeta) N_t + \zeta (B_{t-1} - Q_{t-1} K_{t-1})) + B_t
\] (5.24)

\[
B_t = x E_t A_{t+1} Q_{t+1} ξ K_t + (1 - x) ξ (P_t + B_t).\]
(5.25)

I solve for a second-order approximation of this rational expectations equilibrium around its non-stochastic steady state.

### 5.3 Learning equilibrium

I introduce learning about stock market valuations as in the simple model of Section 4. One slight complication is now that there is a continuum of firms to be priced in the market. In this respect, I retain the belief that the stock price of an individual firm is proportional to firm net worth, as is the case under rational expectations. As such, under \(P\),

\[
P_{jt} = \frac{N_{jt}}{N_t} P_t.
\] (5.26)

But while investors know how to price individual stocks by observing the valuation of the market, they are uncertain about the evolution of the market itself. As in the simple model of the previous section, I impose the same beliefs about aggregate stock prices as in the last section along with the other assumptions (equations (4.25)-(4.28)), including expectations on other variables that are conditionally consistent with outcomes on the equilibrium path: For any variable \(x_t\), any future date \(t + τ\), and any sequence of stock prices and
fundamentals (denoted by \( u_t \)) which is on the equilibrium path, agents’ conditional beliefs coincide with equilibrium outcomes: \( \mathbb{E}_t^P \left[ x_{t+\tau} \mid u_t, P_t, \ldots, u_{t+\tau}, P_{t+\tau} \right] = x_{t+\tau} \) almost surely.

In practice, I solve the model using a two-stage procedure. The first stage is to solve for the policy functions and beliefs under \( P \). The Kalman filtering equations that describe beliefs about stock prices are as follows:

\[
\begin{align*}
\log P_t &= \log P_{t-1} + \hat{\mu}_{t-1} - \frac{\sigma^2_v + \sigma^2_\eta}{2} + z_t \\
\hat{\mu}_t &= \hat{\mu}_{t-1} - \frac{\sigma^2_v}{2} + g z_t
\end{align*}
\]

where \( \hat{\mu}_t \) is the mean belief about the trend in stock price growth, and \( z_t \) is the forecast error. Under the subjective beliefs \( P \), it is normally distributed white noise. I impose that beliefs about any other endogenous variable are consistent with model outcomes conditional on the evolution of stock prices, and so beliefs and policy functions can be calculated much in the same way as under rational expectations, taking \( z_t \) as an exogenous shock process. The market clearing condition for stocks does not enter this first stage of the problem. Adding it would effectively impose that beliefs about stock prices, too, be consistent with equilibrium outcomes - and the solution would collapse to the rational expectations equilibrium. Now if \( x_t \) is the set of model variables and \( u_t \) the set of exogenous shocks, solving this first stage leads to a policy function \( x_t = h (x_{t-1}, u_t, z_t) \).

The second stage of the model consists in finding the value for \( z_t \) which leads to market clearing in the stock market and thereby establishes equilibrium. This results in a mapping from the state variables and exogenous shocks to the perceived forecast error \( r : (x_{t-1}, u_t) \mapsto z_t \). Clearly, this function generally does not make \( z_t \) an iid disturbance in equilibrium. This is why agents make systematic forecast errors. The complete solution of the model is given by \( x_t = h (x_{t-1}, u_t, r (x_{t-1}, u_t)) \). A complete description of a second-order perturbation of this solution is contained in Appendix C.

The fact that I solve for a policy function in the first stage does not mean that agents necessarily have to know and understand all the equations in the model. It is enough that they are endowed with beliefs that are consistent with equilibrium outcomes conditional on any realisation of stock prices. This requires no more knowledge than under rational expectations.

Some readers might also object that the belief system about stock prices is misspecified in the sense that agents are never able to learn the true law of motion for \( P_t \) (where \( P_t \) is not a random walk and depends on several state variables). However, this is a deliberate choice. The chosen form of subjective beliefs captures several aspects of reality: People think that stock prices are well approximated by random walks; still, they try to identify predictability in prices; their subjective expectations seem to ignore the degree of mean reversion in returns observed in the data, instead overly extrapolating past observation; and they think that the laws governing prices are changing over time, so that observations
from the distant past are only of limited usefulness in predicting the future.

5.4 Frictionless benchmark

Because my aim is to gauge the importance of financial frictions for business cycle analysis, I will also use a benchmark model without financial frictions to which the model can be compared to. This benchmark model will be identical to the rational expectations model above, except that intermediate firms are now owned by households, and face no frictions in accessing external finance.

5.5 Choice of parameters

I partition the set of parameters into two groups. The first set of parameters is calibrated to first moments, and the second set is estimated by simulated method of moments.

5.5.1 Calibration

The capital share in production is set to \( \alpha = 0.33 \), implying a labour share in output of two thirds. The depreciation rate \( \delta = 0.025 \) corresponds to 10\% annual depreciation. The non-stochastic trend productivity growth rate \( G \) is set to its post-war average of 1.64\% annually. The persistence of the temporary component of productivity is set to 0.95.

The discount factor of the household is set such that the steady-state interest rate matches the average annual real return on Treasury bills of 2\%, implying a discount factor \( \beta = 0.9991 \). The firm owners’ discount factor is set such that in the non-stochastic steady state, bonds and stocks have the same return \( (\hat{\beta} = 0.995) \). The elasticity of substitution between varieties of the final consumption good, as well as that between varieties of labour used in production, is set to \( \sigma = \sigma_w = 4 \). The Frisch elasticity of labour supply is set to three, implying \( \phi = 0.33 \).

The strength of monetary policy reaction to inflation is set to \( \phi_m = 1.5 \), while the degree of nominal rate smoothing is set to \( \rho_i = 0.5 \). The inflation target is set to \( \pi^* = 0 \). The persistence of the inflation target is set to \( \rho_n = 0.99 \).

Four parameters describe the structure of financial constraints: \( x \), the probability of restructuring after default; \( \xi \), the tightness of the borrowing constraint; \( \omega \), the equity received by new firms relative to average equity; and \( \gamma \), the rate of firm exit and entry. I calibrate the restructuring rate \( x \) to equal 9.3\%. This is the fraction of US business bankruptcy filings in 2006 which filed for Chapter 11 instead of Chapter 7, and which subsequently emerged from bankruptcy with an approved restructuring plan. I have to restrict myself to 2006 because it is the only year for which this number can be constructed. Public bankruptcy data shows that the fraction of Chapter 11 cases as opposed to Chapter 7 cases fluctuates around 28\%, and several papers analysing sub-samples of Chapter 11 filings
arrive at confirmation rates between 29% and 64%, which suggests that the 2006 number is reasonable (a sensitivity check is included in Section 6.5).\footnote{Data on bankruptcies by chapter are available at \url{http://www.uscourts.gov/Statistics/BankruptcyStatistics.aspx}. Data on Chapter 11 outcomes are analysed in various samples by Flynn and Crewson (2009), Warren and Westbrook (2009), Lawton (2012), and Altman (2014).}

The remaining three parameters are chosen such that the non-stochastic steady state of the model jointly matches the US average investment share in output of 20%, average debt-to-equity ratio of 1:1 (as recorded in the Fed Flow of Funds), and average quarterly P/D ratio of 139 (taken from the S&P500). The parameter values thus are $\gamma = 0.010$, $\xi = 0.38$, and $\omega = 0.55$. Intuitively, a larger value of $\xi$ relaxes the borrowing constraint, leading to higher leverage. A larger value of $\omega$ means that new firms enter with more equity, increasing aggregate net worth and capital. Since returns on capital are diminishing, this raises the investment share in output. A higher rate of exit $\gamma$ makes firms shorter-lived, reducing the firm’s value relative to its current dividend payments, thereby lowering the P/D ratio.

### 5.5.2 Estimation

The remaining parameters are the standard deviations of the technology and monetary shocks ($\sigma_A, \sigma_\pi$), the size of investment adjustment costs ($\psi$), the degree of nominal price and wage rigidities ($\kappa, \kappa_w$), the fraction of dividends paid out as earnings by continuing firms ($\zeta$), and the learning gain ($g$). Since my goal is to see how well the model can do in terms of matching both business cycle and asset pricing facts, I estimate these parameters to minimise the distance to a set of second moments pertaining to both. I use the variances of the nominal interest rate, the inflation rate, real output, consumption, investment, employment, as well as real dividend payments and stock prices in the data.\footnote{All variables are at quarterly frequency and HP-detrended. I use CPI inflation as the inflation measure and the Federal Funds rate as the nominal interest rate. Employment is total non-farm payroll employment. Consumption is the sum of services and non-durable private consumption. Investment is the sum of private non-residential fixed investment and durable consumption. Output is the sum of consumption and investment. Dividend payments are the four-quarter moving average of S&P 500 dividends and stock prices is the S&P500 index.}

The set of estimated parameters $\theta$ solves

$$\min_{\theta \in \mathcal{A}} \left( m(\theta) - \hat{m} \right)^\top W \left( m(\theta) - \hat{m} \right)$$

where $m(\theta)$ are moments obtained from model simulation paths with 50,000 periods, $\hat{m}$ are the estimated moments in the data, and $W$ is a weighting matrix.\footnote{I choose $W = \text{diag}\left( \hat{\Sigma} \right)^{-1}$ where $\hat{\Sigma}$ is the covariance matrix of the data moments, estimated using a Newey-West kernel with optimal lag order. This choice of $W$ leads to a consistent (albeit not fully efficient) estimator that places more weight on moments which are more precisely estimated in the data.}

I also impose that $\theta$ has to lie in a subset $\mathcal{A}$ of the parameter space which rules out deterministic oscillations of impulse response functions.\footnote{To be precise, $\theta \notin \mathcal{A}$ if there exists an impulse response of stock prices with positive peak value also having a negative value of more than 20% of the peak value.} Parameters outside this region would fit the moments well but can be ruled out from prior knowledge about the shape of impulse responses.
Table 3: Estimated parameters.

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
<th>s.e.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_A$</td>
<td>0.00685</td>
<td>(0.00150)</td>
</tr>
<tr>
<td>$\sigma_\pi$</td>
<td>8.94·10^{-4}</td>
<td>(2.55·10^{-4})</td>
</tr>
<tr>
<td>$\psi$</td>
<td>10.6</td>
<td>(3.10)</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.809</td>
<td>(0.123)</td>
</tr>
<tr>
<td>$\kappa_w$</td>
<td>0.943</td>
<td>(0.0233)</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>0.8209</td>
<td>(0.0197)</td>
</tr>
<tr>
<td>$g$</td>
<td>0.00460</td>
<td>(9.76·10^{-4})</td>
</tr>
</tbody>
</table>

Parameters as estimated by simulated method of moments. Asymptotic standard errors in parentheses.

3 summarises the results of the estimation. It selects a rather high degree of adjustment costs (although imprecisely estimated) and of nominal rigidities. The fraction of earnings paid out as dividends is fitted to more than 80%, higher than the actual historical average for the S&P500 at about 50%. This suggests that my assumption about the behaviour of dividends is overly simplistic. This is a commonly encountered problem (Covas and Den Haan, 2012) which I have to leave for future research. Finally, a low estimate for the learning gain $g$ implies that agents believe the predictability of stock prices to be small.

6 Results

6.1 Business cycle and asset price moments

To get a better understanding of the quantitative properties of the model, Table 4 reviews key business cycle moments as well as the statistics describing stock price volatility and return predictability across model specifications. The moments for the estimated learning model are in Column (1), while Columns (2) and (3) contain the corresponding moments for the model under rational expectations and the frictionless benchmark. Here, the parameters are held constant at the same values as for the learning model. By nature of the estimation, the learning model has the best fit across Columns (1) to (3). The comparison serves to single out the contribution of learning and financial frictions to the fit. In Column (4), I also let the rational expectations version of the model compete with the learning model: I re-estimate the same model parameters ($\sigma_A, \sigma_\pi, \psi, \kappa, \kappa_w$) using the same objective function as before, minimising the distance between the moments in the data and in the model.\(^{22}\)

\(^{22}\)The re-estimated parameters are $\sigma_A = .00664$, $\sigma_\pi = .00112$, $\psi = .0557$, $\zeta = .446$, $\kappa = .797$ and $\kappa_w = .915$. Standard errors around the estimates are large.
Table 4: Comparing moments in the data and across model specifications.

<table>
<thead>
<tr>
<th>moment</th>
<th>data</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
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<td>1.43%</td>
<td>1.41%</td>
<td>1.76%</td>
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<tr>
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<td>1.02</td>
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<td>.36</td>
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<tr>
<td>dividends</td>
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<td>2.99</td>
<td>3.07</td>
<td>2.36</td>
<td>-</td>
</tr>
<tr>
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<td>2.63</td>
<td>1.81</td>
<td>.21</td>
<td>-</td>
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<td>.408</td>
<td>.131</td>
<td>.037</td>
<td>-</td>
</tr>
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<td></td>
<td>σ(\frac{R_{stock, t,t+1}}{D_t})</td>
<td>.335</td>
<td>.150</td>
<td>.011</td>
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<td>-.240</td>
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<td>-</td>
</tr>
<tr>
<td>predictability</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>.907</td>
<td>.432</td>
<td>.758</td>
<td>-</td>
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Quarterly US data 1962Q1-2012Q4. \( \pi_t \) is quarterly CPI inflation. \( i_t \) is the Federal Funds rate. \( L_t \) is total non-farm payroll employment. Consumption \( C_t \) consists of services and non-durable private consumption. Investment \( I_t \) consists of private non-residential fixed investment and durable consumption. Output \( Y_t \) is the sum of consumption and investment. Dividends \( D_t \) are four-quarter moving averages of S&P 500 dividends. The stock price index \( P_t \) is the S&P 500.

The first row reports the standard deviation of detrended output. By this measure, output fluctuations under learning are double the size of those under rational expectations; in other words, learning adds considerable endogenous amplification to the model. In fact, the standard deviations of shocks in Column (4) needed to generate the same amount of output volatility are much larger.

The following rows report the relative volatilities of key macroeconomic time series. The learning model matches all of them well. Shutting down learning leads to a sharp drop in the volatility of investment and a corresponding increase in the volatility of consumption. This is because the estimated learning model features a rather high level of investment adjustment costs to match investment volatility. Without high adjustment costs, the learning model would transform the high degree of observed asset price volatility into a counterfactually high degree of investment volatility, i.e. the amplification mechanism would be too strong. When the amplification is shut down in Columns (2) and (3), investment becomes very smooth and as a flip side, consumption is prevented from being smoothed. Looking at inflation and interest rates, shutting down learning increases their...
volatility relative to output, although this is mostly due to the decrease in output volatility itself. The volatility of employment and dividends is not much affected. Finally, the re-estimated model in Column (4) is able to match all moments just as well as under learning with the exception of dividends.

Big differences appear in the asset price statistics. The learning model is able to approach the volatility of prices and returns to a degree which is impossible to achieve even in the re-estimated RE model.\textsuperscript{23} The ability to generate realistic stock price volatility is of course at the heart of the amplification mechanism, since stock market valuations enter firms’ borrowing constraints. Moreover, returns are negatively predicted by the P/D ratio.

At the one year horizon, the predictability is very similar to that found in the data. At the five-year horizon however, it is too high. This is also reflected in the fact that the P/D ratio decays too fast, as documented in the last row. In a sense, this is not really surprising: The learning model has only one parameter (the learning gain $g$) to match all statistics pertaining to stock prices. In particular, it can be shown that a higher gain increases volatility, but reduces persistence of prices. A richer belief specification could possibly achieve a better fit, but at the expense of introducing additional parameters.

\textbf{6.2 Impulse response functions}

Looking at impulse response functions reveals some of the workings of the amplification mechanism at play. Figure 6 plots the impulse responses to a persistent productivity shock. Red solid lines represent the learning equilibrium, blue dashed lines represent the rational expectations version, and black thin lines represent the frictionless benchmark. Looking at the response of output $Y_t$, one can see quite clearly how the rational expectations version generates only a slightly amplified reaction to the shock compared to the frictionless benchmark. With learning, however, output rises by almost double the amount and the response is hump-shaped. Consumption $C_t$ and investment $I_t$ also exhibit much stronger responses, and employment $L_t$ also rises considerably (without learning, it actually contracts). This amplification is due to the large rise in the stock prices $P_t$, even though the degree of dependence of borrowing constraints on stock prices (measured by $x$) is less than 10\%. Under rational expectations, stock prices stay virtually flat, thus producing almost no amplification through the equity price channel. The behaviour of dividends $D_t$ reveals that initially, dividends fall, partly offsetting the rise of beliefs, but then overshoot their rational expectations counterpart.

The feedback loop is also present after an interest rate shock. Figure 7 plots the response to a temporary reduction in the nominal interest rate. Again, all macroeconomic aggregates rise substantially more under learning than under both rational expectations and the frictionless benchmark. The monetary stimulus increases stock prices and thus relaxes

\textsuperscript{23}In fact, it would be possible to obtain an even better fit of the asset pricing moments, but at the expense of business cycle moments.
credit constraints. The consequent increase in investment demand further raises immediate inflationary pressure, which in the model leads to the central bank undoing its interest rate reaction, acting according to its interest rate rule. Here, asset prices undershoot after the stimulus and remain depressed for a long period of time, leading to lower output, inflation and employment with a trough at about 16 quarters after the stimulus. This boom and bust pattern is not present under rational expectations.

6.3 Does learning matter?

The discussion so far has mainly focused on how large swings in asset prices lead to large swings in real activity through their effect on credit constraints. This raises the question of whether learning is necessary for the story at all - maybe any theory which replicates the same asset price dynamics also replicates the same business cycle outcomes? To answer it, I set up the following experiment. I replace the stock market value $P_t$ in the borrowing constraint (5.25) with an exogenous process $V_t$ that has the same law of motion as the stock price under learning. More precisely, I fit an ARMA(10,5)-process for $V_t$ such that its impulse responses are as close as possible to those of $P_t$ under learning (the exogenous shock in the ARMA-process are the productivity and monetary shocks). I then solve this model, but with rational expectations. If learning only matters because it affects stock price dynamics, then this hypothetical model should have identical dynamics to the model under learning.

Impulse responses to a one-standard deviation innovation in $\varepsilon_{At}$. Stock prices $P_t$, dividends $D_t$, output $Y_t$, investment $I_t$, consumption $C_t$, and employment $L_t$ are in 100*log deviations. The interest rate $i_t$ and inflation $\pi_t$ are in percentage-point deviations.
Figure 7: Impulse responses to a monetary shock.

Impulse responses to a one-standard deviation innovation in $\varepsilon_{mt}$. Stock prices $V_t$, dividends $D_t$, output $Y_t$, investment $I_t$, consumption $C_t$ and employment $L_t$ are in 100*log deviations. The interest rate $i_t$ and inflation $\pi_t$ are in percentage point deviations.

Figure 8 shows that this is not the case. The ARMA-process fits stock prices well: The impulse response of $P_t$ under learning and $V_t$ in the counterfactual experiment are indistinguishable. But after a positive productivity shock, output, investment and consumption rise much more under learning, even though the counterfactual model has the same stock price dynamics by construction. The reason is found in the fact that expectations matter beyond stock prices: For interest rates equilibrating loan demand by firms and supply by households; for inflation and wages, set by forward-looking Calvo price and wage setters; and for the borrowing constraint (5.25) itself, since it depends on the expected liquidation value of capital $E_tQ_{t+1}$. This last channel is in fact crucial for the additional amplification.

Under learning, agents do not understand that the increase in stock prices is temporary. By over-predicting the slackness of borrowing constraints, they also over-predict the amount of investment taking place in the future, and hence the future price of capital. Lenders, predicting a high value of the firm’s capital stock, are then willing to lend more to firms today. Thus, identical firm market value still leads to easier access to credit under learning.

This illustrates how expectations in financial markets, over and above their effect on asset prices, can have important effects on the real economy.

6.4 Relation to survey evidence on expectations

Agents in the learning equilibrium make systematic, predictable forecast errors. The patterns of predictability are testable model implications. As it turns out, they are sur-
Figure 8: Does learning matter?

Solid red line: Impulse response to a one-standard deviation positive productivity shock under learning. Black dash-dotted line: Impulse response to a hypothetical rational expectations model with stock price dynamics identical to those under learning (see text).

Surprisingly consistent with survey data.

Importantly, agents in the model do make systematic forecast errors not only about stock prices, but also about almost all other endogenous model variables. This is despite the fact that, conditional on stock prices, agents’ beliefs are model-consistent. A systematic mistake in predicting stock prices will still translate into a corresponding mistake in predicting the tightness of borrowing constraints, and hence investment, output, and so forth. I can therefore compare forecast errors on many model variables with the data. Predictability in all variables arises from the introduction of only a single parameter (the learning gain), so this is a potentially tough test for the model.

Figure 9 repeats the scatter plot of Section 3, contrasting expected and realised one-year ahead returns in a model simulation. The same pattern as in the data emerges: When the P/D ratio is high, return expectations are most optimistic. In the learning model, this has a causal interpretation: high return expectations drive up stock prices. At the same time, realised future returns are on average low when the P/D ratio is high. This is because the P/D ratio is mean-reverting (which agents do not realise, instead extrapolating past price growth into the future): At the peak of investor optimism, realised price growth is already reversing and expectations are due to be revised downward, pushing down prices towards their long-run mean.

Table 5 compares the correlation of forecast errors with the predictors discussed in Section 3, in the data and the model with learning. Note that the correlations under rational
Expected and realised nominal returns along a simulated path of model with learning. Simulation length 200 periods. Theoretical correlation coefficient for expected returns $\rho = .30$, for realised returns $\rho = -.11$.

Column (1) reports predictability based on the P/D ratio. As already discussed, the correlation is negative, and it is even stronger in the data than in the model. The correlation is also negative for the other macroeconomic aggregates, again with a higher magnitude in the data. Column (2) repeats the exercise for the growth rate of the P/D ratio. This measure positively predicts forecast errors: When the stock market is rising, people are on average not optimistic enough about returns and economic activity. This is because expectations about asset prices (and hence lending conditions) adjust only slowly. Here, the degree of correspondence with the data is striking. Only forecast errors on consumption remain too weakly predictable in the model.

The model also does very well in terms of the Coibion and Gorodnichenko (2010) predictability statistic. As documented in Column (3), forecast errors on macro aggregates are predictable by the direction of the forecast revision, again reflecting slow belief adjustment. This is also true in the model. For stock returns, the model predicts a negative correlation of return forecast errors with the revision, but the CFO survey does not allow for the construction of a corresponding statistic.

Forecast error predictability is illustrated graphically in Figure 10. The solid red line is a standard impulse response function to a technology shock in the learning model. Consider the following thought experiment. Suppose that at the time of the shock, the economy is in steady state and that no further shocks are realised. Under rational expectations, the mean expectation about the future path of the economy at any point in time is then equal to the impulse response itself. Under learning, however, subjective expectations do not coincide with the impulse response. The green dashed line of Figure 5 depicts the mean
Table 5: Forecast errors under learning and in the data.

<table>
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<td>model data</td>
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<tr>
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<td>.10 .05</td>
<td>-.25 -.27</td>
<td>.15 .43</td>
</tr>
</tbody>
</table>

Data as in Table 2. Model correlations are exact theoretical correlations. Stock returns are nominal returns. Unemployment in the model is taken to be $u_t = 1 - L_t$.

Figure 10: Actual versus expected impulse response.


subjective forecast at the peak of the stock market. Agents do not foresee the decline in the stock market, and instead extrapolate high stock price growth into the future. Because stock market valuations matter for access to credit, agents also forecast loose borrowing conditions and are too optimistic about investment and output as well.\textsuperscript{24} The green dashed line (expectations) is above the red solid line (realisation) when the P/D ratio is high: It negatively predicts forecast errors. Next, the change in the P/D ratio predicts forecast errors the other way around, which is illustrated with the blue dotted line. This is the forecast made at a time in which the P/D ratio is rising fast. In this situation, agents under-predict the size of the coming boom in the stock market and real activity. The blue dotted line is below the red solid line: Forecast errors are positive when P/D ratio growth is positive, too.

\textsuperscript{24}Note that the forecasts for output and investment are still downward-sloping as agents are perfectly aware of the mean reversion of productivity. Their long-run forecast has permanently higher stock prices, output and investment because of easier access to credit, while productivity remains at the steady-state level.
6.5 Sensitivity checks

The amount of endogenous amplification under learning relies on the dynamics of stock prices, and also on general equilibrium effects (as was already illustrated in the simplified model). To gain an idea of the sensitivity of the results, Figure 11 plots the volatility of output and stock prices as a function of a set of influential parameters.

Panel (a) shows the role of the restructuring rate $x$ which parametrises the dependency of borrowing constraints on firm market value (as opposed to the liquidation value of its capital). Not surprisingly, this parameter is key in driving amplification under learning. The point $x = 0$ is a special case. At this point, stock prices have no allocative role for the economy. Apart from stock prices then, the model dynamics under learning and rational expectations then coincide perfectly. Another special point is $x = 1$, where borrowing constraints depend exclusively on stock prices. This was the case analysed in the simple model of Section 4. In the full model, this degree of stock price dependency leads to so
much amplification that the belief dynamics become explosive, and no stable equilibrium exists. Another remarkable fact is that the rational expectations solution barely depends on the parameter $x$. This might be one reason why the distinction between market and liquidation value has not featured prominently in the existing literature on firm credit frictions.

Panel (b) shows the dependency on the average tightness of credit frictions $\xi$. Amplification is hump-shaped with respect to this parameter. At $\xi = 0$, no collateral is pledgeable and firms cannot borrow at all. In this case, fluctuations in stock prices do not matter and learning does not introduce any amplification. On the other hand, as pledgeability increases to its maximum value (beyond which a steady state with permanently binding borrowing constraint does not exist), amplification also disappears. This mirrors the analysis of the simplified model: when borrowing constraints relax, general equilibrium effects in the form of wage and interest rate changes offset amplification.

Panels (c) and (d) document the importance of wage rigidity and investment adjustment costs, respectively. Wage rigidity helps to improve the fit of the model, but at the same time it also helps amplification. When credit constraints relax, a larger degree of rigidity mitigates the negative general equilibrium effect on firm profits, raising internal net worth and stock prices. In contrast, adjustment costs dampen amplification since they make investment more costly. The sensitivity to these parameters is much greater under learning than under rational expectations.

7 Implications for monetary policy

In the model with learning, changes in subjective expectations in financial markets lead to large and inefficient asset price and business cycle fluctuations. A natural question, therefore, is whether policy should intervene in order to stabilise asset prices. The question is not so much whether a monetary policy reaction to asset prices is desirable at all in theory, but whether the benefits are predicted to be sizeable enough to warrant the inclusion of such a volatile indicator into a monetary policy framework in practice. This question has been discussed for a long time by academics and policymakers alike, but generally answered in the negative. However, the discussion has generally centred on models in which asset price fluctuations are either efficient, or the inefficient ("bubble") component cannot be reduced by raising interest rates. However, when expectations in financial markets are not rational, the potential benefits of reacting to asset prices are much larger. There is a possibility that the central bank can mitigate excessive fluctuations in subjective expectations, thereby substantially stabilising the business cycle.

The model does not permit solving analytically for optimal monetary policy, but I can numerically evaluate the effect of a class of interest rate rules, augmented with a reaction

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25To be precise, the equilibrium is not first-order stable, so that the perturbation solution method is not feasible. Global stability could still hold in principle.
Table 6: Optimal policy rules.

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<td>w/</td>
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<td>w/o</td>
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<td>%cons. loss</td>
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<td>.309</td>
<td>.028</td>
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</table>

Standard deviations of output, inflation, and interest rates (unfiltered) in percent. Welfare loss based on second order approximation to objective expected conditional welfare.

to asset prices. Consider extending the interest rate rule (5.4) as follows:

\[ i_t = \rho_i i_{t-1} + (1 - \rho_i) \left( \frac{1}{\beta} + \pi^t + \phi_\pi (\pi_t - \pi^*) + \phi_{\Delta Y} (\log Y_t - \log \text{GY}_{t-1}) + \phi_{\Delta P} (\log P_t - \log \text{GP}_{t-1}) \right) \]  

(7.1)

In addition to raising interest rates when inflation is above its target level, the central bank can raise interest rates by \( \phi_{\Delta Y} \) percentage points when real GDP growth is above long-run TFP growth, and by \( \phi_{\Delta P} \) percentage points when stock market growth is above long-run TFP growth. It also rules out monetary policy shocks. I explicitly do not include the levels of output or asset prices or output gap measures. From a theoretical perspective, this would imply that the central bank has more knowledge than the private sector under learning, since the long-run level of asset prices and output is believed to be essentially a random walk. From a practical perspective, imposing a level target for asset prices is an even more audacious measure than a target for price growth.

I compute the parameters for which such a rule maximises conditional welfare (Schmitt-Grohe and Uribe, 2004) (evaluated as a second-order approximation around the non-stochastic steady-state). Under learning, I use objective, i.e. statistical expected welfare, rather than expected welfare under subjective beliefs \( P \). This criterion is paternalistic in the sense that it does not coincide with the policy that agents in the model would prefer the central bank to take. In this sense, I assume that the central bank is able to commit to a rule that goes against what markets think it should do.

Table 6 summarises the key findings. Column (1) reports the baseline model under learning. The bottom row shows the welfare criterion in terms of equivalent steady-state consumption loss. This consumption loss is .74% under learning and an order of magnitude larger than the usual Lucas cost of business cycles. This is partly due to inflation fluctuations with Calvo price setting, but partly due to the fact that under learning, agents make
choices that are optimal under subjective beliefs but highly suboptimal when evaluated under the objective probability measure. Column (2) calculates the welfare-maximising rule that does not include a reaction to asset prices ($\phi_{\Delta P} = 0$). This makes it possible to cut the welfare loss by more than half, to .31%; however, output and stock price volatility remain high. A reaction to stock price growth, however, is able to deliver substantial additional stabilisation. The coefficient $\phi_{\Delta P}$ is large and positive: A quarter-over-quarter increase in stock prices by 1% is met with an interest rate rise of almost 0.4%. This does not mean, however, that the central bank interest rate is itself very volatile. In fact, the standard deviations of output, stock prices, and interest rates are considerably reduced.

By raising interest rates in a stock price boom, the central bank lowers aggregate demand and raises the cost of borrowing for firms, which both act to reduce corporate profits. This curbs investor optimism in the stock market, thus dampening the feedback loop between beliefs, prices, and profits.

This result is specific to the learning equilibrium and is not obtained under rational expectations, as shown in Columns (4) to (6). When learning is shut down, the welfare cost of business cycles for the baseline parameterisation is already much lower to start with. Optimising over the reaction to inflation and output growth alone, the central bank can already effectively stabilise the economy and reduce the welfare loss considerably. Column (6) shows that when $\phi_{\Delta P}$ is unconstrained, then the optimal rule is calculated to include a positive reaction to asset price growth as well. However, the added benefit of doing so is negligible, with the welfare cost virtually unchanged. Hence, under rational expectations, a monetary policy reaction to asset prices carries no benefits.

8 Conclusion

This paper has analysed the implications of a learning-based asset pricing theory in a business cycle model with financial frictions. When firms borrow against the market value of their assets, learning in the stock market interacts with credit frictions to form a two-sided feedback loop between beliefs, stock prices and firm profits that amplifies the asset price dynamics.

I have embedded the mechanism in a dynamic stochastic general equilibrium model with nominal rigidities. Unlike most of the literature on adaptive learning, beliefs retained a high degree of rationality and internal consistency. In the baseline calibration of the model, introducing learning was shown to considerably improve the model’s asset price properties, notably volatility and return predictability, while still matching standard business cycle statistics. At the same time, it leads to a large amount of propagation and amplification of both supply and demand shocks, endogenising up to one third of the volatility of the business cycle.

A natural criticism of a theory based on beliefs other than rational expectations is that it has many degrees of freedom to adjust for almost any fact of choice. However, the present
model only introduces a single degree of freedom, the learning gain $g$, which is calibrated to the relative volatility of stock prices over dividends. Moreover, the predictable biases in forecasts made by agents in the model correspond surprisingly well to the patterns found in actual data from surveys.

The model was also used to study normative implications of learning. In particular, I have revisited the question of whether monetary policy should react systematically to asset prices. I found that a strong reaction to stock price growth is desirable from a welfare perspective when investors in financial markets are learning. In contrast, under rational expectations, such a reaction does not improve welfare, in line with previous findings in the literature. This illustrates the importance of asset pricing for macroeconomics. Assumptions about the source of volatility in financial markets has profound consequences for our policy recommendations.

While this paper mainly argues that improving asset price dynamics in a macroeconomic model can lead to novel insights about business cycles, future work should explore the other direction: Can a model of learning in a production economy teach us something new about asset pricing? I suspect that the endogenous feedback between beliefs and real activity might provide for insights into the predictability of dividend growth vs the predictability of returns. It could also be worthwhile to explore extensions to the learning mechanism (such as a different perceived law of motion for prices, or learning about returns) that might further improve the asset price characteristics, a greater persistence of the P/D ratio and possibly even account for the equity premium.

But the analysis of this model need not be restricted to the role of stock prices affecting investment. Prices of other assets, for example housing, are also known to exhibit patterns that are difficult to reconcile with the simple asset pricing models used commonly in the macro literature. At the same time, they clearly play a crucial role in affecting business cycle fluctuations through credit constraints. A learning-based approach is likely to be fruitful in this context.
References


A Further details on the full model

Retailers

Retailers transform a homogeneous intermediate good into differentiated final consumption goods using a one-for-one technology. The intermediate good trades in a competitive market at the real price $q_t$ (expressed in units of the composite final good). Each retailer enjoys market power in her output market though, and sets a nominal price $p_i^t$ for her production. A standard price adjustment friction à la Calvo means that a retailer cannot adjust her price with probability $\alpha$, which is independent across retailers and across time. Hence, the retailer solves the following optimisation:

$$\max_{P_{it}} \lim_{s \to \infty} \sum_{s=0}^{\infty} \left( \prod_{\tau=1}^{s} \kappa \Lambda_{t+\tau} \right) ((1 + \tau) P_{it} - q_{t+s} P_{t+s}) Y_{it+s}$$

s.t. $Y_{it} = \left( \frac{P_{it}}{P_t} \right)^{-\sigma} \tilde{Y}_t$

where $Q_{t,t+s}$ is the nominal discount factor of households between time $t$ and $t + s$ and $\tilde{Y}_t$ is aggregate demand for the composite final good. Since all retailers that can re-optimise at $t$ are identical, they all choose the same price $P_{it} = P^*_t$. Since I want to evaluate welfare in the model, I cannot log-linearise the first-order conditions of this problem. Their derivation is nevertheless standard (for example Maussner, 2010) and I only report the final equations here:

$$\frac{P^*_t}{P_t} = \frac{1}{1 + \tau \sigma - 1} \frac{\Gamma_{1t}}{\Gamma_{2t}} \quad (A.1)$$

$$\Gamma_{1t} = q_t + \kappa E^P_{t+1} \tilde{Y}_{t+1} \pi^{\sigma}_{t+1} \quad (A.2)$$

$$\Gamma_{2t} = 1 + \kappa E^P_{t+1} \tilde{Y}_{t+1} \pi^{\sigma-1}_{t+1} \quad (A.3)$$

I assume that the government sets subsidies such that $\tau = 1/(\sigma - 1)$ so that the steady-state markup over marginal cost is zero. Inflation and the reset price are linked through the price aggregation equation which can be written as:

$$1 = (1 - \kappa) \left( \frac{P^*_t}{P_t} \right)^{1-\sigma} + \kappa \pi^{\sigma-1}_{t} \quad (A.4)$$

and the Tak-Yun distortion term is

$$\Delta_t = (1 - \kappa) \left( \frac{\Gamma_{1t}}{\Gamma_{2t}} \right)^{-\sigma} + \kappa \pi^{\sigma}_{t} \Delta_{t-1} \quad (A.5)$$

This term $\Delta_t \geq 1$ is the wedge due to price distortions between the amount of intermediate goods produced and the amount of the final good consumed.
Labour agencies

Similarly to retailers, labour agencies transform the homogeneous household labour input into differentiated labour goods at the nominal price \( \tilde{w}_t P_t \) and sell them to intermediate firms at the price \( W_{ht} \). Labour agency \( h \) solves the following optimisation:

\[
\max_{W_{ht}} \mathbb{E}_t \sum_{s=0}^{\infty} \left( \prod_{\tau=1}^{s} \kappa_w \Lambda_{t+\tau} \right) \left( (1 + \tau_w) W_{ht} - \tilde{w}_{t+s} P_{t+s} \right) L_{ht+s} \\
\text{s.t.} \quad L_{ht} = \left( \frac{W_{ht}}{W_t} \right)^{-\sigma_w} \tilde{L}_t
\]

Since all labour agencies that can re-optimise at \( t \) are identical, they all choose the same price \( W_{ht} = W_t^* \). The first-order conditions are analogous to those for retailers. Again, I assume that the government sets taxes such that \( \tau = 1/(\sigma_w - 1) \) so that the steady-state markup over marginal cost is zero. Wage inflation \( \pi_{wt} \) and the Tak-Yun distortion \( \Delta_{wt} \) are defined in the same way as for retailers. Finally, the real wage that intermediate producers effectively pay is

\[
w_t = \frac{W_t}{P_t} = \frac{w_{t-1} \pi_{wt}}{\pi_t} \quad (A.6)
\]

Capital good producers

Capital good producers operate competitively in input and output markets, producing new capital goods using old final consumption goods. For the latter, they have a CES aggregator just like households. Their maximisation programme is entirely intratemporal:

\[
\max_{I_t} Q_t I_t - \left( I_t + \frac{\psi}{2} \left( \frac{I_t}{I_{t-1}} - 1 \right)^2 \right)
\]

In particular, they take past investment levels \( I_{t-1} \) as given when choosing current investment output. Their first-order conditions defines the price for capital goods:

\[
Q_t = 1 + \psi \left( \frac{I_t}{I_{t-1}} - 1 \right) \quad (A.7)
\]
Market clearing

The market clearing conditions are summarised below. Supply stands on the left-hand side, demand on the right-hand side.

\[ Y_t = \int_0^1 Y_{jt}dj = \int_0^1 Y_{it}di \] (A.8)

\[ \dot{Y}_t = \frac{Y_t}{\Delta t} = Ct + It + \frac{\psi}{2} \left( \frac{I_t}{I_{t-1}} - 1 \right)^2 + C_t^e \] (A.9)

\[ L_t = \int_0^1 L_{ht}dh \] (A.10)

\[ \dot{L}_t = \frac{L_t}{\Delta wt} = \int_0^1 L_{jt}dj \] (A.11)

\[ K_t = \int_0^1 K_{jt}dj = (1 - \delta) K_{t-1} + I_t \] (A.12)

\[ 1 = S_{jt}, j \in [0, 1] \] (A.13)

\[ 0 = B_t^q \] (A.14)

B Properties of the rational expectations equilibrium

The rational expectations equilibrium considered here has the following properties that need to be verified. All statements are local in the sense that for each of them, there exists a neighbourhood of the non-stochastic steady-state in which the statement holds.

1. All firms choose the same capital-labour ratio \( K_{jt}/L_{jt} \).

2. The expected return on capital is higher than the internal return on debt: \( E_t R_{t+1}^k > R_t \).

3. At any time \( t \), the stock market valuation \( P_{jt} \) of a firm \( j \) is proportional to its net worth after entry and exit \( \tilde{N}_{jt} \) with a slope that is strictly greater than one.

4. Borrowers never default on the equilibrium path and borrow at the risk-free rate, and the lender only accepts debt payments up to a certain limit.

5. If the firm defaults and the lender seizes the firm, she always prefers restructuring to liquidation.

6. The firm always exhausts the borrowing limit.

7. All firms can be aggregated. Aggregate debt, capital and net worth are sufficient to describe the intermediate goods sector.

I take the following steps to prove existence of this equilibrium. After setting up the firm value functions, Property 1 just follows from constant returns to scale. I then take
Properties 2 and 3 as given and prove 4 to 6. I verify that 3 holds. The aggregation property 7 is then easily verified. I conclude by establishing the parameter restrictions for which 2 holds.

**Value functions**

An operating firm \( j \) enters period \( t \) with a predetermined stock of capital and debt. It is convenient to decompose its value function into two stages. The first stage is given by:

\[
Y_1(K, B) = \max_{N,L,D} \gamma N + (1 - \gamma) (D + Y_2(N - D))
\]

subject to:

\[
\begin{align*}
N &= qY - wL + (1 - \delta) QK - RB \\
Y &= K^\alpha \,(AL)^{1-\alpha} \\
D &= \zeta \,(N - QK + B)
\end{align*}
\]

(I suppress the time and firm indices for the sake of notation.) After production, the firm exits with probability \( \gamma \) and pays out all net worth as dividends. The second stage of the value function consists in choosing debt and capital levels as well as a strategy in the default game:

\[
Y_2(\hat{N}) = \max_{K',B',\text{strategy in default game}} \tilde{\beta} \mathbb{E} [Y_1(K', B'), \text{no default}] + \tilde{\beta} \mathbb{E} [Y_1(K', B^*), \text{debt renegotiated}] + \tilde{\beta} \mathbb{E} [0, \text{lender seizes firm}]
\]

subject to:

\[
K' = N + B'
\]

A firm which only enters in the current period starts directly starts with an exogenous net worth endowment and the value function \( Y_2 \).

**Characterising the first stage**

The first order conditions for the first stage with respect to \( L \) equalises the wage with the marginal revenue: \( w = q \,(1 - \alpha) \,(K/L)^\alpha \,A^{1-\alpha} \). Since there is no firm heterogeneity apart from capital \( K \) and debt \( B \), this already implies Property 1 that all firms choose the same capital-labour ratio. Hence the internal rate of return on capital is also common across firms:

\[
R^k = \alpha q \left( (1 - \alpha) \left( \frac{qA}{w} \right)^{1-\alpha} \right) + (1 - \delta) Q \quad (B.1)
\]
Now taking Property 3 as given (and verifying it later on), $\Upsilon_2$ is a linear function with slope strictly greater than one. Then the following holds for the first-stage value function $\Upsilon_1$:

$$
\Upsilon_1(K, B) = N + (1 - \gamma) (D - N + \Upsilon_2(N - D)) \\
= N + (1 - \gamma) (\Upsilon'_2 - 1) ((1 - \zeta) N + \zeta (QK - B)) \\
> N \\
= R^k K - RB
$$

(B.2)

This property will be used repeatedly in the next step of the proof.

**Characterising the second stage**

The second stage involves solving for the subgame-perfect equilibrium of the default game between borrower and lender. Pairings are anonymous, so repeated interactions are ruled out. Also, only the size $B$ and the interest rate $\bar{R}$ of the loan can be contracted (I omit primes for ease of notation and separate $\bar{R}$ from the risk-free rate $R$). The game is played sequentially:

1. The firm (F) proposes a borrowing contract $(B, \bar{R})$.

2. The lender (L) can accept or reject the contract.
   - A rejection corresponds to setting the contract $(B, \bar{R}) = (0, 0)$.
     Payoff for L: 0. Payoff for F: $\bar{\beta} \mathbb{E}[\Upsilon_1(\bar{N}, 0)]$.

3. F acquires capital and can then choose to default or not.
   - If F does not default, it has to repay in the next period.
     Payoff for L: $\mathbb{E}Q_{1,t+1} \bar{R}B - B$. Payoff for F: $\bar{\beta} \mathbb{E}[\Upsilon_1(K, \frac{\bar{R}}{R}B)]$.

4. If F defaults, the debt needs to be renegotiated. F makes an offer for a new debt level $B^*$.\(^{26}\)

5. L can accept or reject the offer.
   - If L accepts, the new debt level replaces the old one.
     Payoff for L: $\mathbb{E}\Lambda \bar{R} B^* - B$. Payoff for F: $\bar{\beta} \mathbb{E}[\Upsilon_1(K, \frac{\bar{R}}{R}B^*)]$.

6. If L rejects, then she seizes the firm. A fraction $1 - \xi$ of the firm’s capital is lost in the process. Nature decides randomly whether the firm can be “restructured”.

\(^{26}\)That the interest rate on the repayment is fixed is without loss of generality.
• If the firm cannot be restructured, or it can but the lender chooses not to do so, then the lender has to liquidate the firm.
  Payoff for L: \( E\Lambda\xi QK - B \). Payoff for F: 0.

• If the firm can be restructured and the lender chooses to do so, she retains a debt claim of present value \( \xi B \) and sells the residual equity claim in the firm to another investor.
  Payoff for L: \( \xi B + \tilde{\beta}E[T_1(\xi K, \xi B)] - B \). Payoff for F: 0.

Backward induction leads to the (unique) subgame-perfect equilibrium of this game. Start with the possibility of restructuring. The lender L prefers this to liquidation if
  \[
  \xi B + \tilde{\beta}E[T_1(\xi K, \xi B)] \geq E\Lambda\xi QK. \tag{B.3}
  \]
This holds true at the steady state because \( R^k > R \) (Property 2), \( Q = 1 \), \( \tilde{\beta} = \Lambda \) and
  \[
  \xi B + \tilde{\beta}E[T_1(\xi K, \xi B)] > \xi B + \tilde{\beta}E[R^k\xi K - R\xi B] = \tilde{\beta}E[R^k\xi K] > \xi K. \tag{B.4}
  \]
Since the inequality is strict, the statement holds in a neighbourhood around the steady-state as well. This establishes Property 5.

Next, the lender L will accept an offer \( B^* \) if it gives her a better expected payoff (assuming that lenders can diversify among borrowers so that their discount factor is invariant to the outcome of the game). The probability of restructuring is given by \( x \). The condition for accepting \( B^* \) is therefore that
  \[
  E\Lambda\tilde{R}B^* \geq x\left(\xi B + \tilde{\beta}E[T_1(\xi K, \xi B)]\right) + (1 - x)\, E\Lambda\xi QK. \tag{B.5}
  \]
Now turn to the firm F. Among the set of offers \( B^* \) that are accepted by L, the firm will prefer the lowest one, i.e. that which satisfies (B.5) with equality. This follows from \( T_1 \) being a decreasing function of debt. This lowest offer will be made if it leads to a higher payoff than expropriation: \( \tilde{\beta}E\left[T_1\left(K, \frac{\tilde{R}}{R}B^*\right)\right] \geq 0 \). Otherwise, the firm offers zero and the lender seizes the firm.

Going one more step backwards, the firm has to decide whether to declare default or not. It is preferable to do so if the \( B^* \) that the lender will just accept is strictly smaller than \( B \) or if expropriation is better than repaying, \( \tilde{\beta}E\left[T_1\left(K, \frac{\tilde{R}}{R}B\right)\right] \geq 0 \).

What is then the set of contracts which the lender L accepts in the first place? From the perspective of L, there are two types of contracts: those that will not be defaulted on and those that will. If the firm does not default (\( B^* \geq B \)), the lender will accept the contract simply if it pays at least the risk-free rate, \( \tilde{R} \geq R \). If the firm does default (\( B^* < B \), then
the lender accepts if the expected discounted recovery value exceeds the size of the loan, i.e. $\mathbb{E} \tilde{A} \tilde{R} B^* \geq B$.

Finally, let’s consider the contract offer by the firm. Firm F can offer a contract on which it will not default. In this case, it is optimal to offer just the risk-free rate $\tilde{R} = R$. Also note that the payoff from this strategy is strictly positive since

\[
\tilde{\beta} \mathbb{E} [\Upsilon_1 (K, B)] > \tilde{\beta} \mathbb{E} \left[ R^k K - RB \right] = \tilde{\beta} \mathbb{E} \left[ R^k \tilde{N} + \left( R^k - \tilde{R} \right) B \right] > 0.
\] (B.6)

The payoff is also increasing in the size of the loan $B$. So conditional on not defaulting, it is optimal for F to take out the maximum loan size $B = B^*$, and this is preferable to default with expropriation. However, it might also be possible for F to offer a contract that only leads to a default with debt renegotiation. The optimal contract of this type is the solution to the following problem:

\[
\max_{\tilde{R}, B, B^*} \tilde{\beta} \mathbb{E} \left[ \Upsilon_1 \left( \tilde{N} + B, \frac{\tilde{R}}{R} B^* \right) \right]
\]

s.t. $\mathbb{E} \tilde{A} \tilde{R} B^* \geq B$

\[
\mathbb{E} \tilde{A} \tilde{R} B^* = x \left( \xi B + \tilde{\beta} \mathbb{E} \left[ \Upsilon_1 \left( \xi \left( \tilde{N} + B \right), \xi B \right) \right] \right) + \left( 1 - x \right) \mathbb{E} \tilde{A} Q \xi \left( \tilde{N} + B \right)
\]

The first thing to note is that only the product $\tilde{R} B^*$ appears, so the choice of the interest rate $\tilde{R}$ is redundant. Further, $B = B^*$ and $\tilde{R} = R$ solve this problem, and this amounts to the same as not declaring default. This choice solves the maximisation problem above if the following condition is satisfied at the steady state:

\[
\frac{\xi}{R} \left( 1 - x + xR + x \Upsilon_1 \left( \frac{R^k}{R} - 1 \right) \right) < 1
\] (B.7)

For the degree of stock price dependence $x$ sufficiently small, this condition is satisfied. This establishes Properties 4 and 6.

**Linearity of firm value**

Since firms do not default and exhaust the borrowing limit $B^*$, the second-stage firm value can be written as follows:

\[
\Upsilon_2 (\tilde{N}) = \tilde{\beta} \mathbb{E} \left[ \Upsilon_1 \left( \tilde{N} + B, B \right) \right]
\]

where $B = x \left( \xi B + \tilde{\beta} \mathbb{E} \left[ \Upsilon_1 \left( \xi \left( \tilde{N} + B \right), \xi B \right) \right] \right) + \left( 1 - x \right) Q \xi \left( \tilde{N} + B \right)$ (B.8)
We already know that if $\Upsilon_2$ is a linear function, then $\Upsilon_1$ is also linear. The converse also holds: The constraint above together with linearity of $\Upsilon_1$ imply that $B$ is linear in $\bar{N}$, and thus $\Upsilon_2$ is linear, too.

To establish Property 3, it remains to show that the slope of $\Upsilon_2$ is greater than one. This is easy to see in steady state:

\[
\Upsilon'_2 = \beta \Upsilon_1 \frac{(K, B)}{N} \\
= \beta \gamma \left( R^k K - RB \right) + (1 - \gamma) \Upsilon_2 \left( R^k K - RB \right) \\
= \beta (\gamma + (1 - \gamma) \Upsilon'_2) \left( \frac{R^k K}{N} - \frac{B}{N} \right) \\
= (\gamma + (1 - \gamma) \Upsilon'_2) \frac{R + (R^k - R) \frac{B}{N}}{\gamma c_0 > 1} \\
= \gamma c_0 \\
> 1 
\]  

\[(B.10)\]

Finally, the aggregated law of motion for capital and net worth need to be established (Property 7). Denoting again by $\Gamma_t \subset [0,1]$ the indices of firms that exit and are replaced in period $t$, we have:

\[
K_t = \int_0^1 K_{jt} dj = \int_{j \notin \Gamma_t} (N_{jt} - \zeta E_{jt} + B_{jt}) dj + \int_{j \in \Gamma_t} (\omega (N_{jt} - \zeta E_{jt}) + B_{jt}) dj \\
= (1 - \gamma + \gamma \omega) (N_{jt} - \zeta E_{jt}) + B_t 
\]  

\[(B.11)\]

\[
N_t = \int_0^1 N_{jt} dj = R^k K_{t-1} - R_{t-1} B_{t-1} 
\]  

\[(B.12)\]

\[
B_t = \int_0^1 B_{jt} dj = x \xi (B_t + P_t) + (1 - x) \xi E_t \Lambda_{t+1} Q_{t+1} K_t 
\]  

\[(B.13)\]

So far then, all model properties are established except for $R^k > R$.

**Return on capital**

It can now be shown under which conditions the internal rate of return is indeed greater than the return on debt. From the steady-state versions of equations (B.11) and (B.12), it follows that

\[
R^k = R + (G - R (1 - \gamma + \gamma \omega)) \frac{\bar{N}}{K} + Rc (1 - \gamma + \gamma \omega) \frac{\bar{E}}{K}. 
\]  

\[(B.14)\]
Sufficient conditions for $R^k > R$ are therefore that $\bar{N}/\bar{K}$ and $\bar{E}/\bar{K}$ are strictly positive and that the following holds:

$$\gamma > \frac{R - G}{G(1 - \omega)} \quad \text{(B.15)}$$

## C Approximation method for the learning equilibrium

The second order perturbation method for the learning equilibrium follows Schmitt-Grohe and Uribe (2004), but has to be adapted to allow for relaxation of the rational expectations assumption. A rational expectations equilibrium can generally be described as a solution $(y_t)_{t \in \mathbb{N}}$ to

$$E_t[f(y_{t+1}, y_t, x_t, u_t)] = 0 \quad \text{(C.1)}$$

where $E_t$ is the expectations operator with respect to the probability measure and filtration induced by exogenous stochastic disturbances $u_t$. These disturbances are of dimensionality $n_u$, independent and identically distributed, of zero mean and variance $\sigma^2 \Sigma_u$. The solution $y_t$ is of dimensionality $n$, as is the image of $f$. $x_t$ denotes a vector of predetermined state variables of dimensionality $n_x < n$: $x_{i,t} = y_{i(1),t-1}$ for an injective $i : \{1..n\} \to \{1..n_x\}$, or simply $x_t = C y_{t-1}$ for an appropriate matrix $C$. One is interested in finding a policy function that generates solutions of the form

$$y_t = g(x_t, u_t, \sigma) \quad \text{(C.2)}$$

Perturbation methods for approximating the policy function to higher orders are straightforward. They compute Taylor expansions of $g$, typically around a non-stochastic steady state of the model, i.e. a constant solution $\bar{y}$ for $\sigma = 0$ such that $f(\bar{y}, \bar{y}, \bar{x}, 0) = 0$ and hence $g(\bar{y}, 0, 0) = \bar{y}$.

In a learning equilibrium, (C.1) does not fully characterise the equilibrium because the probability measure used by agents to form expectations does not coincide with the actual probability measure of the model. The stock price in the model of this paper is determined by the usual market-clearing condition, but agents think it is determined by random unforecastable shocks that are not necessarily related to the rest of the economy. A model described by (C.1) cannot contain a shock that is perceived as exogenous but at the same time determined endogenously.

The model in this paper belongs to a class that can be written as follows:

$$E_t^P[f(y_{t+1}, y_t, x_t, u_t, z_t)] = 0 \quad \text{(C.3)}$$

$$E_t^P[\phi(y_{t+1}, y_t, x_t, u_t, z_t)] = 0 \quad \text{(C.4)}$$

Here, the probability measure $P$ denotes beliefs for which the disturbances $z_t$ (of dimensionality $n_z$) are perceived as exogenous, independent and identically distributed with zero mean and variance $\sigma^2 \Sigma_z$. They are also perceived as independent of $u_t$, although
this can be relaxed. These disturbances have the interpretation of forecast errors. The iid assumption then amounts to imposing that agents holding the belief $P$ think their forecasts cannot be improved upon. As before, $f$ is of dimensionality $n$. The system (C.3) is assumed to have a unique solution for each initial condition $x_t$ and path of disturbances $u_t$ and $z_t$ which can be described by a subjective policy function:

$$y_t = h(x_t, u_t, z_t, \sigma)$$  \hspace{1cm} (C.5)

Contrary to agents’ beliefs, $z_t$ is not an exogenous disturbance, but determined endogenously by the second set of equilibrium conditions (C.4). The function $\phi$ is of dimension $n_z$. This set of conditions is not known to agents. The actual probability measure $P_0$, induced by (C.3)-(C.4) and the disturbances $u_t$, is thus different from $P$. Under $P_0$, $z_t$ is a function of the state and the fundamental disturbances:

$$z_t = r(x_t, u_t, \sigma)$$  \hspace{1cm} (C.6)

This leads to the objective policy function:

$$g(x_t, u_t, \sigma) = h(x_t, u_t, r(x_t, u_t, \sigma), \sigma)$$  \hspace{1cm} (C.7)

All functional forms are assumed to be such that the functions $h$ and $r$ are uniquely determined.

In the case of the model of this paper, lagged belief updating requires two pseudo-disturbances, $n_z = 2$. Agents cannot update their beliefs about future stock prices at the same time as they observe current prices, yet by observing the price they can infer the current forecast error. This implies the following subjective belief equations that are part of (C.3):

$$\log P_t = \log P_{t-1} + \hat{\mu}_{t-1} - \frac{\sigma^2_n + \sigma^2_v}{2} + z_{1t}$$  \hspace{1cm} (C.8)

$$\hat{\mu}_t = \hat{\mu}_{t-1} - \frac{\sigma^2_v}{2} + g z_{2t}$$  \hspace{1cm} (C.9)

The conditions (C.4) that pin down the values for the forecast errors $z_t$ in equilibrium are then described as follows:

$$P_t - \tilde{\beta} \mathbb{E}^P_t [D_{t+1} + P_{t+1}] = 0$$  \hspace{1cm} (C.10)

$$z_{2t} - z_{1t-1} = 0$$  \hspace{1cm} (C.11)

Going back to the general case, the goal is to derive an accurate second-order approxim-
ation of the objective policy function $g$ around the non-stochastic steady state:

$$g (x_t, u_t, \sigma) \approx g (\bar{x}, 0, 0) + g_x (x_t - \bar{x}) + g_x u_t + \frac{1}{2} g_{xx} [(x_t - \bar{x}) \otimes (x_t - \bar{x})] + \frac{1}{2} g_{xu} [(x_t - \bar{x}) \otimes u_t] + \frac{1}{2} g_{uu} [u_t \otimes u_t] + \frac{1}{2} g_{\sigma \sigma} \sigma^2$$  \hspace{1cm} (C.12)

The first step in deriving the approximation consists in calculating this approximation for the subjective policy function $h$. This can be done using standard methods as described in Schmitt-Grohe and Uribe (2004) and implemented e.g. in Dynare. The second step consists in finding the derivatives of the function $r$ in (C.6). Substituting it into the equilibrium conditions (C.4) gives:

$$0 = \Phi (x, u, \sigma) = \mathbb{E}_t^P \left[ \phi \left( y', y, x, u, z \right) \right] = \mathbb{E}_t^P \left[ \phi \left( h (Ch (x, u, z, \sigma), u', z', \sigma), x, u, z \right) \right] = \mathbb{E}_t^P \left[ \phi \left( h (Ch (x, u, r (x, u, \sigma), \sigma), u', z', \sigma), x, u, r (x, u, \sigma) \right) \right] \hspace{1cm} (C.13)$$

Here, I drop time subscripts and denote by a prime variables at $t + 1$. Note that the term $z'$ must not be substituted out when the expectation is taken under $\mathcal{P}$. Doing so would imply that agents know the true relationship between $z_{t+1}$ and the model variables instead of taking it as an exogenous disturbance. Total differentiation at the non-stochastic steady state leads to the following first-order derivatives:

$$0 = \frac{d\Phi}{dx} (\bar{x}, 0, 0) = (\phi_y' h_x C + \phi_y) (h_x + h_z r_x) + \phi_x + \phi_z r_x \hspace{1cm} (C.14)$$
$$0 = \frac{d\Phi}{du} (\bar{x}, 0, 0) = (\phi_y' h_x C + \phi_y) (h_u + h_z r_u) + \phi_u + \phi_z r_u \hspace{1cm} (C.15)$$
$$0 = \frac{d\Phi}{d\sigma} (\bar{x}, 0, 0) = (\phi_y' h_x C + \phi_y) (h_\sigma + h_z r_\sigma) + \phi_z r_\sigma \hspace{1cm} (C.16)$$

Since the existence of a unique solution for $r$ is assumed, the first derivatives can be solved for. I also assume that the equilibrium conditions imply that $\bar{z} = 0$ at the steady-state. This means that in the absence of shocks, agents make no forecast errors under learning. Define the matrix $A = (\phi_y' h_x C + \phi_y) h_z + \phi_z$. Then the first-order derivatives of $r$ are
given by:

\[ r_x = -A^{-1} \left( (\phi_y' g_x C + \phi_y) h_x + \phi_x \right) \quad (C.17) \]
\[ r_u = -A^{-1} \left( (\phi_y' g_x C + \phi_y) h_u + \phi_u \right) \quad (C.18) \]
\[ r_\sigma = 0 \quad (C.19) \]

Up to first order, the existence and uniqueness of the function \( r \) is equivalent to invertibility of the matrix \( A \). The first-order derivatives of the actual policy function \( g \) can be obtained by applying the chain rule. The certainty-equivalence property holds for the subjective policy function \( h \), hence \( h_\sigma = 0 \). This implies that \( r_\sigma = 0 \) and \( g_\sigma = 0 \) as well, so certainty equivalence also holds under learning.

The second-order calculations are similar, if more tedious. The second-order derivative of \( \Phi \) with respect to \( x \) is:

\[ 0 = \frac{d^2 \Phi}{dx^2} (\bar{x}, 0, 0) = (\phi_y' g_x C + \phi_y) (h_{xx} + 2h_{xz} [I_n \otimes r_x] + h_{zz} [r_x \otimes r_x]) \]
\[ + \phi_y' h_{xx} [C g_x \otimes C g_x] + B_{xx} + Ar_{xx} \quad (C.20) \]

This equation is \( n_2 \times n_2 \)-dimensional and linear in \( r_{xx} \), and thus can be solved easily. As in first order, only invertibility of the matrix \( A \) is required for a unique local solution under learning. I have collected all cross-derivatives of \( \phi \) inside the matrix \( B_{xx} \) (of size \( n_z \times n_z^2 \)), which contains only first-order derivatives of the policy functions:

\[ B_{xx} = \phi_y' [h_x C g_x \otimes h_x C g_x] + \phi_y y [g_y \otimes g_y] + \phi_{xx} + \phi_{zz} [r_x \otimes r_x] \]
\[ + 2\phi_y' y [h_x C g_x \otimes g_x] + 2\phi_y y [h_x C g_x \otimes I_n] + 2\phi_{yy} [h_x C g_x \otimes r_x] \]
\[ + 2\phi_{yy} [g_y \otimes I_n] + 2\phi_{yy} [g_y \otimes r_x] + 2\phi_{yx} [I_n \otimes r_x] \quad (C.21) \]

The formulae to solve for \( r_{xu} \) and \( r_{uu} \) are analogous. It remains to look at the derivatives involving \( \sigma \). This simplifies considerably because the first derivatives of the policy functions \( g \) and \( h \) with respect to \( \sigma \) are zero. The cross-derivative of \( \Phi \) with respect to \( x \) and \( \sigma \) thus reads:

\[ 0 = \frac{d^2 \Phi}{dx d\sigma} (\bar{x}, 0, 0) = (\phi_y' g_x C + \phi_y) (h_{x\sigma} + h_x r_{x\sigma}) + \phi_y r_{x\sigma} \quad (C.22) \]

But because \( h_{x\sigma} = 0 \) as under rational expectations, \( r_{x\sigma} = 0 \) holds as well. The same applies to \( r_{u\sigma} = 0 \). Finally, the second derivative with respect to \( \sigma \) involves the variance of the disturbances:

\[ 0 = \frac{d^2 \Phi}{d\sigma^2} (\bar{x}, 0, 0) = \phi_y' (h_{\sigma\sigma} + h_{uu} \text{vec} (\Sigma_u) + h_{x\sigma} \text{vec} (\Sigma_z)) \]
\[ + \phi_y' (\text{vec} (h_x' \Sigma_u h_u) + \text{vec} (h_x' \Sigma_z h_z)) \]
\[ + (\phi_y' g_x C + \phi_y) (h_{\sigma\sigma}) + Ar_{\sigma\sigma} \quad (C.23) \]
Again, this can be solved for $r_{\sigma \sigma}$ when $A$ is invertible. Note that the perceived variance $\Sigma_z$ appears in the calculation because it matters for expectations for the future (unless $\phi_{y'y} = 0$). This variance is not necessarily equal to the objective variance of $z$.

The second-order derivatives of the actual policy function $g$ are calculated easily once those of $r$ are known:

$$g_{xx} = h_{xx} + 2h_{xz} [I_{nx} \otimes r_x] + g_{zz} [r_x \otimes r_x] + g_{z} r_{xx}$$  \hspace{1cm} (C.24)

and analogously for $g_{xu}$, $g_{uu}$ and $g_{\sigma \sigma}$. The cross-derivatives $g_{u \sigma}$ and $g_{x \sigma}$ are zero.