Open Loop Equilibria and Perfect Competition in Option Exercise Games*

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Abstract

The investment boundaries defined by Grenadier (2002) for an oligopoly investment game determine equilibria in open loop strategies. As closed loop strategies, they are not equilibria, because any firm by investing sooner can preempt the investments of other firms and expropriate the growth options. The perfectly competitive outcome is produced by closed loop strategies that are mutually best responses. In this equilibrium, the option to delay investment has zero value, and the simple NPV rule is followed by all firms.

This paper analyzes oligopoly investment under uncertainty, assuming capital investment is irreversible and capital stocks are instantaneously adjustable upwards at a fixed price of capital. This oligopoly model is analyzed by Baldursson (1998) and Grenadier (2002). A similar model is analyzed by Leahy (1993) and Baldursson and Karatzas (1997) under the assumption of perfect competition and by Abel and Eberly (1996) and Merhi and Zervos (2005) under the assumption of monopoly. The investment policies described by all of these authors are singular, meaning that the investment rate is zero almost everywhere and undefined when investment occurs. The oligopoly model has important implications for the value of the option to delay investment – and hence the cost of ignoring this option and using the simple NPV rule for project choice – and may also be useful for understanding the dynamics of risk and return in equilibrium (see Novy-Marx (2007)).

Baldursson's (1998) equilibrium concept is Nash, and strategies are stochastic processes adapted to the exogenous process that influences demand. This is an "open loop" concept, in the sense that there is no feedback from the investment of any firm to the investment of any other firm. It appears that Grenadier presents an equilibrium in closed loop strategies, but this is misleading. We show that his equilibrium is also open loop.

The distinction between open loop and closed loop (or feedback) strategies is well understood in the context of deterministic oligopoly investment games. See Fershtman and Muller (1984), Reynolds (1987), Tirole (1988), or Fudenberg and Tirole (1992). Equilibria in open loop strategies are unattractive because they fail subgame or Markov perfection. Open loop strategies are commitments to invest, depending on the history of demand in the stochastic context, regardless of the investments of other firms, even though there is no device in the game to make such commitments credible. For example, in an open loop equilibrium, if one firm deviates to invest more than the equilibrium strategy specifies, driving the price down, other firms ignore this and continue to invest as they would have. This is inconsistent with subgame perfection.[1]

There are difficulties in even defining the game in closed loop form. To do so would seem to require an extension of Simon and Stinchcombe's (1989) analysis of deterministic continuous-time games with finite action sets to stochastic continuous-time games with continuum action sets. However, it is possible to show that the closed loop "trigger strategies" of Grenadier (2002) are not mutually best responses. By investing earlier, any firm can preempt the investments of other firms. To do so reduces the aggregate value of growth options but allows the preempting firm to expropriate growth options. We show that this tradeoff favors preemption.

Trigger strategies employing the perfectly competitive trigger (i.e., following the simple NPV rule) are mutually best responses. If one firm's strategy is to invest enough to ensure that aggregate industry capital equals the capital stock of a perfectly competitive industry, then any other firm might as well employ the same strategy. This is an extreme form of Reynolds' (1987) observation in a deterministic model with quadratic adjustment costs that closed loop equilibria involve higher steady-state capital stocks than open loop equilibria, because "the preemptive or strategic element of investment behavior in the feedback Nash equilibrium influences the long run market outcome."

Perfect competition is of course also the outcome of Bertrand competition, so one might conjecture that playing the perfectly competitive trigger is implicitly competition in prices. We believe that this is the wrong interpretation. The game is one of competition in quantities (capital stocks). However, modeling time as continuous means that firms can instantaneously respond to others' investment choices. The basic assumption of the model is that

[1] More precisely, it is inconsistent with subgame perfection if each firm observes the output price and hence has at least partial information about other firms' outputs.

arbitrarily large investments can be made instantaneously with no adjustment cost other than the fixed price of capital. Thus, at each instant in time, the game can be viewed as one of Stackelberg competition, in which each firm chooses its investment with all other firms instantaneously following. Naturally, each firm aspires to be the Stackelberg leader. A stable point, perhaps the only stable point, of this joint Stackelberg leadership is perfect competition.

The first author would like to note that his prior sole-authored working paper on this topic was excessively critical of Grenadier's concept of a myopic firm. That concept is indeed useful — to derive an open loop equilibrium. We prove that conditions similar to those in Grenadier's Proposition 3 are sufficient conditions for an open loop equilibrium.

The proof of open loop equilibrium is in Section 1. Section 2 discusses the difficulties with defining the game in closed loop form, the fact that the trigger strategies of Grenadier (2002) are not best responses to each other, and the fact that playing the perfectly competitive triggers are mutually best responses. Section 3 briefly concludes.

1 Open Loop Equilibrium

There are n firms in the industry. There is a constant required rate of return r. The capital stock of firm i at date t is denoted by Q_{it}, and we set $Q_{-it} = \sum_{j \neq i} Q_{jt}$. The cost of a unit of capital is normalized to 1. Capital does not depreciate, and investment is irreversible.

Consider a filtered probability space $(\Omega, (\mathrm{F}_t)_{t \geq 0}, \mathrm{P})$ and a one dimensional standard Brownian motion B on the probability space. Let X be a solution of a stochastic differential equation

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t\,. \tag{1}$$

Assume $\sigma(X_t) \neq 0$ for all t almost surely. Define the running maximum

$$X_t^* = \max_{0 \le s \le t} X_s \,.$$

Assume the operating cash flow rate of firm i at date t depends on X_t, Q_{it} and Q_{-it} and is increasing in X_t and Q_{it}. Denote it by $\pi(X_t, Q_{it}, Q_{-it})$. Denote marginal operating cash flow by

$$\zeta(x, q_i, q_{-i}) = \frac{\partial}{\partial q_i} \pi(x, q_i, q_{-i}) \,.$$

Assume $\zeta_{q_{-i}} \ge \zeta_{q_i}$ and $\zeta_{q_{-i}}(x, 0, q_{-i}) \le 0$, where the subscripts denote partial derivatives. Though it is not required, we could take

$$\pi(x, q_i, q_{-i}) = P(x, q_i + q_{-i}) q_i - C(q_i)$$

for some functions P and C. In that case, $P_q \le 0$ and $C' \ge 0$ imply $\zeta_{q_{-i}} \ge \zeta_{q_i}$.

Assume π is twice continuously differentiable, increasing and concave in q_i. Assume further that ζ is increasing in x and that the integrability constraint

$$\mathrm{E} \int_0^\infty e^{-rt} \sup_{a \le q_{-i} \le b} |\zeta(X_t, q_i, q_{-i})| dt < \infty \,. \tag{2}$$

is satisfied for each fixed triple (q_i, a, b).[2]

Denote the initial capital stock of firm i by q_{i0}. The set of admissible open loop strategies of firm i is

$$\mathrm{A}(q_{i0}) = \{(Q_{it})_{t \ge 0} \mid \text{nondecreasing, left-continuous, } \mathrm{F}_t\text{-adapted}, Q_{i0} \ge q_{i0}\} \,. \tag{3}$$

If each firm $j \neq i$ plays an open loop strategy, then the stochastic process Q_{-i} is an exogenous F_t-adapted process from the point of view of firm i. Firm i chooses $Q_i \in \mathrm{A}(q_{i0})$

[2]The integrability constraint is used to deduce convergence of expectations in the proof of Proposition 1. It can be replaced by $\zeta \ge 0$, using the monotone convergence theorem in the proof.

to maximize

$$\Pi(Q_i, Q_{-i}) = \mathrm{E} \int_0^\infty \left(e^{-rt} \pi(X_t, Q_{it}, Q_{-it}) \, \mathrm{d}t - \mathrm{d}Q_{it} \right) . \tag{4}$$

An open loop equilibrium — i.e., a Nash equilibrium in open loop strategies — is an n-tuple $(Q_1^*, \ldots, Q_n^*)$ of admissible strategies such that, for each i,

$$Q_i^* \in \operatorname{argmax}_{Q_i \in \mathrm{A}(q_{i0})} \Pi(Q_i, Q_{-i}^*) . \tag{5}$$

The function m described in the proposition below should be interpreted as a marginal value function (marginal with respect to q_i). It is also the value function of the optimal stopping problem (9) defined below. The hypotheses of the proposition are similar to those in Grenadier's Proposition 3, though Grenadier's assumptions regard the functions

$$(x, q) \mapsto m(x, q, (n-1)q) \qquad \text{and} \qquad q \mapsto X(q, (n-1)q) ,$$

i.e., the values of m and X along a ray in the (q_i, q_{-i}) domain, whereas our assumptions concern m and X on their entire domains. The conclusion differs regarding the nature of equilibrium — open loop instead of closed loop. The proposition is proven in Appendix A.

Proposition 1. *Suppose there exist functions $m(x, q_i, q_{-i})$ and $X(q_i, q_{-i})$ satisfying the following conditions:*

1. *$X(q_i, q_{-i})$ is differentiable in q_{-i} and continuous in q_i.*

2. *$X(q_i, q_{-i})$ is increasing in both arguments.*

3. *m is bounded from below for each fixed pair (q_i, q_{-i}), twice continuously differentiable with respect to x, and once continuously differentiable with respect to (q_i, q_{-i}).*

4. *m is monotonically increasing in x for $x \leq X(q_i, q_{-i})$.*

5. *m solves the PDE*

$$\mu m_x + \frac{1}{2}\sigma^2 m_{xx} - rm + \zeta = 0 \tag{6}$$

on the region $x < X(q_i, q_{-i})$.

6. *$m(X(q_i, q_{-i}), q_i, q_{-i}) = 1$ (value matching).*

7. *$m_x(X(q_i, q_{-i}), q_i, q_{-i}) = 0$ (smooth pasting).*

Then the following are true:

(A) Myopic Optimality. If $Q_{jt} = q_{j0}$ for all $j \neq i$ and all $t \geq 0$, then

$$Q_{it} = \inf\{q_i \geq q_{i0} \mid X_t^* \leq X(q_i, q_{-i0})\} \tag{7}$$

maximizes (4) over $Q_i \in \mathrm{A}(q_{i0})$.

(B) Symmetric Open Loop Equilibrium. Suppose $q_{i0} = q_{j0}$ for all i and j. Then $(Q_1^*, \ldots, Q_n^*)$ is an open loop equilibrium, where, for each i,

$$Q_{it}^* = \inf\{q_i \geq q_{i0} \mid X_t^* \leq X(q_i, (n-1)q_i)\} \ . \tag{8}$$

Assumptions 3–7 ensure that the function m fulfills the criteria of a Hamilton-Jacobi-Bellman verification theorem for the optimal stopping problem

$$\min_{\tau} E\left[\int_0^{\tau} e^{-rt}\zeta(X_t, q_i, q_{-i})dt + e^{-r\tau}\right] . \tag{9}$$

More precisely, m is the value function of this stopping problem, and the hitting time of the boundary $X(q_i, q_{-i})$ is an optimal stopping rule. The optimal stopping problem can be interpreted as the problem of a firm to optimally install a unit of capital under the myopic assumption that rival firms will never do so and that no further unit can be installed. In the formulation (9), the firm minimizes the opportunity cost (the foregone marginal cash flow) of not investing plus the discounted cost of investing.

The smooth pasting condition deserves a comment. Heuristically it can be derived by the envelope theorem. Namely let $M(x, y)$ be the expected value in (9) when $X_0 = x$ and the stopping time is the first hitting time of y. By the value matching condition, $M(y, y) = 1$, so differentiating this with respect to y yields $M_x(y, y) + M_y(y, y) = 0$. However, if y^* is optimal, then $M_y(\cdot, y^*)$ should equal zero. So $M_x(y^*, y^*) = 0$ at the optimal boundary y^*.

There is a large literature on the connection between singular stochastic control problems and optimal stopping problems, starting with Karatzas and Shreve (1984). The equivalence between the control problem and the stopping problems here is the same as in Theorem 1 of Bank (2005). What is new about our proof, as far as we know, is the solution of the stopping problems in the presence of the exogenous singular process Q_{-i}.

Now we apply Proposition 1 to the linear model studied by Baldursson (1997) and the constant elasticity example considered by Grenadier (2002). Grenadier and Baldursson present the value of $X(q_i, q_{-i})$ when $q_{-i} = (n-1)q_i$. The entire function $X(q_i, q_{-i})$ is presented below.

Constant Elasticity

Assume $\pi(x, q_i, q_{-i}) = P(x, q_i + q_{-i})q_i$ with $P(x, q) = xq^{-1/\gamma}$. Assume $\gamma > 1$. Assume X is a geometric Brownian motion with parameters μ and σ; i.e., $\mu(x) = \mu x$ and $\sigma(x) = \sigma x$ with a slight abuse of notation. Assume $r > \mu$ and $\beta > \gamma$, where

$$\beta = \frac{-\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2}}{\sigma^2}\,. \tag{10}$$

These restrictions on the parameters imply that the model satisfies our assumptions regarding π. We have

$$\zeta(x, q_i, q_{-i}) = \left(1 - \frac{q_i}{\gamma(q_i + q_{-i})}\right) x(q_i + q_{-i})^{-1/\gamma}\,.$$

Proposition 2. *In the constant elasticity model, there is a unique pair (m, X) satisfying the conditions of Proposition 1,*[3] *and*

$$X(q_i, q_{-i}) = \frac{\beta}{\beta - 1} \left(\frac{\gamma}{\gamma - q_i/(q_i + q_{-i})} \right) (r - \mu)(q_i + q_{-i})^{\frac{1}{\gamma}}, \tag{11a}$$

$$m(x, q_i, q_{-i}) = \frac{\zeta(x, q_i, q_{-i})}{r - \mu} - \frac{\zeta(X(q_i, q_{-i}), q_i, q_{-i})}{(r - \mu)\beta} \left(\frac{x}{X(q_i, q_{-i})} \right)^{\beta}. \tag{11b}$$

Proof. The general solution to the PDE (6) is

$$m(x, q_i, q_{-i}) = \frac{\zeta(x, q_i, q_{-i})}{r - \mu} + Ax^{\beta} + Bx^{\beta'}$$

with β given by (10), and $\beta' = 1 - \beta - 2\mu/\sigma^2 < 0$. Assumptions 3 and 4 imply m is bounded in x on the interval $(0, X(q_i, q_{-i}))$, so $B = 0$. Solving conditions 6 and 7 in the unknowns A and $X(q_i, q_{-i})$ yields (11). □

Linear Demand

Assume $\pi(x, q_i, q_{-i}) = P(x, q_i + q_{-i})q_i - cq_i$, with $P(x, q) = x - bq$, for constants b and c. Assume X is a geometric Brownian motion with parameters μ and σ. Assume $\beta > 2$, where β is defined in (10). Then the model satisfies our assumptions regarding π. We have $\zeta(x, q_i, q_{-i}) = x - 2bq_i - bq_{-i} - c$.

Proposition 3. *In the linear model, there is a unique pair (m, X) satisfying the conditions of Proposition 1, and*

$$X(q_i, q_{-i}) = \frac{\beta}{\beta - 1} \left(\frac{r - \mu}{r} \right) (c + r + 2bq_i + bq_{-i}), \tag{12a}$$

$$m(x, q_i, q_{-i}) = \frac{-2bq_i - bq_{-i} - c}{r} + \frac{x}{r - \mu} - \frac{x^{\beta}}{(r - \mu)\beta X(q_i, q_{-i})^{\beta - 1}}. \tag{12b}$$

Proof. The general solution to the PDE (6) is:

$$m(x, q_i, q_{-i}) = \frac{\zeta(x, q_i, q_{-i})}{r} + Ax^{\beta} + Bx^{\beta'}$$

[3]Uniqueness of m is on the domain $\{(x, q_i, q_{-i}) \mid x \leq X(q_i, q_{-i})\}$.

with β as given in (10) and $\beta' = 1 - \beta - 2\mu/\sigma^2 < 0$. For the same reasons as in the constant elasticity case, $B = 0$. Solving the smooth pasting and the value matching condition for A and X yields (12). □

2 Closed Loop Strategies and Best Responses

First, we admit we do not know how to define this game in closed loop form. There are substantial complications in doing so. If the capital stock processes were absolutely continuous instead of singular, one would view the investment rate $z_{it} = \mathrm{d}Q_{it}/\mathrm{d}t$ as the decision variable of firm i at date t. If each z_{it} were required to depend on the history of $(X, Q_1, \ldots, Q_n)$ prior to t in a sufficiently regular way, then the capital stock processes

$$Q_{it} = q_{i0} + \int_0^t z_{it}\,\mathrm{d}t$$

would be well defined. With singular controls, one could view the action of firm i at any date t as being the Lebesgue-Stieltjes differential $\mathrm{d}Q_{it}$ of its capital stock process Q_i; however, these differentials are meaningful only in integrated form. An alternate view is that the action of firm i at date t is its total capital Q_{it}, chosen subject to the constraint that capital is irreversible: $Q_{it} \geq \sup_{s<t} Q_{is}$. However, this suffers from the general problem with continuous-time games that what seem to be well defined strategies may not produce well defined outcomes (see Simon and Stinchcombe (1989)). For example, the formula $Q_{it} = \sup_{s<t} Q_{is}$ seems to specify Q_{it} as a function of the history of play prior to t; yet, every nondecreasing left-continuous process Q_i satisfies the formula. Likewise, the formulas $Q_{it} = \lim_{s\uparrow t} Q_{js}$ and $Q_{jt} = \lim_{s\uparrow t} Q_{is}$ seem to define each firm's capital stock at time t as a function of the other firm's prior investments, but these formulas are satisfied by every left-continuous $Q_i = Q_j$. Thus, formulas such as these — and one could give an arbitrary

number of similar examples — are very far from uniquely specifying how the game is to be played. In order to define the game, some rules must be constructed to allow one to map such formulas, or whatever strategies are allowed, into unique outcomes. Simon and Stinchcombe (1989) accomplish this for deterministic games with finite action sets. Generalizing their work to the present context, and then finding equilibria, would seem to be substantial tasks.

Though we do not know how to define strategies in general, there are some combinations of decision rules that clearly produce well-defined outcomes. Grenadier's Proposition 1 states: "Each firm's investment strategy is characterized by increasing output incrementally whenever $X(t)$ rises to the trigger function $\bar{X}(q_i, Q_{-i})$."[4] Though Grenadier's statement is not a precise description of a strategy, it seems reasonable to take its meaning to be

$$Q_{it} = \inf \{q_i \geq q_{i0} \mid X_t^* \leq X(q_i, Q_{-it})\} \ . \tag{13}$$

Note that Q_{it} is allowed to depend on the contemporaneous Q_{-it}. This seems reasonable for all $t > 0$ if we restrict to left-continuous paths.[5] Decision rules of the form (13) do not necessarily produce well-defined outcomes. For example, in a two-firm game, if both firms play (13) for $X(q_i, q_{-i}) = q_i + q_{-i}$, then the division of output between the two firms is not defined. However, if all firms play (13) for the open loop equilibrium investment boundaries in the constant elasticity and linear examples — i.e., for X defined in (11a) or (12a) — then the capital paths of all firms are well defined. In fact, the strategies (13) with the open loop equilibrium investment boundaries produce the open loop equilibrium capital processes.

If all firms play (13) for an increasing $X(\cdot)$ and the paths are well defined, then any firm can preempt the investments of other firms by investing aggressively itself. The following

[4] $\bar{X}(q_i, Q_{-i})$ equals the myopic trigger $X(q_i, Q_{-i})$ from Proposition 1 (see Grenadier's Proposition 2).

[5] If $X_0 > X(q_{i0}, q_{-i0})$, then (13) implies a jump at time 0. It allows this jump to depend on simultaneous jumps of other firms. This seems unreasonable, but one could view it as a reduced form for nearly instantaneous reactions. Simon and Stinchcombe (1989) discuss this issue.

proposition shows that the open loop equilibrium boundary in the linear model does not define a closed loop equilibrium, because preemption is a profitable deviation (the strategies (14) and (15b) are the strategies asserted by Grenadier to constitute an equilibrium in the linear model).

Proposition 4. *Suppose X is a geometric Brownian motion with drift μ and volatility σ. Assume $\pi(x, q_i, q_{-i}) = (X - b(q_i + q_{-i}) - c)q_i$ for constants $b > 0$ and $c \geq 0$. Assume $q_{i0} = q_{j0}$ for all i and j, and define $q_0 = \sum_{i=1}^n q_{i0}$. Assume $\beta > 2$, where β is defined in* (10). *Assume*

$$(\forall\, j \neq i) \quad Q_{jt} = \inf\left\{q_j \geq q_{j0} \mid X_t^* \leq \frac{\beta}{\beta - 1}\left(\frac{r-\mu}{r}\right)(c + r + 2bq_j + bQ_{-jt})\right\}. \tag{14}$$

Define

$$\tau = \inf\left\{t \geq 0 \mid X_t^* \geq \frac{\beta}{\beta - 1}\left(\frac{r-\mu}{r}\right)\left(c + r + \frac{(n+1)bq_0}{n}\right)\right\}.$$

There exists $\alpha > 1$ such that the open loop strategy

$$Q_{it}^\alpha = \begin{cases} q_{i0} & \text{for } t \leq \tau\,, \\ \inf\left\{q_i \geq \alpha q_{i0} \mid X_t^* \leq \frac{\beta}{\beta-1}\left(\frac{r-\mu}{r}\right)\left(c + r + \frac{n+\alpha}{\alpha} bq_i\right)\right\} & \text{for } t > \tau\,, \end{cases} \tag{15a}$$

produces higher expected discounted cash flows for firm i than does the closed loop strategy

$$Q_{it} = \inf\left\{q_i \geq q_{i0} \mid X_t^* \leq \frac{\beta}{\beta - 1}\left(\frac{r-\mu}{r}\right)(c + r + 2bq_i + bQ_{-it})\right\}. \tag{15b}$$

Proof. The unique n–tuple $(Q_1, \ldots, Q_n)$ of stochastic processes satisfying (14) and (15b) is the open loop equilibrium $(Q_1^*, \ldots, Q_n^*)$ defined in Proposition 1 and Proposition 3. Let $\alpha \geq 1$. The unique n–tuple $(Q_1, \ldots, Q_n)$ of stochastic processes satisfying (14) and (15a) is $(Q_1^\alpha, \ldots, Q_n^\alpha)$, where Q_i^α is defined in (15a) and

$$(\forall\, j \neq i) \quad Q_{jt}^\alpha = \begin{cases} q_{j0} & \text{for } t < \tau\,, \\ Q_{it}^\alpha/\alpha & \text{for } t \geq \tau\,. \end{cases} \tag{16}$$

To see this, note that for $t \geq \tau$, (14) implies

$$Q_{jt} = \max\left\{q_{j0}, \frac{1}{2b}\left[\frac{\beta-1}{\beta}\left(\frac{r}{r-\mu}\right)X_t^* - c - r - bQ_{-jt}\right]\right\}.$$

The equality of the Q_j for $j \neq i$ implies $Q_{-jt} = (n-2)Q_{jt} + Q_{it}^\alpha$. Making this substitution, we have

$$Q_{jt} = \max\left\{q_{j0}, \frac{1}{nb}\left[\frac{\beta-1}{\beta}\left(\frac{r}{r-\mu}\right)X_t^* - c - r - bQ_{it}^\alpha\right]\right\}. \tag{17a}$$

Moreover, (15a) implies, for $t \geq \tau$,

$$Q_{it}^\alpha = \alpha \max\left\{q_{i0}, \frac{1}{(n+\alpha)b}\left[\frac{\beta-1}{\beta}\left(\frac{r}{r-\mu}\right)X_t^* - c - r\right]\right\}. \tag{17b}$$

Substituting (17b) in (17a) yields (16).

Note that, for $\alpha = 1$, $Q_j^\alpha = Q_j^*$ for all $j = 1, \ldots, n$. Define $F(\alpha) = \Pi(Q_i^\alpha, Q_{-i}^\alpha)$, where $\Pi(\cdot)$ is the expected discounted cash flow defined in (4). The claim is that $F(\alpha) > F(1)$ for some $\alpha > 1$. We show in Appendix B that the right-hand derivative of F at $\alpha = 1$ is positive. Thus, $F(\alpha) > F(1)$ for all sufficiently small $\alpha > 1$. □

The preemption strategy (15a) involves a limited amount of preemption: jumping to a market share of $\alpha/(n+\alpha-1)$ and maintaining that market share forever. For some parameter values in the linear model, expropriating all of the growth options is a profitable deviation from (15b). To explain what it means to expropriate all of the growth options, consider the constant elasticity example and the boundary (11a). Note that

$$\frac{q_i}{q_i + q_{-i}} \to 0 \qquad \Rightarrow \qquad X(q_i, q_{-i}) \to \frac{\beta}{\beta-1}(r-\mu)(q_i+q_{-i})^{\frac{1}{\gamma}}. \tag{18}$$

The limit of $X(q_i, q_{-i})$ displayed in (18) is the perfectly competitive investment boundary defined by Leahy (1993). It is the boundary at which a firm with infinitesimal market share would invest. Thus, if some firm j plays (13), i.e.,

$$Q_{jt} = \inf\{q_j \geq q_{j0} \mid X_t^* \leq X(q_j, Q_{-jt})\}, \tag{19}$$

then the behavior of firm j will approach that of a perfectly competitive firm as its market share decreases. If firm i invests sufficiently aggressively that it deters the investments of other firms, then the market shares of other firms will gradually decline towards zero, and their behavior under the decision rule (19) with boundary (11a) will approach that of perfect competition. Thus, aggregate output and price will approach the perfectly competitive output and price, and the value of industry growth options will eventually be destroyed. In exchange for this diminution of industry growth options, the preempting firm can expropriate all growth options to itself. We have calculated, though it is not presented here, that expropriating all of the growth options is a profitable deviation from (15b) in the linear model for some parameter values. Paulsen (2006) shows that preempting for a finite period of time is a profitable deviation in the constant elasticity model for some parameter values.

The perfectly competitive boundary is immune to preemption. Suppose, in the constant elasticity example, that each firm plays

$$Q_{it} = \inf\left\{ q_i \geq q_{i0} \mid X_t^* \leq \frac{\beta}{\beta - 1}(r - \mu)(q_i + Q_{-it})^{\frac{1}{\gamma}} \right\}. \tag{20}$$

Given symmetric initial conditions and initial industry capital q_0, equation (20) holds for each i whenever industry capital $Q_t = \sum_{i=1}^{n} Q_{it}$ satisfies

$$Q_t = \inf\left\{ q \geq q_0 \mid X_t^* \leq \frac{\beta}{\beta - 1}(r - \mu)q^{\frac{1}{\gamma}} \right\}.$$

Thus, (20) suffers from the problem discussed in the first paragraph of this section: It does not produce well-defined individual firm capital processes. However, it does produce well-defined individual firm values, which is the key requirement for choosing among strategies. Because industry growth options have zero value when investing at the perfectly competitive boundary, it does not matter how growth is distributed among the firms. Moreover, if all other firms play (20), then it is optimal for each individual firm to play (20), because the price process is unaffected by an individual firm playing (20) when other firms also play (20), and the investments from playing (20) are zero NPV when the price process is taken as given. The only suboptimal decision a firm could make when other firms play (20) is to invest *before* the perfectly competitive boundary is reached, and this does not occur when a firm plays (20). Thus, the strategies (20) are mutually best responses.

Though the strategies (20) are choices of quantities, not prices, the outcome is like Bertrand in that the "economic value added" of each firm is zero. Related to this is another feature the equilibrium shares with Bertrand: the strategies are weakly dominated. Investing zero at all times is as valuable as making zero NPV investments, and it is superior to playing (20) if other firms play investment strategies that are less aggressive than (20).

3 Conclusion

Open loop equilibria have an extreme Cournot nature: Each firm optimizes taking the entire output process of each other firm as given. They fail subgame perfection, because if a firm invests aggressively the game will reach a node from which the given output processes of other firms will not be part of a Nash equilibrium starting from that node. The closed loop strategies (13) have the potential to form subgame perfect equilibria, because each firm reacts to the investments of others. These strategies have a Stackelberg flavor, because all firms react to the investments of any firm like Stackelberg followers, and hence each firm is like a Stackelberg leader. A stable point of this joint Stackelberg leadership is perfect competition.

The closed loop strategies (13) employing the myopic (open loop equilibrium) boundary do not form an equilibrium, because any firm by investing more can cause other firms to invest less, like a Stackelberg leader, and hence expropriate some of the growth options.

We proved that this preemptive investment is a profitable deviation in the linear model. Paulsen (2006) shows the same for the constant elasticity model, for some parameter values.

It is an open question whether the perfectly competitive boundary is the unique boundary such that closed loop strategies of the form (13) are impervious to preemption. If so, the perfectly competitive boundary would be the unique boundary such that the strategies (13) could constitute an equilibrium.

It seems likely that there would be other closed-loop equilibria, if the strategy spaces and mapping from strategy n–tuples to outcomes could be specified. There should be equilibria in punishment strategies, in which firms invest less than the perfectly competitive amount and threaten to punish any firm that deviates. These strategies are *not* of the form: invest when $X_t = X(Q_{it}, Q_{-it})$ for an increasing function X, because strategies of this form prescribe less investment when competitors invest more and hence do not allow for punishment.

A Proof of Proposition 1

Define

$$\bar{m}(x, q_i, q_{-i}) = \begin{cases} m(x, q_i, q_{-i}) & \text{if } x \leq X(q_i, q_{-i})\,, \\ 1 & \text{otherwise}\,. \end{cases} \tag{A.1}$$

Lemma 1. *$\bar{m}_{q_{-i}}(x, q_i, q_{-i}) = 0$ for $x \geq X(q_i, q_{-i})$.*

Proof. By the value-matching condition, we have

$$\bar{m}(X(q_i, q_{-i}), q_i, q_{-i}) = 1\,.$$

Differentiating this equation with respect to q_{-i} yields

$$\bar{m}_x(X(q_i, q_{-i}), q_i, q_{-i}) X_{q_{-i}}(q_i, q_{-i}) + \bar{m}_{q_{-i}}(X(q_i, q_{-i}), q_i, q_{-i}) = 0\,.$$

As the first term vanishes by the smooth-pasting condition, we get

$$\bar{m}_{q_{-i}}(X(q_i, q_{-i}), q_i, q_{-i}) = 0\,.$$

This proves the claim for $x = X(q_i, q_{-i})$. For $x > X(q_i, q_{-i})$, $\bar{m}(x, q_i, z) = 1$ for z in a neighborhood of q_{-i} by definition of $\bar{m}$. Hence, differentiation yields the result. □

Lemma 2. *We have (for all $x \neq X(q_i, q_{-i})$)*

$$-r\bar{m} + \mu\bar{m}_x + \frac{1}{2}\sigma^2\bar{m}_{xx} \geq -\zeta$$

with equality for $x < X(q_i, q_{-i})$.

Proof. For $x < X(q_i, q_{-i})$ equality holds by Assumption 5 of Proposition 1. Assumptions 4 and 7 imply that $m_{xx} \leq 0$ for $x = X(q_i, q_{-i})$. Therefore

$$-rm + \mu m_x \geq -\zeta$$

at $x = X(q_i, q_{-i})$. Using the smooth pasting condition we get

$$-r\bar{m} = -rm \geq -\zeta$$

at $x = X(q_i, q_{-i})$. Note that ζ increases in x while $\bar{m}$ remains constant, so

$$-r\bar{m} + \mu\bar{m}_x + \frac{1}{2}\sigma^2\bar{m}_{xx} = -r\bar{m} + 0 + 0 \geq -\zeta$$

for $x > X(q_i, q_{-i})$. □

Consider the myopic problem indexed by $y = q_{i0}$ and $z = q_{-i0}$:

$$\max_{Q_i \in \mathrm{A}(y)} \quad \mathrm{E}\int_0^\infty e^{-rt}\left(\pi(X_t, Q_{it}, z)\,\mathrm{d}t - \mathrm{d}Q_{it}\right) . \tag{A.2}$$

Related to this problem is the following optimal stopping problem: Choose a stopping time τ to maximize

$$\mathrm{E}\left[\int_\tau^\infty \mathrm{e}^{-rt}\zeta(X_t, y, z)\,\mathrm{d}t - \mathrm{e}^{-r\tau}\right] = \mathrm{E}\left[\int_\tau^\infty \mathrm{e}^{-rt}\left(\zeta(X_t, y, z) - r\right)\,\mathrm{d}t\right] . \tag{A.3}$$

To interpret the optimal stopping problem, notice that a small investment ϵ at time τ increases expected discounted revenues by approximately

$$\epsilon \cdot \mathrm{E}\int_\tau^\infty \mathrm{e}^{-rt}\zeta(X_t, y, z)\,\mathrm{d}t\,,$$

and has expected discounted cost equal to $\epsilon \cdot \mathrm{E}\left[\mathrm{e}^{-r\tau}\right]$. The optimal stopping time is therefore the optimal time to make a investment of size $\epsilon \approx 0$. Subtracting $\mathrm{E}\left[\int_0^\infty \mathrm{e}^{-rt}\zeta(X_t, y, z)\,\mathrm{d}t\right]$ and multiplying by (-1) converts the problem of maximizing (A.3) to the following equivalent problem:

$$\min_\tau \quad \mathrm{E}\left[\int_0^\tau \mathrm{e}^{-rt}\zeta(X_t, y, z)\,\mathrm{d}t + \mathrm{e}^{-r\tau}\right] . \tag{A.4}$$

This can be interpreted as minimizing the opportunity cost of a unit of capital.

We also consider the equilibrium problem indexed by $y = q_{i0}$ and $z = q_{-i0}$. In this problem, we assume the aggregate capital of firms $j \neq i$ is

$$L_{zt} = \inf \left\{ q \geq z \mid \max_{0 \leq s \leq t} X_s \leq X(q/(n-1), q) \right\} . \tag{A.5}$$

The optimization problem for firm i that we study is:

$$\max_{Q_i \in \mathrm{A}(y)} \quad \mathrm{E} \int_0^\infty e^{-rt} \left(\pi(X_t, Q_{it}, L_{zt}) \, \mathrm{d}t - \mathrm{d}Q_{it} \right) . \tag{A.6}$$

The related optimal stopping problem is: Choose a stopping time τ to maximize

$$\mathrm{E} \left[\int_\tau^\infty \mathrm{e}^{-rt} \zeta(X_t, y, L_{zt}) \, \mathrm{d}t - \mathrm{e}^{-r\tau} \right] = \mathrm{E} \left[\int_\tau^\infty \mathrm{e}^{-rt} \left(\zeta(X_t, y, L_{zt}) - r \right) \, \mathrm{d}t \right] , \tag{A.7}$$

which is equivalent to:

$$\min_\tau \quad \mathrm{E} \left[\int_0^\tau \mathrm{e}^{-rt} \zeta(X_t, y, L_{zt}) \, \mathrm{d}t + \mathrm{e}^{-r\tau} \right] . \tag{A.8}$$

For any y and z, define

$$\tau_{yz} = \inf\{ t \mid X_t > X(y, z) \} . \tag{A.9}$$

Lemma 3. *τ_{yz} solves the myopic stopping problem* (A.4).

Proof. By an approximation argument as in Øksendal (2002) (see Theorem 10.4.1), we can assume that $\bar{m}$ is twice continuously differentiable with respect to x. Let τ be an arbitrary stopping time. Applying Itô's rule to $\mathrm{e}^{-r(t \wedge \tau)} \bar{m}(X_{t \wedge \tau}, y, z)$ yields:

$$\begin{aligned} \mathrm{e}^{-r(t \wedge \tau)} \bar{m}(X_{t \wedge \tau}, y, z) = \bar{m}(X_0, y, z) &+ \int_0^{t \wedge \tau} \mathrm{e}^{-rs} \bar{m}_x \sigma \, \mathrm{d}B_s \\ &+ \int_0^{t \wedge \tau} \mathrm{e}^{-rs} (-r\bar{m} + \mu \bar{m}_x + \frac{1}{2} \sigma^2 \bar{m}_{xx}) \, \mathrm{d}s . \end{aligned} \tag{A.10}$$

Applying Lemma 2 to (A.10), we get

$$\begin{aligned} \mathrm{e}^{-r(t \wedge \tau)} \bar{m}(X_{t \wedge \tau}, y, z) \quad \geq \quad & \bar{m}(X_0, y, z) + \int_0^{t \wedge \tau} \mathrm{e}^{-rs} \bar{m}_x \sigma \, \mathrm{d}B_s \\ & - \int_0^{t \wedge \tau} \mathrm{e}^{-rs} \zeta_s \, \mathrm{d}s , \end{aligned}$$

with equality for $\tau \le \tau_{yz}$. We cannot directly take expectations on both sides as we do not know whether the stochastic integral is a martingale or just a local martingale. So let $\tau_k \le k$ be a localizing sequence for the stochastic integral. That is, $\tau_k \uparrow \infty$ and the stopped integrals are martingales. Taking expectations on both sides and using Doob's optional sampling theorem yields, for each k,

$$\mathrm{E}\left[\mathrm{e}^{-r(\tau_k\wedge\tau)}\bar{m}(X_{\tau_k\wedge\tau}, y, z)\right] \;\ge\; \bar{m}(X_0, y, z) - \mathrm{E}\left[\int_0^{\tau_k\wedge\tau} \mathrm{e}^{-rs}\zeta_s\,\mathrm{d}s\right].$$

Observe that $\bar{m}$ is bounded from above by 1 and bounded from below by Assumption 3 in Proposition 1, whereas the integrals involving ζ are uniformly integrable by the integrability assumption (2). Taking the limit $k \to \infty$ we get

$$\begin{aligned}\bar{m}(X_0, y, z) &\le \lim_{k\to\infty} \mathrm{E}\left[\mathrm{e}^{-r(\tau_k\wedge\tau)}\bar{m}(X_{\tau_k\wedge\tau}, y, z)\right] + \lim_{k\to\infty} \mathrm{E}\left[\int_0^{\tau_k\wedge\tau} \mathrm{e}^{-rs}\zeta_s\,\mathrm{d}s\right] \\ &= \mathrm{E}\left[\mathrm{e}^{-r\tau}\bar{m}(X_\tau, y, z)\right] + \mathrm{E}\left[\int_0^{\tau} \mathrm{e}^{-rs}\zeta_s\,\mathrm{d}s\right],\end{aligned}$$

or

$$\begin{aligned}\bar{m}(X_0, y, z) &\le \mathrm{E}\left[\int_0^{\tau} \mathrm{e}^{-rs}\zeta_s\,\mathrm{d}s\right] + \mathrm{E}\left[\mathrm{e}^{-r\tau}\bar{m}(X_\tau, y, z)\right] \\ &\le \mathrm{E}\left[\int_0^{\tau} \mathrm{e}^{-rs}\zeta_s\,\mathrm{d}s\right] + \mathrm{E}\left[\mathrm{e}^{-r\tau}\right],\end{aligned}$$

with equality for $\tau = \tau_{yz}$.

□

Lemma 4. *If $y = z/(n-1)$, then τ_{yz} solves the equilibrium stopping problem (A.8).*

Proof. We proceed as in the proof of Lemma 3 with the difference that now $dL_{zt} \ne 0$. Let τ be an arbitrary stopping time. Applying Itô's rule to $\mathrm{e}^{-r(t\wedge\tau)}\bar{m}(X_{t\wedge\tau}, y, L_{z,t\wedge\tau})$ yields:

$$\begin{aligned}\mathrm{e}^{-r(t\wedge\tau)}\bar{m}(X_{t\wedge\tau}, y, L_{t\wedge\tau,z}) &= \bar{m}(X_0, y, z) + \int_0^{t\wedge\tau} \mathrm{e}^{-rs}\bar{m}_x\sigma\,\mathrm{d}B_s \\ &+ \int_0^{t\wedge\tau} \mathrm{e}^{-rs}(-r\bar{m} + \mu\bar{m}_x + \frac{1}{2}\sigma^2\bar{m}_{xx})\,\mathrm{d}s + \int_0^{t\wedge\tau} \mathrm{e}^{-rs}\bar{m}_{q_{-i}}\,\mathrm{d}L_{zs}\,. \quad \text{(A.11)}\end{aligned}$$

Note that L increases only when $X_t = X(L_{zt}/(n-1), L_{zt})$. We have $L_{zt}/(n-1) \geq z/(n-1) = y$. By monotonicity of X, it follows that L increases only when $X_t \geq X(y, L_{zt})$. In this case, Lemma 1 implies

$$\bar{m}_{q_{-i}}(X_t, y, L_{zt})\,\mathrm{d}L_{zt} = 0\,. \tag{A.12}$$

Applying (A.12) and Lemma 2 to (A.10), we get

$$\begin{aligned}
\mathrm{e}^{-r(t\wedge\tau)}\bar{m}(X_{t\wedge\tau}, y, L_{z,t\wedge\tau}) \quad &\geq \quad \bar{m}(X_0, y, z) + \int_0^{t\wedge\tau} \mathrm{e}^{-rs}\bar{m}_x\sigma\,\mathrm{d}B_s \\
&\quad - \int_0^{t\wedge\tau} \mathrm{e}^{-rs}\zeta_s\,\mathrm{d}s\,,
\end{aligned}$$

with equality for $\tau \leq \tau_{yz}$. As in the proof of Lemma 3 we take a localizing sequence $\tau_k \leq k$. Taking expectations on both sides yields for each k.

$$\mathrm{E}\left[\mathrm{e}^{-r(\tau_k\wedge\tau)}\bar{m}(X_{\tau_k\wedge\tau}, y, L_{z,\tau_k\wedge\tau})\right] \geq \bar{m}(X_0, y, z) - \mathrm{E}\left[\int_0^{\tau_k\wedge\tau} \mathrm{e}^{-rs}\zeta_s\,\mathrm{d}s\right]\,.$$

Observe that $\bar{m}$ is bounded from above by 1 and $\zeta(X_s, y, L_{zs}) \leq \zeta(X_s, 0, L_{zs}) \leq \zeta(X_s, 0, z)$ which is integrable by assumption (2). So applying Fatou's lemma yields:

$$\begin{aligned}
\bar{m}(X_0, y, z) \quad &\leq \quad \limsup_{k\to\infty}\mathrm{E}\left[\mathrm{e}^{-r(\tau_k\wedge\tau)}\bar{m}(X_{\tau_k\wedge\tau}, y, L_{z,\tau_k\wedge\tau})\right] + \limsup_{k\to\infty}\mathrm{E}\left[\int_0^{\tau_k\wedge\tau} \mathrm{e}^{-rs}\zeta_s\,\mathrm{d}s\right] \\
&\leq \quad \mathrm{E}\left[\mathrm{e}^{-r\tau}\bar{m}(X_\tau, y, L_{z\tau})\right] + \mathrm{E}\left[\int_0^{\tau} \mathrm{e}^{-rs}\zeta_s\,\mathrm{d}s\right]\,;
\end{aligned} \tag{A.13}$$

i.e.,

$$\begin{aligned}
\bar{m}(X_0, y, z) \quad &\leq \quad \mathrm{E}\left[\int_0^{\tau} \mathrm{e}^{-rs}\zeta_s\,\mathrm{d}s\right] + \mathrm{E}\left[\mathrm{e}^{-r\tau}\bar{m}(X_\tau, y, L_{z\tau})\right] \\
&\leq \quad \mathrm{E}\left[\int_0^{\tau} \mathrm{e}^{-rs}\zeta_s\,\mathrm{d}s\right] + \mathrm{E}\left[\mathrm{e}^{-r\tau}\right]\,.
\end{aligned}$$

Note that $L_{zt} = z$ for $t \leq \tau_{yz}$ when $z = (n-1)y$, so for $\tau = \tau_{yz}$ the value $\bar{m}$ is also bounded from below by Assumption 3 in Proposition 1. Using this and the integrability assumption (2) the right hand side in (A.13) converges in L^1 and we get equality for $\tau = \tau_{yz}$. □

The next lemma shows, for a firm with capital stock y, that it is optimal to wait at least until X_t hits $X(y,(n-1)y)$, even if other smaller firms invest earlier.

Lemma 5. *Suppose $y > z/(n-1)$. For any stopping time τ,*

$$E\left[\int_0^\tau \mathrm{e}^{-rt}\zeta(X_t,y,L_{zt})\,\mathrm{d}t + \mathrm{e}^{-r\tau}\right] \geq E\left[\int_0^{\hat{\tau}} \mathrm{e}^{-rt}\zeta(X_t,y,L_{zt})\,\mathrm{d}t + \mathrm{e}^{-r\hat{\tau}}\right],$$

where $\hat{\tau} = \tau \vee \tau_{y,(n-1)y}$.

Proof. For $k = 1,2,\ldots$, define the following convex combinations of y and $z/(n-1)$:

$$y_k = y\left[1 - \left(\frac{n-1}{n}\right)^{k-1}\right] + \frac{z}{n-1}\left(\frac{n-1}{n}\right)^{k-1}.$$

Define $z_k = (n-1)y_k$. Note $z_1 = z$. Set $\tau_0 = 0$ and $\tau_k = \tau_{y_k,z_k}$ for $k \geq 1$. Note that $\tau_k \uparrow \tau_{y,(n-1)y}$.

We will first show, for $k \geq 1$,

$$\tau_{k-1} \leq t \leq \tau_k \quad \Rightarrow \quad \zeta(X_t,y_k,z_k) \geq \zeta(X_t,y,L_{zt})\,. \tag{A.14}$$

In the case $k=1$, (A.14) follows from the fact that $L_{zt} = z_1 = z$ for $t \leq \tau_1$ and the fact that $y_1 < y$ and ζ is decreasing in its second argument.

For $k > 1$, consider any $t \in [\tau_{k-1},\tau_k]$. Note that $y_k = y + z_{k-1} - z_k \leq y + L_{zt} - z_k$. Hence $\zeta(X_t,y_k,z_k) \geq \zeta(X_t,y+L_{zt}-z_k,z_k)$. Moreover,

$$\begin{aligned}\zeta(X_t,&y+L_{zt}-z_k,z_k) - \zeta(X_t,y,L_{zt}) \\ &= \int_0^{z_k-L_{zt}} \left[\zeta_{q_{-i}}(X_t,y-u,L_{zt}+u) - \zeta_{q_i}(X_t,y-u,L_{zt}+u)\right]\,\mathrm{d}u \geq 0\,,\end{aligned}$$

the inequality following from $\zeta_{q_{-i}} \geq \zeta_{q_i}$ and the fact that $z_k \geq L_{zt}$ for $t \leq \tau_k$. Thus, (A.14) holds for all k.

For $k \geq 0$, define $\hat{\tau}_k = \tau \vee \tau_k$. Note that $\hat{\tau}_k \uparrow \hat{\tau}$. Using (A.14) and the optimality of τ_k in the myopic problem starting from (y_k, z_k) (see Lemma 3), we obtain

$$
\begin{aligned}
0 &\geq \mathrm{E}\left[1_{\{\hat{\tau}_{k-1}<\tau_k\}}\left(\int_{\hat{\tau}_{k-1}}^{\tau_k} \mathrm{e}^{-rt}\zeta(X_t, y_k, z_k)\,\mathrm{d}t + \mathrm{e}^{-r\tau_k} - \mathrm{e}^{-r\hat{\tau}_{k-1}}\right)\right] \\
&\geq \mathrm{E}\left[1_{\{\hat{\tau}_{k-1}<\tau_k\}}\left(\int_{\hat{\tau}_{k-1}}^{\tau_k} \mathrm{e}^{-rt}\zeta(X_t, y, L_{zt})\,\mathrm{d}t + \mathrm{e}^{-r\tau_k} - \mathrm{e}^{-r\hat{\tau}_{k-1}}\right)\right] \\
&= \mathrm{E}\left[\int_{\hat{\tau}_{k-1}}^{\hat{\tau}_k} \mathrm{e}^{-rt}\zeta(X_t, y, L_{zt})\,\mathrm{d}t + \mathrm{e}^{-r\hat{\tau}_k} - \mathrm{e}^{-r\hat{\tau}_{k-1}}\right]. \qquad \text{(A.15)}
\end{aligned}
$$

The equality (A.15) follows from the fact that $\{\hat{\tau}_{k-1} < \tau_k\} = \{\hat{\tau}_{k-1} < \hat{\tau}_k\}$ and the fact that $\tau_k = \hat{\tau}_k$ on this event. When we add the right-hand sides of (A.15) from $k = 1$ to $k = \ell$ for any ℓ, we obtain

$$
0 \geq E\left[\int_{\tau}^{\hat{\tau}_\ell} \mathrm{e}^{-rt}\zeta(X_t, y, L_{zt})\,\mathrm{d}t + \mathrm{e}^{-r\hat{\tau}_\ell} - \mathrm{e}^{-r\tau}\right],
$$

or

$$
E\left[\int_0^{\tau} \mathrm{e}^{-rt}\zeta(X_t, y, L_{zt})\,\mathrm{d}t + \mathrm{e}^{-r\tau}\right] \geq E\left[\int_0^{\hat{\tau}_\ell} \mathrm{e}^{-rt}\zeta(X_t, y, L_{zt})\,\mathrm{d}t + \mathrm{e}^{-r\hat{\tau}_\ell}\right].
$$

The claim now follows by taking the limit $\ell \to \infty$, using the fact that the set

$$
\left\{\int_0^{\hat{\tau}_\ell} \mathrm{e}^{-rt}\zeta(X_t, y, L_{zt})\,\mathrm{d}t \;\middle|\; \ell \in \mathbb{N}\right\}
$$

of random variables is uniformly integrable due to the integrability assumption (2).

□

Lemma 6. *If $y \geq z/(n-1)$, then $\tau_{y,(n-1)y}$ solves the equilibrium stopping problem* (A.8).

Proof. For convenience, denote $\tau_{y,(n-1)y}$ by τ^*. For $y = (n-1)z$ the statement follows from Lemma 4. Suppose $y > (n-1)z$. By virtue of the previous lemma, we can restrict the search for optimal stopping times to those times τ satisfying $\tau \geq \tau^*$. For such a stopping time, the value achieved in (A.8) is

$$E\left[\int_0^{\tau} \mathrm{e}^{-rt}\zeta(X_t, y, L_{zt})\,\mathrm{d}t + \mathrm{e}^{-r\tau}\right] = E\left[\int_0^{\tau^*} \mathrm{e}^{-rt}\zeta(X_t, y, L_{zt})\,\mathrm{d}t + \mathrm{e}^{-r\tau^*}\right]$$
$$+ \mathrm{e}^{-r\tau^*} E\left[\int_{\tau^*}^{\tau} \mathrm{e}^{-r(t-\tau^*)}\zeta(X_t, y, L_{zt})\,\mathrm{d}t + \mathrm{e}^{-r(\tau-\tau^*)}\right]. \quad \text{(A.16)}$$

Thus, minimizing (A.8) is equivalent to minimizing

$$E\left[\int_{\tau^*}^{\tau} \mathrm{e}^{-r(t-\tau^*)}\zeta(X_t, y, L_{zt})\,\mathrm{d}t + \mathrm{e}^{-r(\tau-\tau^*)} \mid \mathrm{F}_{\tau^*}\right]. \quad \text{(A.17)}$$

Recall that $\tau^* = \inf\{t | X_t > X(y, (n-1)y)\}$. Thus,

$$X_{\tau^*} = X(y, (n-1)y) = \max_{0 \le s \le \tau^*} X_s\,.$$

This implies that

$$L_{z\tau^*} = \inf\{q \ge z \mid X_{\tau^*} \le X(q/(n-1), q)\} = (n-1)y\,.$$

Minimizing (A.17) is therefore equivalent to minimizing (A.8) given $z = (n-1)y$, and the solution of this is $\tau_{y,(n-1)y}$; i.e., the minimum value of (A.17) is attained at $\tau = \tau^*$.

□

The following lemma completes the proof of the symmetric open loop equilibrium. If all firms $j \neq i$ choose the processes Q_j^* defined in (8), then $Q_{-i} = L_z$ defined in (A.5), where $z = q_{-i0}$. For convenience, set $\tau_y = \tau_{y,(n-1)y}$. Note that Q_i^* defined in (8) and τ_y satisfy

$$\tau_y = \inf\{t \ge 0 \mid Q_{it}^* \ge y\}\,, \quad \text{(A.18)}$$

for $y > q_{i0}$, meaning that τ_y is the investment time of unit number y for the capital process Q_i^*.

Lemma 7. *Assume $q_{i0} = q_{j0}$ for all i and j and Q_j^* is given by (8) for $j \neq i$. Then Q_i^* defined in (8) maximizes (4) on $\mathrm{A}(q_{i0})$.*

Proof. Let $\xi \in A$ be an arbitrary admissible control. We can assume that $\mathrm{E}\int_0^\infty \mathrm{e}^{-rt}\,\mathrm{d}\xi_t < \infty$ as otherwise the firm value would be $-\infty$. Integrating by parts and applying the monotone convergence theorem shows that this implies $\mathrm{E}\int_0^\infty \mathrm{e}^{-rt}\xi_t\,\mathrm{d}t < \infty$, which implies further that $\mathrm{E}\left[\lim_{t\to\infty} \mathrm{e}^{-rt}\xi_t\right] = 0$. For $y \geq q_{i0}$, define

$$\tau_y^\xi = \inf\{t \geq 0 | \xi_t \geq y\}\,,$$

the investment time of unit number y. Then

$$\begin{aligned}
&\mathrm{E}\left[\int_0^\infty \mathrm{e}^{-rt}(\pi(X_t,\xi_t,L_{zt})\,\mathrm{d}t - \,\mathrm{d}\xi_t)\right]\\
&= \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-rt}(\pi(X_t,\xi_t,L_{zt}) - r(\xi_t - q_{i0}))\,\mathrm{d}t - \lim_{t\to\infty} \mathrm{e}^{-rt}\xi_t\right]\\
&= \mathrm{E}\left[\int_0^\infty \left(\mathrm{e}^{-rt}\pi(X_t,q_{i0},L_{zt}) + \int_{q_{i0}}^{\xi_t} \mathrm{e}^{-rt}(\pi_{q_i}(X_t,y,L_{zt}) - r)\,\mathrm{d}y\right)\,\mathrm{d}t\right]\\
&= \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-rt}\pi(X_t,q_{i0},L_{zt})\,\mathrm{d}t + \int_{q_{i0}}^\infty \int_{\tau_y^\xi}^\infty \mathrm{e}^{-rt}(\pi_{q_i}(X_t,y,L_{zt}) - r)\,\mathrm{d}t\,\mathrm{d}y\right]\,,
\end{aligned}$$

where we integrated by parts to obtain the first equality and changed the order of integration to obtain the third. Recalling the definition $\zeta = \pi_{q_i}$, and applying Fubini and the optimality of τ_y for $y > q_{i0}$, we get

$$\begin{aligned}
&\mathrm{E}\left[\int_0^\infty \mathrm{e}^{-rt}(\pi(X_t,\xi_t,L_{zt})\,\mathrm{d}t - \,\mathrm{d}\xi_t)\right] - \mathrm{E}\left[\int_0^\infty \mathrm{e}^{-rt}\pi(X_t,q_{i0},L_{zt})\,\mathrm{d}t\right]\\
&= \int_{q_{i0}}^\infty \mathrm{E}\left[\int_{\tau_y^\xi}^\infty \mathrm{e}^{-rt}(\zeta(X_t,y,L_{zt}) - r)\,\mathrm{d}t\right]\,\mathrm{d}y\\
&\leq \int_{q_{i0}}^\infty \mathrm{E}\left[\int_{\tau_y}^\infty \mathrm{e}^{-rt}(\zeta(X_t,y,L_{zt}) - r)\,\mathrm{d}t\right]\,\mathrm{d}y
\end{aligned}$$

with equality if $\tau_y^\xi = \tau_y$ (equivalently, if $\xi = Q_i^*$).

□

To prove the myopic optimality, note that the investment times

$$\tau_{yq_{-i0}} = \inf\{t \mid X_t > X(y,q_{-i0})\} = \inf\{t \geq 0 \mid Q_{it} \geq y\} \tag{A.19}$$

of the myopic strategy Q_i are optimal stopping times for the myopic stopping problems (see Lemma 3). The same proof as for Lemma 7 shows that Q_i is an optimal investment strategy when the rival firms hold their capital stocks constant.

B Proof of Proposition 4

Set $P_t^\alpha = X_t - b\sum_{j=1}^n Q_{jt}^\alpha$. We have

$$F(\alpha) = E\int_{[0,\tau)} \mathrm{e}^{-rt}[(P_t^\alpha - c)Q_{it}^\alpha - \mathrm{d}Q_{it}^\alpha] + E\int_{[\tau,\infty)} \mathrm{e}^{-rt}[(P_t^\alpha - c)Q_{it}^\alpha - \mathrm{d}Q_{it}^\alpha]\,.$$

The first term is independent of α. We want to take the derivative of the second term with respect to α. For $t \geq \tau$, we have

$$Q_{it}^\alpha = \inf\{q_i \geq \alpha q_{i0} \mid X_t^* \leq A + Bq_i\}\,,$$

where

$$A = \frac{\beta}{\beta-1}\left(\frac{r-\mu}{r}\right)(c+r)\,,$$
$$B = \frac{\beta}{\beta-1}\left(\frac{r-\mu}{r}\right)\left(\frac{n+\alpha}{\alpha}\right)b\,.$$

Also, for $t > \tau$,

$$P_t^\alpha = X_t - C - DQ_{it}^\alpha\,,$$

where

$$C = c\,,$$
$$D = \left(\frac{n+\alpha-1}{\alpha}\right)b\,.$$

We show below that

$$\mathrm{E}\int_{[\tau,\infty)} \mathrm{e}^{-r(t-\tau)}(P_t^\alpha Q_{it}^\alpha\,\mathrm{d}t - \mathrm{d}Q_{it}^\alpha) = -(\alpha-1)q_{i0} + \alpha q_{i0}\left(\frac{X_\tau}{r-\mu} - \frac{C + D\alpha q_{i0}}{r}\right.$$
$$\left. + \frac{X_\tau}{B}\left[\frac{(rB - 2rD + 2\mu D)(A + B\alpha q_{i0})}{r(r-\mu)(\beta-2)B} + \frac{2AD - BC - rB}{r(\beta-1)B}\right]\left(\frac{A + B\alpha q_{i0}}{X_\tau}\right)^{1-\beta}\right. \quad \text{(B.1)}$$

The right-hand derivative of this with respect to α at $\alpha = 1$ (computed using Mathematica and Maple) is

$$\frac{(n-1)\left[(n+1)^2\beta(\beta-1)b^2q_{i0}^2 + 2(n+1)(r+c)\beta b q_{i0} + 2(r+c)^2\right]}{(n+1)^3(\beta-1)(\beta-2)rb} > 0\,.$$

It remains to verify (B.1). Define

$$L_t = \log\left(\frac{A+BQ_{it}}{A+Bq_{i0}}\right)\,,$$
$$Z_t = \log\left(\frac{A+Bq_{i0}}{X_t}\right)\,,$$
$$Y_t = \log\left(\frac{A+BQ_{it}}{X_t}\right)\,.$$

Note that Z is a Brownian motion with drift, and $\mathrm{d}Z_t = -\left(\mu - \frac{1}{2}\sigma^2\right)\,\mathrm{d}t - \sigma\,\mathrm{d}B_t$. We have

$$A + BQ_{it} = \max(A + Bq_{i0}, X_t^*)\,.$$

It follows that

$$L_t = \max\left(0, \max_{0\le s\le t} -Z_s\right)\,.$$

Hence, $Y_t = L_t + Z_t$ is a Brownian motion (with drift) reflected at zero — see, e.g., Harrison (1985). Moreover, L increases only when $Y = 0$, and Y_s is a sufficient statistic for the F_s–conditional distribution of the increment $L_t - L_s$ for any $t > s$. Note that $L_t - Z_t = 2L_t - Y_t$. Also,

$$\mathrm{E}\int_0^\infty \mathrm{e}^{-rt}\mathrm{e}^{-(Z_t - Z_0)}\,\mathrm{d}t = \frac{1}{r-\mu}\,.$$

We have

$$X_t = X_0 \mathrm{e}^{-(Z_t - Z_0)}\,,$$

$$Q_{it} = \frac{A + BQ_{i0}}{B}\mathrm{e}^{L_t - L_0} - \frac{A}{B}\,, \quad L_0 = 0\,,$$

$$X_t Q_{it} = \frac{X_0(A + BQ_{i0})}{B}\mathrm{e}^{(L_t - Z_t)-(L_0 - Z_0)} - \frac{AX_0}{B}\mathrm{e}^{-(Z_t - Z_0)}\,,$$

$$Q_{it}^2 = \frac{A^2}{B^2} - \frac{2A(A + BQ_{i0})}{B^2}\mathrm{e}^{L_t - L_0} + \left(\frac{A + BQ_{i0}}{B}\right)^2 \mathrm{e}^{2(L_t - L_0)}\,,$$

$$\mathrm{d}Q_{it} = \frac{A + BQ_{i0}}{B}\mathrm{e}^{L_t - L_0}\,\mathrm{d}L_t\,.$$

We will calculate the following below:

$$h_1(y) = E\left[\int_s^\infty \mathrm{e}^{-r(t-s)+L_t - L_s}\,\mathrm{d}t \mid Y_s = y\right]\,,$$

$$h_2(y) = E\left[\int_s^\infty \mathrm{e}^{-r(t-s)+(2L_t - Y_t)-(2L_s - Y_s)}\,\mathrm{d}t \mid Y_s = y\right]\,,$$

$$h_3(y) = E\left[\int_s^\infty \mathrm{e}^{-r(t-s)+2L_t - 2L_s}\,\mathrm{d}t \mid Y_s = y\right]\,,$$

$$h_4(y) = E\left[\int_s^\infty \mathrm{e}^{-r(t-s)+L_t - L_s}\,\mathrm{d}L_t \mid Y_s = y\right]\,.$$

In terms of these functions,

$$E\int_0^\infty \mathrm{e}^{-rt} Q_{it}\,\mathrm{d}t = \frac{A + BQ_{i0}}{B} h_1(Y_0) - \frac{A}{rB}\,,$$

$$E\int_0^\infty \mathrm{e}^{-rt} X_t Q_{it}\,\mathrm{d}t = \frac{X_0(A + BQ_{i0})}{B} h_2(Y_0) - \frac{AX_0}{(r-\mu)B}\,,$$

$$E\int_0^\infty \mathrm{e}^{-rt} Q_{it}^2\,\mathrm{d}t = \frac{A^2}{rB^2} - \frac{2A(A + BQ_{i0})}{B^2} h_1(Y_0) + \left(\frac{A + BQ_{i0}}{B}\right)^2 h_3(Y_0)\,,$$

$$E\int_0^\infty \mathrm{e}^{-rt}\,\mathrm{d}Q_{it} = \frac{A + BQ_{i0}}{B} h_4(Y_0)\,.$$

It is shown below that

$$h_1(y) = \frac{1}{r} + \frac{1}{r(\beta-1)} \mathrm{e}^{-\beta y}\,, \tag{B.2a}$$

$$h_2(y) = \frac{1}{r-\mu} + \frac{1}{(r-\mu)(\beta-2)} \mathrm{e}^{(1-\beta)y}\,, \tag{B.2b}$$

$$h_3(y) = \frac{1}{r} + \frac{2}{r(\beta-2)} \mathrm{e}^{-\beta y}\,, \tag{B.2c}$$

$$h_4(y) = \frac{1}{\beta-1} \mathrm{e}^{-\beta y}\,. \tag{B.2d}$$

Straightforward algebra then yields (B.1).

To calculate h_1–h_4, use the fact that each of the following is a martingale, and hence the ds and dL_s terms of its differential vanish:

$$\int_0^s \mathrm{e}^{-rt+L_t}\,\mathrm{d}t + \mathrm{e}^{-rs+L_s} h_1(Y_s)\,,$$

$$\int_0^s \mathrm{e}^{-rt+2L_t-Y_t}\,\mathrm{d}t + \mathrm{e}^{-rs+2L_s-Y_s} h_2(Y_s)\,,$$

$$\int_0^s \mathrm{e}^{-rt+2L_t}\,\mathrm{d}t + \mathrm{e}^{-rs+2L_s} h_3(Y_s)\,,$$

$$\int_0^s \mathrm{e}^{-rt+L_t}\,\mathrm{d}L_t + \mathrm{e}^{-rs+L_s} h_4(Y_s)\,.$$

We have

$$\mathrm{d}Y = -\left(\mu - \frac{1}{2}\sigma^2\right)\,\mathrm{d}t - \sigma\,\mathrm{d}B + \mathrm{d}L\,,$$

so the ds terms vanishing implies

$$1 - rh_1 - \left(\mu - \frac{1}{2}\sigma^2\right) h_1' + \frac{1}{2}\sigma^2 h_1'' = 0\,, \tag{B.3a}$$

$$1 - (r-\mu)h_2 - \left(\mu + \frac{1}{2}\sigma^2\right) h_2' + \frac{1}{2}\sigma^2 h_2'' = 0\,, \tag{B.3b}$$

$$1 - rh_3 - \left(\mu - \frac{1}{2}\sigma^2\right) h_3' + \frac{1}{2}\sigma^2 h_3'' = 0\,, \tag{B.3c}$$

$$-rh_4 - \left(\mu - \frac{1}{2}\sigma^2\right) h_4' + \frac{1}{2}\sigma^2 h_4'' = 0\,. \tag{B.3d}$$

Equating the coefficients of dL to zero at $y = 0$ yields the boundary conditions:

$$h_1(0) + h_1'(0) = 0\,,$$

$$h_2(0) + h_2'(0) = 0\,,$$

$$2h_3(0) + h_3'(0) = 0\,,$$

$$1 + h_4(0) + h_4'(0) = 0\,.$$

Boundary conditions at $y = \infty$ are obtained by noting that $L_t - L_s \downarrow 0$ pointwise as $y = Y_s \uparrow \infty$ and using the dominated convergence theorem. This yields

$$\lim_{y\to\infty} h_1(y) = \frac{1}{r}\,,$$

$$\lim_{y\to\infty} h_2(y) = \frac{1}{r-\mu}\,,$$

$$\lim_{y\to\infty} h_3(y) = \frac{1}{r}\,,$$

$$\lim_{y\to\infty} h_4(y) = 0\,.$$

The general solutions of the differential equations (B.3) subject to the boundary conditions at infinity are

$$h_1(y) = \frac{1}{r} + A_1 \mathrm{e}^{-\beta y}\,,$$

$$h_2(y) = \frac{1}{r-\mu} + A_2 \mathrm{e}^{(1-\beta) y}\,,$$

$$h_3(y) = \frac{1}{r} + A_3 \mathrm{e}^{-\beta y}\,,$$

$$h_4(y) = A_4 \mathrm{e}^{-\beta y}\,.$$

for constants A_i. Imposing the boundary conditions at zero yields (B.2).

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