Contracts versus Salaries in Matching: Comment

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Abstract

In this note, I extend the work of Echenique (Amer. Econ. Rev. 102(1): 594-601, 2012) to show that under the assumption of unilaterally substitutable preferences a matching market with contracts may be embedded into a matching market with salaries. In particular, my result applies to the recently studied problem of cadet-to-branch matching.

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In Echenique (2012) it is shown that the matching with contracts model of Hatfield and Milgrom (2005) can be "reduced" to the job matching model with salaries of Kelso and Crawford (1982) in the following sense: Under the assumption of substitutability in markets with contracts, there exists an embedding which assigns to each market with contracts a corresponding market with salaries such that the set of stable allocations of the market remains invariant under the embedding. Furthermore the gross substitutability condition is satisfied in the market with salaries. This result was extended by Kominers (2012) to many-to-many models of matching with contracts.

The results of Echenique and of Kominers show that, under the assumption of substitutability, the matching with contracts model is essentially not more general than the Kelso-Crawford model. Nevertheless substitutable preferences are not the most general domain of preferences for which the theory of matching with contracts can be expressed. Hatfield and Kojima (2010) have pointed out that the essential results of the theory - the existence of a worker-optimal stable allocation, the rural hospitals theorem, group-strategy-proofness of the worker-optimal stable mechanism for the workers and weak Pareto optimality of the worker-optimal stable allocation for the workers -

can be proved under the strictly weaker assumption of unilateral substitutes respectively under the assumption of unilateral substitutes and the law of aggregate demand. Furthermore, in recent applications of the theory the weaker assumption of unilateral substitutes plays a crucial role (Sönmez and Switzer, 2013). To summarize this development I cite Aygün and Sönmez (2012):

"This isomorphism [of Echenique] is considered to be a highly negative result since it reduces the scope of Hatfield and Milgrom (2005) to that of Kelso and Crawford (1982). Fortunately this restrictive "equivalence" between the two models breaks under two weaker conditions, bilateral substitutes and unilateral substitutes, introduced by Hatfield and Kojima (2010). The significance of these weaker substitutes conditions was further increased when Sönmez and Switzer (2013) introduced a brand new market design application of matching with contracts, cadet-branch matching, which satisfies the unilateral substitutes condition but not the substitutes condition. "

In this note, I extend the result of Echenique (2012) to unilaterally substitutable preferences. In particular, my result applies to the cadet-to-branch matching problem as formulated by Sönmez and Switzer (2013).[1] My result is the most general one can hope for in the sense that for bilateral substitutes an embedding is not possible (Echenique, 2012).

The embedding in this note differs in two crucial aspects from the embedding of Echenique (2012) and that of Kominers (2012). The embedding does not map contracts of a market with contracts to salaries in a market with salaries. This approach would run into problems when firms' preferences over contracts are not substitutable because then a firm's ranking of contracts with a given worker depends on its contracts with other workers.[2] Instead I define the embedding in terms of allocations, i.e. I construct a one-to-one mapping of allocations in a market with contracts to allocations in a market with salaries such that the set of stable allocations remains invariant under the mapping.

Second, I allow that firms in the market with salaries may be indifferent between different salary schedules. This is crucial because under strict preferences and gross substitutability the set of stable allocations of a matching market with salaries has a lattice structure (Blair, 1988; Hatfield and Milgrom, 2005) whereas the set of stable allocations of a matching market with contracts and unilaterally substitutable preferences does not need to have

[1] In a recent paper, Kominers and Sönmez (2013) introduce "matching problems with slot-specific priorities" which generalize cadet-to-branch matching. My result does not apply to this new framework.

[2] Hatfield and Kojima (2010) call the property that a firm's ranking of contracts with a given worker is independent of its other contracts "Pareto separability". Kominers (2012) stresses the necessity of Pareto separability for the embedding method employed by him and by Echenique (2012).

a lattice structure (Hatfield and Kojima, 2010). Thus an embedding into a market where preferences of firms are strict and gross substitutable is impossible. Having indifferences in firms' preferences permits the embedding of a market with contracts and unilaterally substitutable preferences into a market with salaries where gross substitutability of preferences is satisfied.

1 Model

1.1 Matching with Contracts

Let F and W be two finite disjoint sets of agents. We call members of F *firms* and members of W *workers*. A *matching market with contracts* is a tuple $(X, (u_i)_{i \in F \cup W})$ consisting of a finite set X of *contracts*, where each contract $x \in X$ is assigned one firm x_F and one worker x_W, a utility function $u_f : 2^X \to \mathbb{R}$ for each firm $f \in F$ and a utility function $u_w : X \cup \{\emptyset\} \to \mathbb{R}$ for each worker $w \in W$. For a set of contracts $Y \subseteq X$ we denote the set of contracts in Y that involve agent $i \in F \cup W$ by Y_i and the set of firms resp. workers involved in contracts in Y by $F(Y)$ resp. $W(Y)$. In the following we assume that there are *no consumption externalities*, i.e. that for every firm $f \in F$ and set of contracts $Y \subseteq X$, we have $u_f(Y) = u_f(Y_f)$ and for every worker $w \in W$ and contract $x \in X$ we have that $x \notin X_w$ implies $u_w(x) = u_w(\emptyset)$. Furthermore we assume that agents' utilities are one-to-one[3] over contracts in which they are involved in, i.e. for every worker w and contracts $x, y \in X_w$ we have that $x \neq y$ implies $u_w(x) \neq u_w(y)$ and for every firm f and sets of contracts $Y, Z \subseteq X$ we have that $Y_f \neq Z_f$ implies $u_f(Y) \neq u_f(Z)$.

For every firm $f \in F$ the utility function u_f induces a choice function $C_f : 2^X \to 2^X$ which selects from every set of contracts the utility maximizing set of contracts

$$C_f(Y) = \text{argmax}_{Z \subseteq Y_f} u_f(Z).$$

We require that C_f satisfies

- *unilateral substitutability* (Hatfield and Kojima, 2010):
  for all $Y \subseteq X$, $z, z' \in X \setminus Y$ with $z_W \notin W(Y)$,

$$z \in C_f(Y \cup \{z, z'\}) \Rightarrow z \in C_f(Y \cup \{z\}).$$

[3]The assumption of one-to-one utilities (strict preferences) for firms in matching markets with contracts is not without problems and is criticized by Aygün and Sönmez (2013). In Section 3 I will come back to this point and discuss how my result can be extended to the case where firms' choice functions and not (one-to-one) utility function are the primitive of the model. For the time being I restrict myself to the case of one-to-one utility functions which was the case considered by Echenique (2012). But I note that a more general result without this assumption can be obtained.

A set of contracts $Y \subseteq X$ with $|Y_w| \leq 1$ for each worker $w \in W$ is called an *allocation*. An allocation Y is *individually rational* if for each firm $f \in F$ we have $C_f(Y) = Y_f$ and for each $y \in Y$ we have $u_{y_W}(y) \geq u_{y_W}(\emptyset)$. An allocation Y is *blocked* if there is a firm f and an allocation $Z \subseteq X_f$, $Z \neq C_f(Y)$, such that $Z = C_f(Y \cup Z)$ and for every $z \in Z$ we have $u_{z_W}(z) \geq \max_{x \in Y \cup \{\emptyset\}} u_{z_W}(x)$. An allocation that is individually rational and not blocked is called *stable*. We denote the set of all stable allocations in $(X, (u_i)_{i \in F \cup W})$ by $\mathcal{S}(X, (u_i)_{i \in F \cup W})$. An allocation Y is *(Pareto)-dominated* by an allocation Z if for every agent $i \in F \cup W$ we have $u_i(Z_i) \geq u_i(Y_i)$ with at least one strict inequality. We call an undominated allocation *Pareto-optimal* and denote the set of all Pareto-optimal and individually rational allocations in $(X, (u_i)_{i \in F \cup W})$ by $\mathcal{P}(X, (u_i)_{i \in F \cup W})$. Note that $\mathcal{S}(X, (u_i)_{i \in F \cup W}) \subseteq \mathcal{P}(X, (u_i)_{i \in F \cup W})$.

1.2 Matching with Salaries

We consider a set of firms F and a set of workers W. A *matching market with salaries* is a tuple $(S, (v_i)_{i \in F \cup W})$ consisting of a set $S \subseteq \mathbb{R}_+$ of *salaries*, a utility function $v_f : 2^{W \times S} \to \mathbb{R}$ for each firm $f \in F$ and a utility function $v_w : (F \times S) \cup \{\emptyset\} \to \mathbb{R}$ for each worker $w \in W$. We make the following assumptions on utilities: For each worker $w \in W$ the utility function v_w is increasing in salaries, i.e. for $f \in F$ and $s_{fw}, s'_{fw} \in S$ we have

$$s_{fw} < s'_{fw} \Rightarrow v_w(f, s_{fw}) < v_w(f, s'_{fw}).$$

For each firm $f \in F$ the utility function v_f is non-increasing in salaries, i.e. for each set of workers $A \subseteq W$ and vectors of salaries $s_f = (s_{fw})_{w \in A}$, $s'_f = (s'_{fw})_{w \in A}$ we have

$$s_f \leq s'_f \Rightarrow v_f(\{(w, s'_{fw}) : w \in A\}) \leq v_f(\{(w, s_{fw}) : w \in A\}).$$

As pointed out in the introduction, the assumption of non-increasing utilities in salaries is crucial for the embedding result and differs from Echenique (2012) who requires one-to-oneness of firms' utilities.

Furthermore we assume that for each firm f there is a demand function $D_f : S^W \to 2^W$ that assigns to a vector of salaries $s_f = (s_{fw})_{w \in W}$ a utility maximizing set of workers

$$D_f(s_f) \in \operatorname{argmax}_{A \subseteq W} v_f(\{(w, s_{fw}) : w \in A\})$$

such that D_f satisfies

- *gross substitutability* (Kelso and Crawford, 1982):
  for all $s_f \leq s'_f$ and every worker $w \in W$ with $s_{fw} = s'_{fw}$,

$$w \in D_f(s_f) \Rightarrow w \in D_f(s'_f),$$

- *assumption (A)*:
  for every worker $w \in W$ there exists a vector of salaries $s_f \in S^W$ such that,
$$w \notin D_f(s_f).$$

Assumption (A) (together with gross substitutability) guarantees that a salary for a worker can be raised to a level at which he is no longer demanded by firm f. This assumption is implicitly made by Kelso and Crawford (1982) and used in their proof of the existence of firm-optimal stable allocations via the "salary adjustment process".

An allocation in the market $(S, (v_i)_{i \in F \cup W})$ is a set $Y \subseteq F \times W \times S$ such that for $(f, w, s), (f', w', s') \in Y$ with $(f, w, s) \neq (f', w', s')$ we have $w \neq w'$. We can represent an allocation by a *matching* $\mu : W \to F \cup \{\emptyset\}$ together with a salary schedule $s = (s_{\mu(w)w})_{w:\mu(w) \neq \emptyset}$. An allocation (μ, s) is *individually rational* if for every firm f and $A \subseteq \mu^{-1}(f)$ we have

$$v_f(\{(w, s_{fw}) : w \in \mu^{-1}(f)\}) \geq v_f(\{(w, s_{fw}) : w \in A\})$$

and for every worker w we have

$$v_w(\mu(w), s_{\mu(w)w}) \geq v_w(\emptyset)$$

(here and in the following we use the notational convention that $v_w(\emptyset, s_{\emptyset w}) \equiv v_w(\emptyset)$). An allocation (μ, s) is *blocked* if there is a firm f, a set of workers $A \subseteq W$ and a salary schedule $(s'_{fw})_{w \in A}$ such that

$$v_f(\{(w, s'_{fw}) : w \in A\}) \geq v_f(\{(w, s_{fw}) : w \in \mu^{-1}(f)\})$$

and for every worker $w \in A$ we have

$$v_w(f, s'_{fw}) \geq v_w(\mu(w), s_{\mu(w)w})$$

with at least one strict inequality. An allocation that is individually rational and not blocked is called *(core)-stable*. We denote the set of all stable allocations in $(S, (v_i)_{i \in F \cup W})$ by $\mathcal{S}(S, (v_i)_{i \in F \cup W})$. An allocation (μ, s) is *(Pareto)-dominated* by an allocation (μ', s') if for every firm f we have

$$v_f(\{(w, s'_{fw}) : w \in \mu'^{-1}(f)\}) \geq v_f(\{(w, s_{fw}) : w \in \mu^{-1}(f)\})$$

and for for every worker w we have

$$v_w(\mu'(w), s'_{\mu(w)w}) \geq v_w(\mu(w), s_{\mu(w)w})$$

with at least one strict inequality. We call an undominated allocation *Pareto-optimal* and denote the set of all Pareto-optimal and individually rational allocations in $(S, (v_i)_{i \in F \cup W})$ by $\mathcal{P}(S, (v_i)_{i \in F \cup W})$. Note that $\mathcal{S}(S, (v_i)_{i \in F \cup W}) \subseteq \mathcal{P}(S, (v_i)_{i \in F \cup W})$.

2 Embedding

Now I state and prove the embedding result:

Theorem. *Let $(X, (u_i)_{i \in F \cup W})$ be a matching market with contracts such that agents' utility functions are one-to-one over contracts in which they are involved in and firms' choices are unilaterally substitutable. Then there exist*

- *for each worker $w \in W$ a utility function $v_w : (F \times \mathbb{N}) \cup \{\emptyset\} \to \mathbb{R}$ that is increasing in salaries,*
- *for each firm $f \in F$ a utility function $v_f : 2^{W \times \mathbb{N}} \to \mathbb{R}$ that is non-increasing in salaries and yields a demand function which satisfies gross substitutability and assumption (A), and*
- *a one-to-one mapping[4] $g : 2^X \to 2^{F \times W \times \mathbb{N}}, Y \mapsto (\mu, s)$ with $\mu^{-1}(f) = W(Y_f)$ for each $f \in F$*

such that

1. *the set of Pareto optimal and individually rational allocations remains invariant under the mapping, i.e.*
$$g(\mathcal{P}(X, (u_i)_{i \in F \cup W})) = \mathcal{P}(\mathbb{N}, (v_i)_{i \in F \cup W}),$$
and
2. *the set of stable allocations remains invariant under the mapping, i.e.*
$$g(\mathcal{S}(X, (u_i)_{i \in F \cup W})) = \mathcal{S}(\mathbb{N}, (v_i)_{i \in F \cup W}).$$

Proof. In the following we assume that for every agent $i \in F \cup W$ the utility function u_i only takes on integer values and is normalized such that $u_i(\emptyset) = 0$. By finiteness of X this is without loss of generality. First we will define utilities in the market with salaries. Worker w's utility of working for f under salary s_{fw} will simply be s_{fw}. Firm f's utility of hiring workers $A \subseteq W$ under salaries $(s_{fw})_{w \in A}$ will be the utility of hiring the same workers in the market with contracts under contracts which guarantee worker $w \in A$ utility of at least s_{fw}. If there are no contracts which guarantee these utility levels or if it is not individually rational for the firm to sign them then the firm will obtain utility -1 from hiring the workers under these salaries and the firm

[4] Here we embed into a market with an infinite salary set. By finiteness of X it would also be possible to embed into a market with a finite salary set instead by taking for instance the smallest $S \subset \mathbb{N}$ such that $g(2^X) \subseteq 2^{F \times W \times S}$.

will never demand the workers under these salaries. Formally for each firm $f \in F$, we define a mapping $h_f : 2^{W\times\mathbb{N}} \setminus \{\emptyset\} \to 2^X$ by

$$h_f(\{(w, s_{fw}) : w \in A\}) = C_f\left(\{x \in X_f : x_W \in A, u_{x_W}(x) \geq s_{fx_W}\}\right).$$

For an allocation (μ, s) we set $h(\mu, s) := \bigcup_{f\in F} h_f(\{(w, s_{fw}) : w \in \mu^{-1}(f)\})$. For every firm f we define a utility function by

$$v_f(\{(w, s_{fw}) : w \in A\}) := \begin{cases} u_f(h_f(\{(w, s_{fw}) : w \in A\})) & \text{if } A = W(h_f(\{(w, s_{fw}) : w \in A\})) \\ 0 & \text{if } A = \emptyset \\ -1 & \text{else} \end{cases}$$

and for every worker w we define

$$v_w(f, s_{fw}) := s_{fw}, \quad v_w(\emptyset) = 0.$$

By construction, worker w's utility function v_w is increasing in salaries, firm f's utility v_f is non-increasing in salaries and yields a well-defined demand function which is given by

$$D_f(s_f) = W(C_f\left(\{x \in X_f : u_{x_W}(x) \geq s_{fx_W}\}\right)).$$

If $s_{fw} > \max_{x\in X} u_w(x)$ then clearly $w \notin D_f(s_f)$. Thus, D_f satisfies assumption (A).

Next we show that for every firm f the demand D_f satisfies gross substitutability. Let $s_f, s'_f \in \mathbb{N}^W$ such that $s_f \leq s'_f$ and $s_{f\bar{w}} = s'_{f\bar{w}}$. We have to show that if there is a $z \in C_f\left(\{x \in X_f : u_{x_W}(x) \geq s_{fx_W}\}\right)$ with $z_W = \bar{w}$ then there exist a $z' \in C_f\left(\{x \in X_f : u_{x_W}(x) \geq s'_{fx_W}\}\right)$ with $z'_W = \bar{w}$. Defining the sets

$$\begin{aligned} Y &:= \left\{x \in X_f : x_W \neq \bar{w}, u_{x_W}(x) \geq s'_{fx_W}\right\}, \\ Z_1 &:= \left\{x \in X_f : x_W \neq \bar{w}, s'_{fx_W} > u_{x_W}(x) \geq s_{fx_W}\right\}, \\ Z_2 &:= \{x \in X_f : x_W = \bar{w}, u_{\bar{w}}(x) \geq s_{f\bar{w}}\} \end{aligned}$$

this can be reformulated as follows: If there is a $z \in Z_2$ such that $z \in C_f(Y \cup Z_1 \cup Z_2)$ then there is a $z' \in Z_2$ such that $z' \in C_f(Y \cup Z_2)$. Let $z \in C_f(Y \cup Z_1 \cup Z_2) \cap Z_2$. Then by the weak axiom of revealed preferences: $z \in C_f(Y \cup Z_1 \cup \{z\})$. As $z_W = \bar{w} \notin W(Y \cup Z_1)$, unilateral substitutability implies that $z \in C_f(Y \cup \{z\})$. Again by WARP, $C_f(Y \cup Z_2) \cap Z_2 = \emptyset$ would imply $z \notin C_f(Y \cup \{z\})$ contradicting $z \in C_f(Y \cup \{z\})$. Thus there exists a $z' \in C_f(Y \cup Z_2) \cap Z_2$.

Next we define the embedding $g : 2^X \to 2^{F\times W\times\mathbb{N}}$ and show that it has the required properties. For each $Y \in \mathcal{P}(X, (u_i)_{i\in F\cup W})$ we define

$$g(Y) = \{(x_F, x_W, u_{x_W}(x)) : x \in Y\}.$$

All other allocations $Y \notin \mathcal{P}(X, (u_i)_{i \in F \cup W})$ are mapped to arbitrary but distinct allocations $g(Y) \notin g(\mathcal{P}(X, (u_i)_{i \in F \cup W}))$. By construction, we have for each $Y \in \mathcal{P}(X, (u_i)_{i \in F \cup W})$ that $h(g(Y)) = Y$ and that $v_i(g(Y)_i) = u_i(Y_i)$ for each $i \in F \cup W$. Thus, it remains to show that $g(\mathcal{P}(X, (u_i)_{i \in F \cup W})) = \mathcal{P}(\mathbb{N}, (v_i)_{i \in F \cup W})$. First assume that $Y \in \mathcal{P}(X, (u_i)_{i \in F \cup W})$. If there would be a $(\mu, s) \in \mathcal{P}(\mathbb{N}, (v_i)_{i \in F \cup W})$ which Pareto-dominates $g(Y)$ then by construction of h, $h(\mu, s)$ would Pareto-dominate $h(g(Y)) = Y$ contradicting $Y \in \mathcal{P}(X, (u_i)_{i \in F \cup W})$. On the other hand let us assume that (μ, s) is individually rational and $(\mu, s) \notin g(\mathcal{P}(X, (u_i)_{i \in F \cup W}))$. We distinguish two cases, either $h(\mu, s) \notin \mathcal{P}(X, (u_i)_{i \in F \cup W})$ or $h(\mu, s) \in \mathcal{P}(X, (u_i)_{i \in F \cup W})$. In the first case, some $Y \in \mathcal{P}(X, (u_i)_{i \in F \cup W})$ Pareto-dominates $h(\mu, s)$. But then $g(Y)$ Pareto-dominates (μ, s) and $(\mu, s) \notin \mathcal{P}(\mathbb{N}, (v_i)_{i \in F \cup W})$. In the second case, $g(h(\mu, s))$ and (μ, s) match the same workers and firms, yield the same utility for firms and differ only in that $g(h(\mu, s))$ yields bigger or equal salaries for workers in comparison to (μ, s). Thus $g(h(\mu, s))$ Pareto-dominates (μ, s) and $(\mu, s) \notin \mathcal{P}(\mathbb{N}, (v_i)_{i \in F \cup W})$. □

3 Discussion

Choice functions versus one-to-one utility functions

Aygün and Sönmez (2013) criticize the prevalent use of the assumption of strict preferences (one-to-one utility functions) for firms in the literature on matching with contracts. They argue that this assumption has strong implications on the choice structure induced by these preferences (a version of the strong axiom of revealed preferences must hold under strict preferences) while this additional structure is not necessary to prove the results obtained in the literature. Instead of using strict preferences for firms, Aygün and Sönmez (2013) propose to work directly with firms' choice functions and impose a version of the weak axiom of revealed preferences which they call the "Irrelevance of Rejected Contracts (IRC)"[5] on firms' choices. In Aygün and Sönmez (2012) it is shown that the results, obtained by Hatfield and Kojima (2010) under the assumption of strict preferences for firms, can also be obtained under the weaker assumption of IRC for firms' choices. Sönmez and Switzer (2013) use a market with choice functions that satisfy unilateral substitutes and the IRC condition to model the matching between cadets and branches at the United States Military Academy. In the following I

[5]The choice function C_f satisfies *irrelevance of rejected contracts (IRC)* (Aygün and Sönmez, 2013) if for all $Y \subseteq X$, $z \in X \setminus Y$,

$$z \notin C_f(Y \cup \{z\}) \Rightarrow C_f(Y) = C_f(Y \cup \{z\}).$$

sketch how my result can be rephrased in the setting proposed by Aygün and Sönmez (2013).

Assume that we are given a market with contracts where each worker w has a one-to-one utility function $u_w : X_w \cup \{\emptyset\} \to \mathbb{R}$ and each firm f has a choice function $C_f : 2^{X_f} \to 2^{X_f}$ which satisfies IRC and unilateral substitutability. We again define workers' utilities in a corresponding market with salaries by $v_w(f, s_{fw}) = s_{fw}$. Instead of defining firms' utilities we now specify for each firm f a choice function $\tilde{C}_f : 2^{F\times W\times\mathbb{N}} \to 2^{F\times W\times\mathbb{N}}$ for the market with salaries: For each firm f and set $Y \subseteq F \times W \times \mathbb{N}$ we define a corresponding set of contracts $h_f(Y) \subseteq X$ via

$$h_f(Y) := \{x \in X_f : \exists (f, w, s_{fw}) \in Y \text{ such that } u_w(x) \geq s_{fw}\}$$

and let (f, w, s_{fw}) be chosen from Y, i.e. $(f, w, s_{fw}) \in \tilde{C}_f(Y)$, if a contract that guarantees worker w a utility of at least s_{fw} is chosen from the corresponding set $h_f(Y)$ in the market with contracts. If several salaries correspond to a contract chosen from $h_f(Y)$ then the highest salary among them is selected. In other words, for $(f, w, s'_{fw}) \in Y$ we have

$$\begin{aligned}(f, w, s_{fw}) \in \tilde{C}_f(Y) \Leftrightarrow &(\exists x \in C_f(h_f(Y)) \text{ such that } u_w(x) \geq s_{fw} \\ &\text{and } \nexists (f, w, s'_{fw}) \in Y \text{ with } u_w(x) \geq s'_{fw} > s_{fw}).\end{aligned}$$

Stability for the market with salaries is now defined in the same way as for the market with contracts interpreting the market with salaries as a market with contracts with contract set $F \times W \times \mathbb{N}$, firms' choice functions $\tilde{C}_f$ and workers' utility functions $\tilde{u}_w(f, w, s_{fw}) \equiv v_w(f, s_{fw}) = s_{fw}$. The choice function $\tilde{C}_f$ induces a demand function given by

$$D_f(s_f) := W(\tilde{C}_f(\{(f, w, s_{fw}) : w \in W\})) = W(C_f(\{x \in X_f : u_{x_W}(x) \geq s_{f x_W}\})).$$

With the same argument (replace WARP by the IRC condition) as in the proof of the theorem one can show that D_f is gross substitutable. An embedding is defined in essentially the same way as in the theorem. Each stable allocation $Y \subseteq X$ is mapped to the set $g(Y) = \{(x_F, x_W, u_{x_W}(x)) : x \in Y\}$ and unstable allocations are mapped to arbitrary distinct allocations in the market with salaries. One can check that the set of stable allocations remains invariant under this embedding.

Salary adjustment process

My assumptions on firms' utilities in the market with salaries differ from the quasi-linearity in salaries assumed by Kelso and Crawford (1982). As Echenique (2012) has pointed out, quasi-linearity is not necessary for the existence proof of stable allocation with the salary adjustment process of Kelso and Crawford (1982). Instead the weaker assumptions of strictly decreasing utility in salaries for firms together with assumption (A) is sufficient. In the theorem however, I only assumed non-increasing utilities in salaries. Nevertheless, the salary adjustment process of Kelso and Crawford also terminates under the weaker assumptions of non-decreasing utilities in salaries and assumption (A). The terminal allocation is not necessarily stable because it can fail to be Pareto optimal. But the salary adjustment process can be augmented by a final step where after the termination of the process each worker's salary is raised to the highest salary among the salaries which give a firm the same utility. Depending on which worker's salary is raised first, second, third etc. this might lead to different allocations. But each of these allocations is stable.

Bilateral substitutes

The notion of embeddability employed in this paper is slightly different from the one used by Echenique (2012). Nevertheless, it remains true that bilateral substitutability[6] is not sufficient for the embeddability of a market with contract in a market with salaries. This can be shown with the same example - originally due to Hatfield and Kojima (2010) - which is also used in Echenique (2012). Consider a market given by firms $F = \{f, f'\}$, workers $W = \{w, w'\}$, contracts $X = \{x, \tilde{x}, z, \tilde{z}, z'\}$, where x and $\tilde{x}$ involve f and w, z and $\tilde{z}$ involve f and w', z' involves f' and w', and utilities which induce the following preferences:

$\succeq_f$	$\succeq_{f'}$	$\succeq_w$	$\succeq_{w'}$
$\{x, z\}$	$\{z'\}$	$\tilde{x}$	z
$\{\tilde{z}\}$	$\emptyset$	x	z'
$\{\tilde{x}\}$		$\emptyset$	$\tilde{z}$
$\{x\}$			$\emptyset$
$\{z\}$			
$\emptyset$			

One readily checks that the preferences satisfy the bilateral substitutes condition. The two stable allocations in the market are $\{x, z\}$ and $\{\tilde{x}, z'\}$. Now assume for the sake of contradiction that there is a matching market with salaries $(S, (v_i)_{i \in F \cup W})$ such that workers' utility functions are strictly increasing in salaries, firms' utility functions are non-increasing in salaries and yield a demand which satisfies gross substitutability and assumption

[6]The choice function C_f satisfies *bilateral substitutability* (Hatfield and Kojima, 2010) if for all $Y \subseteq X$, $z, z' \in X \setminus Y$ with $z_W, z'_W \notin W(Y)$,

$$z \in C_f(Y \cup \{z, z'\}) \Rightarrow z \in C_f(Y \cup \{z\}).$$

(A). Assume furthermore that there are matchings $\mu, \bar{\mu}$ in $(S, (v_i)_{i \in F \cup W})$ together with salaries $s, \bar{s} \in S^{F \times W}$ such that $\mu^{-1}(f) = W(\{x, z\}) = \{w, w'\}$, $\mu^{-1}(f') = \emptyset$, $\bar{\mu}^{-1}(f) = W(\tilde{x}) = \{w\}$ and $\bar{\mu}^{-1}(f') = W(z') = \{w'\}$ and

$$\begin{aligned} D_f(s_f) = \{w, w'\}, &\quad D_f(\bar{s}_f) = \{w\} \\ D_{f'}(s_{f'}) = \emptyset, &\quad D_{f'}(\bar{s}_{f'}) = \{w'\} \\ v_w(f, s_{fw}) &> \max\{v_w(f', s_{f'w}), v_w(\emptyset)\} \\ v_w(f, \bar{s}_{fw}) &> \max\{v_w(f', \bar{s}_{f'w}), v_w(\emptyset)\} \\ v_{w'}(f, s_{fw'}) &> \max\{v_{w'}(f', s_{f'w'}), v_{w'}(\emptyset)\} \\ v_{w'}(f', \bar{s}_{f'w'}) &> \max\{v_{w'}(f, \bar{s}_{fw'}), v_{w'}(\emptyset)\} \end{aligned}$$

Since v_f is non-increasing in salaries and D_f satisfies gross substitutability we must have $s_{fw'} < \bar{s}_{fw'}$. If additionally, $\bar{s}_{f'w'} \leq s_{f'w'}$ then since $v_{w'}$ is increasing in salaries we would have

$$v_{w'}(f, \bar{s}_{fw'}) > v_{w'}(f, s_{fw'}) > v_{w'}(f', s_{f'w'}) \geq v_{w'}(f', \bar{s}_{f'w'})$$

contradicting $v_{w'}(f, \bar{s}_{fw'}) < v_{w'}(f', \bar{s}_{f'w'})$. Thus we must have $\bar{s}_{f'w'} > s_{f'w'}$. But since $v_{f'}$ is non-increasing in salaries this would imply

$$v_{f'}(\{w'\}, \bar{s}_{f'w'}) \leq v_{f'}(\{w'\}, s_{f'w'}) < v_{f'}(\emptyset)$$

contradicting $D_{f'}(\bar{s}_{f'}) = \{w'\}$. Thus there cannot be an embedding of the market with contracts into a market with salaries.

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