Potential Competitors in Preemption Games*

Catherine Bobtcheff† Thomas Mariotti‡

First draft: July 2007

Abstract

The purpose of this paper is to study the adoption of a new technology by a firm when the competitor comes into play at a random date that can be seen as her birth date. The presence of a competitor is thus only revealed when she invests. We show that there exists a unique Bayesian equilibrium that can be split into three stages. No firm will invest before a threshold T_1^* even if she is born before. After another threshold T_2^* that is strictly less than the date that maximizes the expected payoff function, any firm immediately invests at birth. In between, the equilibrium is in mixed strategies. On $[T_1^*, T_2^*]$, newborn firms compete with firms that were born earlier.

Keywords:
JEL Classification:

---

*

†Toulouse School of Economics (LERNA/CNRS).
‡Toulouse School of Economics (GREMAQ/CNRS and IDEI) and CEPR.

1. Introduction

Timing games of entry in which different players strategically compete to adopt a new technology have been first modeled by Reinganum (1981). In a setting where the adoption by one firm decreases the profit of the other and where each firm precommits to an investment date, the equilibrium consists in a sequential adoption. Fudenberg and Tirole (1985) by relaxing the precommitment assumptions show that two types of equilibria may emerge: one in which the two firms simultaneously invest and another one in which the two firms invest at two different dates, but their rents are equalized. This latter equilibrium is referred to as preemption equilibrium. These two pioneer papers have been the starting point to many extensions.

The purpose of this paper is to study the adoption of a new technology by a firm that does not know whether she faces a competitor. This situation arises each time a firm undertakes an innovative project: before an investment occurs, she does not know whether she is the only one who has had the idea. In research also, any researcher is afraid somebody develops the same idea and publishes the paper first. In fact, two opposite effects come into play: on the one hand, each firm wants to wait until the technology is really efficient (or the paper very well written), but on the other hand, any firm fears somebody invests before. In our model, each player comes into play at a random date that can be seen as her birth date and the presence of a competitor is only revealed when she invests. Intuitively, the outcome of such a game allows to link the two seminal models of Reinganum and Fudenberg and Tirole. On the one hand, if you know nobody will never threat you, you retrieve Reiganum's results; on the other hand, Fudenberg and Tirole's results apply if you know a competitor is already born for sure. In the case we propose to study where the firm does not know whether her competitor is already born, each firm is tempted to delay entry since she might be alone on the market. But as time passes, the probability that a competitor is already born increases and any supplementary waiting may reveal to be a disaster.

From a theoretical viewpoint, our work is related to the continuous time game literature. Fudenberg and Tirole (1985) already noted the difficulty to solve timing game in continuous time. They artificially enlarged the probability space by introducing the probability of investing given that at least one player will invest. Simon and Stinchcombe (1989) explain how a continuous time game can be obtained as the limit of a discrete time game. They need however to put restrictions on the form of the game in order their results to apply. In our model however, we do not meet this difficulty since the probability that the two players simultaneously move equals 0.

Before we turn to the analysis of our model, we rapidly relate our paper to the literature focusing on such preemption games. Hoppe (2002) proposes a complete review of literature on the timing of new technology adoption. Usually in these types of game, there is either a first mover advantage or a second mover advantage. We are here interested with the former. Dutta, Lach and Rustichini (1995) model the strategic behavior of firms in the development phase of R&D when firms choose the quality of the good they produce. They exhibit two kinds of equilibria: a preemption equilibrium as in Fudenberg and Tirole (1985) and a maturation equilibrium in which one firm enters optimally with a more developed technology and the other enters earlier and takes the advantage of the temporary monopoly position. Hoppe and Lehmann-Grube (2005) extend the initial model of Fudenberg and Tirole by introducing the possibility to have access to a better technology in case investment occurs late. They are thus brought to work in the case where the discounted expected profit of the leader is not hump-shaped but rather has several local maxima. They show that the equilibrium outcome is unique: either payoffs are equalized (in a preemption equilibrium) or there is a second mover advantage. Argenziano and Schmidt-Dengler (2007) also propose an extension of the model of Fudenberg and Tirole to N players and highlight the presence of clusters in some cases: several players players invest at the same date. Brunnermeier and Morgan (2006) propose a model of clock games that are characterized by a waiting motive (the more a firm waits, the more valuable the market) and a preemption motive. At the beginning of the game, there is a time interval during which every player receives a signal on the exit payoff. In equilibrium, either all players exit immediately, or they all wait the same time before they exit. The larger the interval, the longer the delay before players exit. Hendricks (1992) extends the initial two players model by differentiating the two firms and adding incomplete information. Each firm is either an innovator or an imitator in which case she will never invest first. In this model, the longer the firm waits, the more likely the other player is an imitator. Hendricks characterizes the equilibrium: if t is small, there is no investment; if t is high enough and lower that the date that maximizes the leader's expected profit, there exists a mixed strategy equilibrium. No investment takes place later. Our paper is different at least for two reasons. First, our two firms are symmetric and second, in our setting, the longer you wait, the more likely your competitor is already born. Therefore, the equilibrium we characterize can be split into three stages according to the birth date. First no firm will invest before a threshold T_1^* even if she is born before. Second all the firms born before a date T_2^* that is lower that the date that maximizes the leader expected profit will have invested at T_2^*, according to a mixed strategy equilibrium on $[T_1^*, T_2^*]$. This means that on $[T_1^*, T_2^*]$, competition takes place between new born firms and firms that were born earlier. And finally, a firm born after T_2^* will immediately invest. Indeed, when t is high the probability of being preempted is too high relative to the gain in the expected profit, an immediate investment turns thus out to be the equilibrium.

The paper is organized as follows. We present the model in section 2, and section 3 is devoted to the analysis of the equilibrium.

2. The Model

Time is continuous, and indexed by $t \geq 0$. There are two players, 1 and 2. In what follows, i refers to an arbitrary player and j to the other player. Each player i comes into play at some random date $\tau^i \geq 0$, which we shall refer to as her date of birth. In an investment timing game, τ^i may represent the date at which player i discovers a new idea or the existence of an investment opportunity.

Actions and Payoffs. Both players are risk-neutral and discount future utilities at rate $r > 0$. Each player i has a single opportunity to make a move at some time $t \geq \tau^i$. If player i moves first, at time t, she obtains a payoff $L(t)$ evaluated in terms of time 0 utilities, while player j obtains a payoff 0. If players i and j move simultaneously, they both obtain a strictly negative payoff $S(t)$. This simple payoff structure may for instance arise in an investment timing game where two firms contemplate investing on a market that can accommodate only one of them because Bertrand competition would otherwise draw profits to 0. Denoting by

$P(t)$ the date t flow monopoly profit, and by $I(t)$ the cost of investing at date t, one then has

$$L(t) = \int_t^{\infty} e^{-rs} P(s)\, ds - e^{-rt} I(t) \tag{1}$$

for all $t \geq 0$. We shall maintain the following assumption throughout the paper.

Assumption 1. *L is twice continuously differentiable. Moreover, there are times $T_2 > T_1 > 0$ such that the following holds*:

$$L(t) < 0 \ \ \text{if} \ \ t \in [0, T_1), \tag{2}$$

$$L(t) > 0 \ \ \text{if} \ \ t \in (T_1, \infty), \tag{3}$$

$$\dot{L}(t) > 0 \ \ \text{if} \ \ t \in [0, T_2), \tag{4}$$

$$\dot{L}(t) < 0 \ \ \text{if} \ \ t \in (T_2, \infty), \tag{5}$$

$$\ddot{L}(t) \leq 0 \ \ \text{if} \ \ t \in [T_1, T_2]. \tag{6}$$

Thus L vanishes only at T_1, $L(T_1) = 0$, reaches its unique maximum at T_2, $\dot{L}(T_2) = 0$, and is concave over $[T_1, T_2]$.

An example that fits these assumptions is an investment timing game in which the monopoly profit flow is given by $P(t) = e^{\mu t}$ for all $t \geq 0$ and the investment cost is given by $I > 1/(r - \mu)$. Formula (1) then yields

$$L(t) = e^{-rt} \left(\frac{e^{\mu t}}{r - \mu} - I \right)$$

for all $t \geq 0$, and times T_1 and T_2 are respectively defined by $e^{\mu T_1} = (r - \mu) I$ and $e^{\mu T_2} = rI$. It is straightforward to verify that L is concave over $[T_1, T_2]$.

Times T_1 and T_2 have straightforward game-theoretic interpretations. Suppose that both players are born at date 0, as in a standard preemption game. Then, at time T_1, each player is indifferent between moving first and becoming a leader, or not moving and becoming a follower, as both options yield her a payoff 0. This corresponds to the unique equilibrium outcome of the complete information preemption model of Fudenberg and Tirole (1985), in which a single player moves at time T_1. Time T_2, by contrast, is the time at which it would be optimal for each player to move if she believed that she were not threatened by her opponent's moving first. When the players are two firms which can invest on a market that can accommodate only one of them, this corresponds to the unique equilibrium outcome of the complete information precommitment model of Reinganum (1981), in which one firm invests at time T_2 and the other firm stays out of the market.[1]

[1]To see that, note that if firm i stays out of the market, it is optimal for firm j to invest at time T_2. Conversely, suppose that firm j commits to invests at time T_2. Investing at any time $t > T_2$ cannot a best

Information. Each player i's date of birth τ^i is exponentially distributed with rate $\lambda > 0$, and independent of the date of birth τ^j of the other player. The date of birth of each player is her private information, or type. In particular, when a player is born, she does not know whether the other player is already born. Nor does she observe when the other player is born when this event occurs after her own birth. It is in that sense that competition is only potential in our model. The only information that accrues to players in the course of the game is whether and when the other player makes a move, which ends the game given our assumption on payoffs.

Compared to previous analyses of preemption games, a distinctive feature of our model is that a player in this game never knows whether she indeed has an opponent, except when it is too late and she has been preempted. However, as time goes by, each player finds it less and less likely that she has no opponent, which drives many of our results. This model can be seen as a perturbation of two limit situations. Taking $\lambda = 0$ leads to the precommitment model of Reinganum (1981), because each player, if ever born, believes that she will never have an opponent. Taking $\lambda = \infty$ leads to the preemption model of Fudenberg and Tirole (1985), because it is common knowledge from date 0 on that each player has an opponent.

3. Equilibrium

In this section, we explicitly characterize a symmetric perfect Bayesian equilibrium of the game. We first informally describe the main features of this equilibrium, leaving the details of its construction to a technical section.

3.1. Overview

Consider first some type $\tau^i \in [T_2, \infty)$ of player i, and suppose that player j has not moved yet at time τ^i. Then, since L is strictly decreasing over $[T_2, \infty)$, τ^i has no incentive to delay her move, if she believes that there is a probability 0 that she will thereby tie with player j. This is for instance the case if the following conditions hold:

- C.1 any $\tau^j \in [0, T_2)$ moves with probability 1 before time T_2,
- C.2 any $\tau^j \in [T_2, \infty)$ moves immediately at time τ^j.

C.1 ensures that τ^i faces no competition from any $\tau^j \in [0, T_2)$. C.2 along with the fact that the distribution of each player's type is atomless implies that τ^i has a probability 0 of tying with some $\tau^j \in [T_2, \infty)$. Hence, provided C.1 is satisfied, we have constructed an equilibrium over $[T_2, \infty)$ such that each player born in this time interval moves immediately at birth. It should be noted that this also describes the off-equilibrium path behavior of any player: at any date $t > T_2$ at which no player has moved yet, it is optimal for any player born before date t to move immediately.

---

response for firm i, because she would earn no profits. Investing at time $t \le T_2$ yields firm i a payoff

$$\int_t^{T_2} e^{-rs} P(s)\, ds - e^{-rt} I(t),$$

which is maximal for $t = T_2$ and equal there to $-e^{-rT_2} I(T_2)$. Hence firm i is better off staying out of the market altogether, and (T_2, ∞) is an equilibrium in precommitment strategies

What happens before time T_2? Suppose for the sake of the argument that player j always moves immediately at birth. This means that the distribution of player j's moving times is exponential with parameter λ. As a result of this, any type $\tau^i \in [0, T_2]$ of player i faces a simple problem. Indeed, if she decides to invest at some time $t \in [\tau^i, T_2]$, she obtains an expected payoff equal to

$$e^{-\lambda(t-\tau^i)}L(t) \tag{7}$$

in terms of time 0 utilities. Now, let time T_2^* be implicitly defined by

$$\frac{\dot{L}(T_2^*)}{L(T_2^*)} = \lambda. \tag{8}$$

That T_2^* is uniquely defined by (8) and lies in (T_1, T_2) follows from Assumption 1. It is straightforward to check that the maximum of (7) over $t \in [\tau^i, \infty)$ is attained at T_2^* if $\tau^i \leq T_2^*$ and at τ^i if $\tau^i > T_2^*$. At time T_2^*, the growth rate of moving first, $\dot{L}(T_2^*)/L(T_2^*)$, is equal to the rate at which the other player moves, λ. Hence, the marginal benefit of delaying one's move by an infinitesimal amount of time dt, $\dot{L}(T_2^*)dt$, exactly compensates the corresponding expected marginal loss, $\lambda L(T_2^*)dt$. Now consider any type $\tau^i \in [T_2^*, T_2)$ of player i, and suppose that player j has not moved yet at time τ^i. Then, if the following conditions hold:

C.1* any $\tau^j \in [0, T_2^*)$ moves with probability 1 before time T_2^*,

C.2* any $\tau^j \in [T_2^*, \infty)$ moves immediately at time τ^j,

τ^i has no incentive to delay her move. Provided C.1* is satisfied, this implies as above that there exists an equilibrium over $[T_2^*, T_2)$ such that each player born in this time interval moves immediately at birth. The intuition for this result is more subtle than for the time interval $[T_2, \infty)$, where the gains of moving first are strictly decreasing. By contrast, there are gains from waiting to move over $[T_2^*, T_2)$, as L is strictly increasing over that range. Yet, these gains are offset by the probability that the other player invests, as $\dot{L} < \lambda L$ over (T_2^*, T_2). Competition in $[T_2^*, T_2)$ takes place at the margin. Indeed, if a player is born in this time interval, she knows, whenever C.1* and C.2* hold, that she is the first player to be born. It is her fear that an opponent born just after her may actually preempt her by moving first that leads her to move immediately at birth. As over the interval $[T_2, \infty)$, these strategies also describe the off-equilibrium path behavior of any player. Under C.1* and C.2*, the probability that a player moves after time T_2^* is simply $e^{-\lambda T_2^*}$.

Immediately moving at birth cannot be part of an equilibrium before time T_2^*. Indeed, one of the player would then have an incentive to wait until T_2^* before moving, which would then in turn lead the other player to delay slightly her move. We conjecture that players born before T_2^* choose to move according to a mixed strategy. For a player i of type 0, this mixed strategy will be represented by a cumulative distribution function G_0 over $[0, T_2^*]$ (see for instance Pitchik (1981)). For any $t \in [0, T_2^*]$, $G_0(t)$ represents the probability that player i of type 0 has moved by time t. To define the corresponding mixed strategy for a player i of type $\tau^i \in [0, T_2^*)$, one rolls over the cumulative distribution function G_0 by defining $G_{\tau^i}(t)$ using Bayes' rule,

$$G_{\tau^i}(t) = \frac{G_0(t) - G_0(\tau^i)}{1 - G_0(\tau^i)} \tag{9}$$

for all $t \in [\tau^i, T_2^*]$. It should be noted that G_{τ^i} represents both the mixed strategy of type τ^i of player i, and the probability that a player i of type $\tilde{\tau}^i \in [0, \tau^i]$ moves between times τ^i and t conditional on not having moved before time τ^i. Indeed, by (9),

$$\frac{G_0(t) - G_0(\tau^i)}{1 - G_0(\tau^i)} = \frac{G_{\tilde{\tau}^i}(t) - G_{\tilde{\tau}^i}(\tau^i)}{1 - G_{\tilde{\tau}^i}(\tau^i)} \tag{10}$$

for all $\tilde{\tau}^i \in [0, \tau^i]$. That this is consistent with equilibrium behavior reflects the fact that all types in $[0, T_2^*]$ face the same payoff function. This implies that our preemption game with incomplete information does not exhibit a single-crossing condition. As a result of this, a type $\tilde{\tau}^i$ who has not yet moved by time $\tau^i \in [\tilde{\tau}^i, T_2^*)$ need not behave from this time on differently from type τ^i.

To complete the description of the equilibrium, we need to characterize the cumulative distribution function of moving times of type 0, G_0. We conjecture that the support of the corresponding probability measure consists in an interval $[T_1^*, T_2^*]$, for some $T_1^* \in (T_1, T_2^*)$, and that G_0 is continuous over $[0, T_2^*]$, so that in particular $G_0(T_1^*) = 0$ and $G_0(T_2^{*-}) = 1$. This last condition implies that C.1* holds: all players born before T_2^* move with probability 1 before that time. Given the continuation equilibrium characterized by C.2*, this implies that the equilibrium payoff of type 0 of player i is

$$e^{-\lambda T_2^*} L(T_2^*). \tag{11}$$

Since type 0 of player i must be indifferent between moving at T_1^* or at T_2^*, and no one moves before T_1^*, T_1^* is implicitly defined by

$$L(T_1^*) = e^{-\lambda T_2^*} L(T_2^*). \tag{12}$$

Let $\mathcal{G}(t)$ be the unconditional equilibrium probability that player j moves before time t. In the conjectured equilibrium, $\mathcal{G}(t) = 0$ for all $t \in [0, T_1^*]$, as player j moves with probability 1 after T_1^*, and $\mathcal{G}(t) = 1 - e^{-\lambda t}$ for all $t \in [T_2^*, \infty)$, as player j moves with probability 1 before T_2^* when born before that time, and moves immediately at birth after that time. Player i of type 0 is indifferent between moving at any time $t \in [T_1^*, T_2^*]$ if and only if

$$[1 - \mathcal{G}(t)] L(t) = L(T_1^*) \tag{13}$$

for all $t \in [T_1^*, T_2^*]$. Special cases of (13) are (11), since $\mathcal{G}(T_2^*) = 1 - e^{-\lambda T_2^*}$, and (12), since $\mathcal{G}(T_1^*) = 0$. For each $t \in (T_1^*, T_2^*)$, (13) can be rewritten in differential form as follows:

$$\frac{\dot{L}(t)}{L(t)} = \frac{\dot{\mathcal{G}}(t)}{1 - \mathcal{G}(t)}. \tag{14}$$

Equation (14) says that, at any time $t \in (T_1^*, T_2^*)$, the marginal benefit of delaying one's move by an infinitesimal amount of time dt, $\dot{L}(t)dt$, exactly compensates the corresponding expected marginal loss, which is equal to the probability that the other player will move in the interval of time $[t, t + dt]$ conditional on not having moved before t, $\dot{\mathcal{G}}(t)dt/[1 - \mathcal{G}(t)]$, multiplied by the lost benefit $L(t)$.

For the distribution $\mathcal{G}$ to be consistent with the postulated equilibrium strategies, it must be equal to the average of the type-dependent distributions of moving times. This implies in particular that the following must hold:

$$\int_0^t G_\tau(t) \lambda e^{-\lambda \tau} \, d\tau = \mathcal{G}(t) \tag{15}$$

for all $t \in [T_1^*, T_2^*]$. Equivalently by (9), this means that G_0 solves the integral equation

$$\int_0^t \frac{G_0(t) - G_0(\tau)}{1 - G_0(\tau)} \lambda e^{-\lambda\tau} \, d\tau = \mathcal{G}(t) \tag{16}$$

where, by (13),

$$\mathcal{G}(t) = 1 - \frac{L(T_1^*)}{L(t)} \tag{17}$$

for all $t \in [T_1^*, T_2^*]$. The construction of the equilibrium will be completed once it will be shown that the integral equation (16) has a positive and increasing solution G_0 that is continuous over $[0, T_2^*]$, and satisfies $G_0(T_1^*) = 0$ and $G_0(T_2^*) = 1$. We turn to this task in the next subsection.

3.2. Solving the Integral Equation

We will say that a measurable function G is *admissible* if $G = 0$ over $[0, T_1^*)$, $0 \leq G < 1$ over $[T_1^*, T_2^*)$, and the function $1/(1 - G)$ is integrable over any interval $[T_1^*, t]$ for $t \in (T_1^*, T_2^*)$. We start with a useful lemma.

Lemma 1. *Any admissible solution G_0 to (16) is continuous over $[0, T_2^*)$, differentiable over (T_1^*, T_2^*), and strictly increasing over $[T_1^*, T_2^*)$.*

From Lemma 1, one can differentiate (16) to obtain

$$\dot{G}_0(t) \int_0^t \frac{1}{1 - G_0(\tau)} \lambda e^{-\lambda\tau} \, d\tau = \dot{\mathcal{G}}(t) \tag{18}$$

for all $t \in (T_1^*, T_2^*)$. To solve equation (18) over the interval (T_1^*, T_2^*), let us introduce an auxiliary function

$$I(t) = \int_0^t \frac{1}{1 - G_0(\tau)} \lambda e^{-\lambda\tau} \, d\tau, \tag{19}$$

which is well-defined for all $t \in [T_1^*, T_2^*)$ as long as G_0 is admissible, which we shall thereafter assume. This allows one to rewrite (18) as

$$\dot{G}_0(t) I(t) = \dot{\mathcal{G}}(t) \tag{20}$$

for all $t \in (T_1^*, T_2^*)$. Lemma 1 implies that I is twice differentiable over (T_1^*, T_2^*), with $\dot{I} > 0$ over this interval. We now eliminate G_0 from (20). From (19), one has $G_0(t) = 1 - \lambda e^{-\lambda t}/\dot{I}(t)$ and therefore $\dot{G}_0(t) = \lambda e^{-\lambda t}[\lambda + \ddot{I}(t)/\dot{I}(t)]/\dot{I}(t)$ for all $t \in (T_1^*, T_2^*)$. Substituting in (20) yields

$$\lambda e^{-\lambda t} \left[\lambda + \frac{\ddot{I}(t)}{\dot{I}(t)} \right] = \frac{\dot{I}(t)}{I(t)} \dot{\mathcal{G}}(t) \tag{21}$$

for all $t \in (T_1^*, T_2^*)$. Define now

$$H(t) = \frac{\dot{I}(t)}{I(t)} \tag{22}$$

for $t \in [T_1^*, T_2^*)$. Lemma 1 implies that H is differentiable over (T_1^*, T_2^*), and it is easy to check that $\ddot{I}/\dot{I} = \dot{H}/H + H$ over this interval. Substituting in (21) then yields the following Bernoulli equation for H:

$$\dot{H}(t) = -\lambda H(t) + \left[\frac{e^{\lambda t}}{\lambda} \dot{\mathcal{G}}(t) - 1 \right] H^2(t) \tag{23}$$

for all $t \in (T_1^*, T_2^*)$. This equation can standardly be transformed into a linear differential equation in $Z = -1/H$ (see for instance Walter (1998, Chapter I, Section 2)),

$$\dot{Z}(t) = \lambda Z(t) + \frac{e^{\lambda t}}{\lambda} \dot{\mathcal{G}}(t) - 1,$$

which can be easily solved to yield the general solution to (23):

$$H(t) = \frac{1}{K_2 \lambda - \mathcal{G}(t) - e^{-\lambda t}} \lambda e^{-\lambda t}, \tag{24}$$

for some constant K_2 yet to be determined. One can check from (8) and (17) that for H to be defined over the whole interval $[T_1^*, T_2^*)$, as required, one must have $K_2 \geq 1/\lambda$, because the denominator of (24) precisely vanishes at T_2^* whenever $K_2 = 1/\lambda$. From (22) and (24), it therefore follows that I is of the form

$$I(t) = K_1 \exp\left(\int_0^t \frac{1}{K_2 \lambda - \mathcal{G}(\tau) - e^{-\lambda \tau}} \lambda e^{-\lambda \tau} \, d\tau \right), \tag{25}$$

for some strictly positive constant K_1 yet to be determined. It is then straightforward to recover G_0 from (19) and (25). This yields

$$G_0(t) = 1 - \frac{K_2 \lambda - \mathcal{G}(t) - e^{-\lambda t}}{K_1 \exp\left(\int_0^t \frac{1}{K_2 \lambda - \mathcal{G}(\tau) - e^{-\lambda \tau}} \lambda e^{-\lambda \tau} \, d\tau \right)} \tag{26}$$

for all $t \in (T_1^*, T_2^*)$. The constants K_1 and K_2 are pinned down by requiring respectively that $G_0(T_1^*) = 0$, which follows from the admissibility of G_0, and $G_0(T_2^{*-}) = 1$, which amounts to say that G_0 is continuous at T_2^*. Imposing the second of these conditions leads to the following lemma.

Lemma 2. *Any admissible solution G_0 to (16) satisfying $G_0(T_2^{*-}) = 1$ is such that*

$$K_2 = \frac{1}{\lambda}. \tag{27}$$

The constant K_1 is then obtained by letting $G_0(T_1^*) = 0$. Using (26) and (27) along with the fact that $\mathcal{G}(T_1^*) = 0$, this yields

$$K_1 = (1 - e^{-\lambda T_1^*}) \exp\left(- \int_0^{T_1^*} \frac{1}{1 - \mathcal{G}(\tau) - e^{-\lambda \tau}} \lambda e^{-\lambda \tau} \, d\tau \right). \tag{28}$$

From the definitions (8) and (17) of T_2^* and $\mathcal{G}$, it is straightforward to check that the mapping $t \mapsto \mathcal{G}(t) + e^{-\lambda t}$ is strictly increasing over $[T_1^*, T_2^*]$. Together with (26), (27) and (28), this confirms that G_0 is strictly increasing over this interval.

We have constructed a positive and increasing function G_0 that is continuous over $[0, T_2^*]$, that satisfies $G_0(T_1^*) = 0$ and $G_0(T_2^*) = 1$, and that solves (18) over (T_1^*, T_2^*). To show that G_0 solves (16) as required, one needs only to integrate (18) by parts, and then impose the condition that $G_0 = \mathcal{G} = 0$ over $[0, T_1^*]$.

3.3. Discussion

We first summarize the results so far obtained. Using (17), (26), (27) and (28), one obtains the following proposition.

Proposition 1. *There exists a perfect Bayesian equilibrium of the game in which:*

(i) At any time $\tau < T_2^*$ such that no player has moved yet, the distribution over $[\tau, T_2^*]$ of the moving times of any player born before or at that time is given by

$$G_\tau = \frac{G_0 - G_0(\tau)}{1 - G_0(\tau)}, \tag{29}$$

where $G_0 = 0$ over $[0, T_1^*]$ and

$$G_0(t) = 1 - \frac{L(T_1^*)/L(t) - e^{-\lambda t}}{1 - e^{-\lambda T_1^*}} \exp\left(- \int_{T_1^*}^{t} \frac{1}{L(T_1^*)/L(\tau) - e^{-\lambda \tau}} \lambda e^{-\lambda \tau} \, d\tau \right) \tag{30}$$

for all $t \in [T_1^*, T_2^*]$.

(ii) At any time $\tau \geq T_2^*$ such that no player has moved yet, any player born before or at that time moves immediately.

To obtain more information on how players compete in this equilibrium, it is useful to determine the shape of the function G_0. One has the following results.

Corollary 1. *G_0 is concave over (T_1^*, T_2^*).*

Corollary 2. $\lim_{t \to T_2^*} \dot{G}_0(t) = 0$.

Corollary 1 implies that the probability that a player born before T_1^* will move in the time interval $[t, t + dt]$ is smaller, the larger is $t \in [T_1^*, T_2^*)$. Strengthening this insight, Corollary 2 implies that the probability that such a player moves in a time interval $[T_2^* - dt, T_2^*]$ is very small, namely of the order $o(dt)$. Similarly, players born between T_1^* and T_2^* are more likely to move close to their dates of birth than close to T_2^*. Overall, competition is fiercer close to T_1^*, and then tends to decrease. This is reflected in the rate with which players moves. To see that, note that since players move independently, the distribution of the first moving time is given by $1 - (1 - \mathcal{G})^2$. It then follows from (14) that over the interval $[T_1^*, T_2^*]$, the corresponding hazard rate is given by $2\dot{L}/L$, which is strictly decreasing as L is positive, strictly increasing and concave over this interval. From T_2^* onward, this hazard rate is constant and equal to 2λ, since players born after T_2^* move immediately at birth.

The players' rate of birth λ plays a crucial role in the analysis. Along with (8), the concavity of L over $[T_1^*, T_2^*]$ implies that T_2^* is a strictly decreasing function of λ, and so is T_1^* by (12). It is straightforward to verify from (17) that a positive shift in λ corresponds to a negative shift in $\mathcal{G}$ in the sense of first-order stochastic dominance, reflecting that players tend to move earlier as the fear of being preempted increases. When λ goes to 0, both T_1^* and T_2^* go to T_2, and the equilibrium payoff of a player born before T_2 goes to $L(T_2)$. This corresponds to the outcome of the precommitment model of Reinganum (1981). By contrast, when λ goes to ∞, both T_1^* and T_2^* go to T_1, and the equilibrium payoff of a player born before T_1 goes to 0. This corresponds to the outcome of the preemption model of Fudenberg

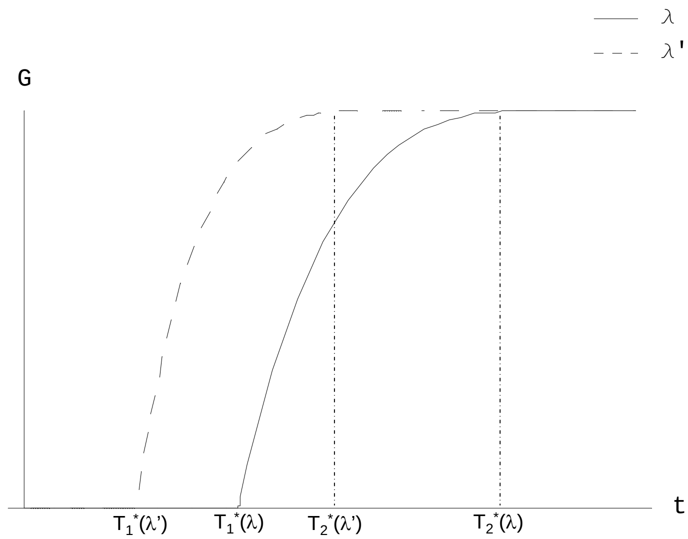

Figure 1: Function G_0 with $\lambda' > \lambda$.

and Tirole (1985). In this latter case, the distribution of the first moving time converges weakly to a Dirac mass at T_1 and all rents are dissipated in the limit.

It should be noted that, because both the distribution of player's types and that of their moving times are atomless, the probability of a joint move by both players is equal to 0 for any value of λ. Thus coordination failures play no role in our analysis, besides the fact that they are assumed to be always detrimental to both players. By contrast, models that use a discrete-time game with very short time lags to represent continuous time, following Simon and Stinchcombe (1989), explicitly need to rule out the possibility of coordination failures, for instance through the use of an ad-hoc randomization device (see for instance Dutta, Lach and Rustichini (1995), Hoppe and Lehmann-Grube (2005), or Argenziano and Schmidt-Dengler (2007)). In the alternative formulation of Fudenberg and Tirole (1985), coordination failures are crucial for determining moving intensities off the equilibrium path, although they cannot arise in equilibrium.

An interesting quantity to focus on to understand how competition evolves in time is the probability that a player (for instance player i) is not yet born at t given that no investment already occurred. Denoting this probability $q_i(t)$, we have the following result.

Corollary 3. $q_i(t) = e^{-\lambda t} L(t) / L(T_1^*)$.

From Corollary 3 and the definition of T_2^* (equation (8)), we observe that $t \mapsto q_i(t)$ is increasing until $t = T_2^*$, and is decreasing afterwards. Remember that T_2^* is the first date from which an immediate investment is the equilibrium strategy. Following Hendricks (1992), we refer to $q_i(t)$ as the reputation of player i: the reputation of any player is thus maximal at T_2^*. This means that once the reputation of a player begins to decrease, she prefers to move immediately. She is indeed afraid an other player preempts her. The analysis of this function q_i allows to distinguish our model with the one of Hendricks (1992), where the reputation (the probability that a player is an imitator and thus never wants to invest first) is strictly increasing until T_2. Moreover the reputation is a submartingale:

$$\begin{aligned}\mathbb{E}\left[q_i(t+\tau)|\mathcal{F}_t\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\text{player } i \text{ is not yet born in} t+\tau}|\mathcal{F}_{t+\tau}\right]|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\text{player } i \text{ is not yet born in } t+\tau}|\mathcal{F}_t\right] \\ &< \mathbb{E}\left[\mathbb{1}_{\text{player } i \text{ is not yet born in } t}|\mathcal{F}_t\right] \\ &= q_i(t).\end{aligned}$$

This means that, as t increases, it is increasingly difficult to maintain a reputation of not being born. Therefore, players are less ready to wait to invest and immediate move occurs from T_2^* on. A last point that deserves attention in our setting concerns the growth rate of $q_i(t)$. As $\dot{q}(t)/q(t) = -\lambda + \dot{L}(t)/L(t)$, the larger the λ, the less rapid the growth rate of reputation. Indeed a large λ means that the birth rate is large and thus the probability that the opponent is not born does not increase very rapidly.

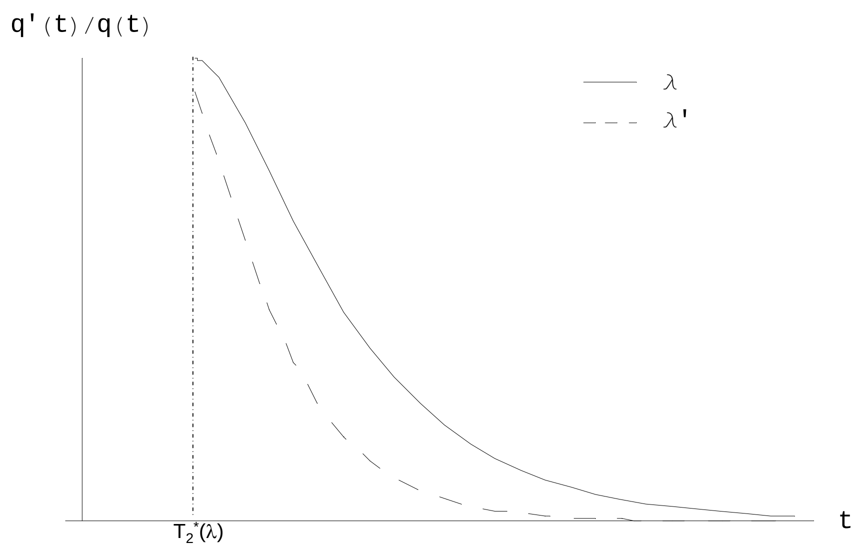

Figure 2: Function $\frac{q'(t)}{q(t)}$ with $\lambda' > \lambda$.

Next proposition deals with a technical result, indeed the uniqueness of the equilibrium we derived in the class of the perfect Bayesian equilibria.

Proposition 2. *There exists a unique perfect Bayesian equilibrium.*

4. Concluding remarks

The purpose of this paper was to study the adoption of a new technology by a firm that does not know when she will face a competitor. We derive the unique Bayesian equilibrium of the game that can be split into three stages according to the birth date. First no firm will invest before a threshold T_1^* even if she is born before. In fact all the firms born before a date T_2^* that is lower that the date that maximizes the leader expected profit will have invested at T_2^*, according to a mixed strategy equilibrium on $[T_1^*, T_2^*]$. And finally, a firm born after T_2^* will immediately invest. Indeed, when t is high the probability of being preempted is too high relative to the gain in the expected profit, an immediate investment turns thus out to be the equilibrium.

Appendix

Proof of Lemma 1. Note first that, by construction, $\mathcal{G}$ is continuous over $[0, T_2^*]$. From (17) along with the fact that $T_2 > T_2^* > T_1^* > T_1$, it is also straightforward to verify that $\mathcal{G}$ is differentiable over (T_1^*, T_2^*), with $\dot{\mathcal{G}} > 0$ over this interval. If G_0 is admissible, one can rewrite (16) as

$$G_0(t) = \frac{\mathcal{G}(t) + \displaystyle\int_0^t \frac{G_0(\tau)}{1 - G_0(\tau)}\, \lambda e^{-\lambda\tau}\, d\tau}{\displaystyle\int_0^t \frac{1}{1 - G_0(\tau)}\, \lambda e^{-\lambda\tau}\, d\tau} \tag{A.1}$$

for all $t \in [T_1^*, T_2^*)$ because, for any such t, the functions within the integrals in (A.1) are integrable over $[0, t]$. This implies that these integrals are themselves continuous functions of $t \in [T_1^*, T_2^*)$. Thus, since $\mathcal{G}$ is continuous over $[T_1^*, T_2^*)$, so is G_0. Note also from (A.1) that $G_0(T_1^*) = 0$, since $\mathcal{G}(T_1^*) = 0$ and $G_0 = 0$ over $[0, T_1^*)$. Hence G_0 is continuous over $[0, T_2^*)$, as claimed. It follows in particular that the functions with the integrals in (A.1) are continuous over $[0, T_2^*)$. This implies that these integrals are themselves differentiable functions of $t \in [0, T_2^*)$. Thus, since $\mathcal{G}$ is differentiable over (T_1^*, T_2^*), so is G_0. One can then differentiate (16) to obtain

$$\dot{G}_0(t) \int_0^t \frac{1}{1 - G_0(\tau)}\, \lambda e^{-\lambda\tau}\, d\tau = \dot{\mathcal{G}}(t) \tag{A.2}$$

for all $t \in (T_1^*, T_2^*)$. Since G_0 is admissible and $\dot{\mathcal{G}} > 0$ over (T_1^*, T_2^*), it follows from (A.2) that G_0 is strictly increasing over $[T_1^*, T_2^*)$. Hence the result. ■

Proof of Lemma 2. Regarding (26), two cases can occur. Either the numerator of the second term on the right-hand side of (26) has a finite limit as t goes to T_2^*. Then $G_0(T_2^{*-}) = 1$ implies that $K_2 = [\mathcal{G}(T_2^*) + e^{-\lambda T_2^*}]/\lambda = 1/\lambda$ by (17). Or the denominator of the second term on the right-hand side of (26) has a infinite limit as t goes to T_2^*. However, this can only happen if the integral within the exponential term diverges at T_2^*, which again implies that $K_2 = [\mathcal{G}(T_2^*) + e^{-\lambda T_2^*}]/\lambda = 1/\lambda$ by (17). The result follows. ■

Proof of Corollary 1. According to (20), $\dot{G}_0 = \dot{\mathcal{G}}/I$ over (T_1^*, T_2^*). It then follows from (17), (25), (27) and (28) that $\dot{G}_0$ is differentiable over this interval. Differentiating (18) and using (20) leads to

$$\ddot{G}_0(t) = \frac{\dot{G}_0(t)}{\dot{\mathcal{G}}(t)} \left[\ddot{\mathcal{G}}(t) - \frac{\dot{G}_0(t)}{1 - G_0(t)}\, \lambda e^{-\lambda t}\right]$$

for all $t \in (T_1^*, T_2^*)$. Since $\dot{\mathcal{G}} > 0$ and $\dot{G}_0 > 0$ over (T_1^*, T_2^*), one needs only to check that $\ddot{\mathcal{G}} \leq 0$ over this interval, which follows immediately from (4), (6) and (17). Hence the result. ■

Proof of Corollary 2. Since $G_0 = 1$ over $[T_2^*, \infty)$, one needs only to check that $\lim_{t \to T_2^{*-}} \dot{G}_0(t) = 0$. According to (8) and (17),

$$\lim_{t \to T_2^{*-}} \dot{\mathcal{G}}(t) = \lambda e^{-\lambda T_2^*} < \infty. \tag{A.3}$$

Thus, given (20), it is enough to establish that $\lim_{t \to T_2^{*-}} I(t) = \infty$, where, by (25) and (27),

$$I(t) = K_1 \exp\left(\int_0^t \frac{1}{1 - \mathcal{G}(\tau) - e^{-\lambda\tau}}\, \lambda e^{-\lambda\tau}\, d\tau\right) \tag{A.4}$$

for all $t \in [T_1^*, T_2^*)$, for some constant $K_1 > 0$. A Taylor–Young expansion to the left of T_2^* yields

$$1 - \mathcal{G}(\tau) - e^{-\lambda\tau} = o(\tau - T_2^*)$$

which follows from the fact that $1-\mathcal{G}(T_2^*)-e^{-\lambda T_2^*}=0$, along with (A.3). Hence

$$\int_0^t \frac{1}{1-\mathcal{G}(\tau)-e^{-\lambda\tau}}\,\lambda e^{-\lambda\tau}\,d\tau=\int_0^t \frac{1}{o(\tau-T_2^*)}\,\lambda e^{-\lambda\tau}\,d\tau$$

diverges to ∞ at T_2^*, which implies the result given (A.4). ■

Proof of Corollary 3. To compute the reputation, we have to update the probability of not being born at some t using Bayes' rule by taking into account the fact that no investment has been undertaken until t:

$$q_i(t)=\frac{e^{-\lambda t}}{e^{-\lambda t}+(1-e^{-\lambda t})\,p},$$

where p is the probability that player i did not invest before t given that she is already born in t. This probability equals

$$p=\frac{1}{1-e^{-\lambda t}}\int_0^t (1-G_s(t))\,\lambda e^{-\lambda s}ds.$$

Using (9), $q_i(t)=e^{-\lambda t}/\left(e^{-\lambda t}+I(t)(1-G_0(t))\right)$. From (19), $I(t)(1-G_0(t))=I(t)\lambda e^{-\lambda t}/\dot{I}(t)=\lambda e^{-\lambda t}/H(t)$. Recalling that $H(t)=\lambda e^{-\lambda t}/\left(1-\mathcal{G}(t)-e^{-\lambda t}\right)$, we finally obtain that

$$q_i(t)=e^{-\lambda t}L(t)/L(T_1^*)\,. \tag{A.5}$$

■

Proof of Proposition 2. We are going to prove the result in different steps.

We first prove that the distribution function $\mathcal{G}$ does not have any atom on $[T_2,+\infty[$. Indeed, if we prove this is the case, the expected payoff of a player investing at t equals

$$(1-\mathcal{G}(t))\,L(t) \tag{A.6}$$

that is strictly decreasing $\forall t\geq T_2$. It is thus not optimal to wait and therefore $\mathcal{G}(t)=1-e^{-\lambda t}$, $\forall t\in[T_2,+\infty[$.

If $\mathcal{G}_i$ has a unique atom on $T>T_2$, then the expected payoff of player j at this date is

$$\left(\mathcal{G}_i(T)-\mathcal{G}_i\left(T^-\right)\right)S(T)+(1-\mathcal{G}_i(T))\,L(T)\,. \tag{A.7}$$

The expected payoff at $t=T-\varepsilon$ equals

$$(1-\mathcal{G}_i(T-\varepsilon))\,L(T-\varepsilon) \tag{A.8}$$

since $T-\varepsilon$ is not an atom. If player j is born before T, she will never invest at $t=T$ since

$$(1-\mathcal{G}_i(T-\varepsilon))\,L(T-\varepsilon)>(1-\mathcal{G}_i(T))\,L(T)\,, \tag{A.9}$$

(remember that $t\mapsto(1-\mathcal{G}(t))\,L(t)$ is decreasing on $[T_2,+\infty[$). Therefore, for ε small enough,

$$(1-\mathcal{G}_i(T-\varepsilon))\,L(T-\varepsilon)>\left(\mathcal{G}_i(T)-\mathcal{G}_i\left(T^-\right)\right)S(T)+(1-\mathcal{G}_i(T))\,L(T)\,, \tag{A.10}$$

and player j has an interest to invest at $t=T-\varepsilon$ rather than $t=T$. Therefore there is no atom at $t=T$ in the distribution of player i.

Let us now focus on player j. Suppose her distribution presents an atom at $t = T$. If player i invests at $t = T$, she gets an expected profit:

$$\begin{aligned}
(1-\mathcal{G}_j(T+\varepsilon))\,L(T+\varepsilon) &\to_{\varepsilon\to 0} (1-\mathcal{G}_j(T))\,L(T) \\
&> \left(\mathcal{G}_j(T)-\mathcal{G}_j\left(T^-\right)\right)S(T)+(1-\mathcal{G}_j(T))\,L(T)\,. \qquad \text{(A.11)}
\end{aligned}$$

Thus player i would like to invest just after time T that does not exist. Therefore the distribution of player j does not have an atom neither. It follows that there is no atom on $[T_2, +\infty[$ and $\forall t \geq T_2$, $\mathcal{G}(t) = 1 - e^{-\lambda t}$.

The second step is to prove that there does not exist any mixed strategy equilibrium on a finite interval in $[t, T_2]$*, with* $t \geq T_2^*$*.* We first assume that there exists a mixed strategy equilibrium on $[t, T_2]$. In this case, $(1-\mathcal{G}(s))\,L(s)$ is constant on $[t, T_2]$. Differentiating with respect to s leads to

$$\lambda > \frac{\dot{L}(s)}{L(s)} = \frac{\dot{\mathcal{G}}_j(s)}{1-\mathcal{G}_j(s)}, \forall s \in [t, T_2]\,. \qquad \text{(A.12)}$$

Integrating this inequality on $[t, T_2]$ leads to

$$1-\mathcal{G}_j(t) > e^{-\lambda t} \qquad \text{(A.13)}$$

but this cannot happen (the number of investment cannot be greater than the number od born people).

The third step consists in proving that there is no atom on $[0, T_2]$*.* We prove this results distinguishing two cases: whether there exists an interval where the distribution function is equal to zero or not.

First, we suppose that there exists an interval $[a, b]$ such that $\mathcal{G}_j(t) = 0, \forall t \in [a, b]$. This means that player i never invests on $[a, b]$. The only way player j has a reason to invest at $t = b$ for instance is that player i invests with a strictly positive probability on $[b, b+\varepsilon)$. Suppose there is an atom at $t = b + \varepsilon$. If we denote $\Delta(\varepsilon)$ the quantity $\mathcal{G}_j(b+\varepsilon) - \mathcal{G}_j(b^-)$, player i's expected payoff by investing at $t = b + \varepsilon$:

$$\begin{aligned}
(1-\mathcal{G}_j(b+\varepsilon))\,L(b+\varepsilon)+\left(\mathcal{G}_j(b+\varepsilon)-\mathcal{G}_j\left(b+\varepsilon^-\right)\right)S(b+\varepsilon) &< (1-\mathcal{G}_j(b+\varepsilon))\,L(b+\varepsilon) \\
&= \left(1-\mathcal{G}_j\left(b^-\right)-\Delta(\varepsilon)\right)L(b+\varepsilon) \\
&< \left(1-\mathcal{G}_j\left(b^-\right)\right)L\left(b^-\right), \qquad \text{(A.14)}
\end{aligned}$$

where the first inequality holds since function S is negative and the second one since $\Delta(\varepsilon)$ is large and ε small. Player i has thus not interest to wait until $b + \varepsilon$ and gets a higher expected payoff by investing at $t = b^-$. Therefore there does not exist an atom on the distribution of player i for $t > b$.

Now we suppose that $\mathcal{G}_j$ is not equal to 0 on interval $[a, b]$. In this case, player j also invests with a strictly positive probability on interval $[b, b+\varepsilon]$. If it were not the case, player j's expected payoff of investing at $t = b$ would be

$$\begin{aligned}
(1-\mathcal{G}_i(b))\,L(b)+\left(\mathcal{G}_i(b)-\mathcal{G}_i\left(b^-\right)\right)S(b) &\leq (1-\mathcal{G}_i(b))\,L(b) \\
&= (1-\mathcal{G}_i(b+\varepsilon))\,L(b) \\
&\leq (1-\mathcal{G}_i(b+\varepsilon))\,L(b+\varepsilon)\,, \qquad \text{(A.15)}
\end{aligned}$$

and player j would have an interest to wait until $b + \varepsilon$. Player i thus invests with a strictly positive probability on $[b, b+\varepsilon]$ and we recover in the same case than before: if there is an atom at $b + \varepsilon$ for instance in player i's distribution, player j would invests just before this atom. Therefore, there does not exist any atom on the distribution of the players on $[0, T_2]$.

The fourth step consists in proving that there does not exist any interval on $[0, T_2]$ *on which the distribution function is equal to zero.* Suppose such an interval $[a, b]$ exists where player j does not invest. Player i's expected payoff of investing at $t = a + \varepsilon$ equals

$$\begin{aligned}(1 - \mathcal{G}_j(a - \varepsilon)) L(a - \varepsilon) &< (1 - \mathcal{G}_j(a)) L(b) \\ &= (1 - \mathcal{G}_j(b)) L(b). \end{aligned} \tag{A.16}$$

This means that player i does not invest neither and prefers to wait until $t = b$. But in this case, each player has an incentive to preempt her rival and thus there does not exist an interval where one of the players does not invest. Therefore function $\mathcal{G}_i$ and $\mathcal{G}_j$ are continuous and increasing and thus differentiable almost every where.

The last step consists in obtaining the shape of $t \mapsto (1 - \mathcal{G}(t)) L(t)$ *on* $[T_2^*, T_2]$. First as there does not exist a mixed strategy equilibrium on $[T_2^*, T_2]$, $(1 - \mathcal{G}(t)) L(t)$ is not constant on this interval. Second, if $t \mapsto (1 - \mathcal{G}(t)) L(t)$ on $[T_2^*, T_2]$ were increasing on an interval $[a, b] \subset [T_2^*, T2]$, then the players born between a and b would wait until b to invest and b would therefore be an atom. But we proved that this cannot happen. Therefore, $t \mapsto (1 - \mathcal{G}(t)) L(t)$ is a decreasing function on $[T_2^*, T_2]$

All these steps allow us to conclude that the equilibrium we derived is the unique one in the class of perfect Bayesian equilibria. ■

References

[1] Argenziano, R., and P. Schmidt-Dengler (2007): "N-Player Preemption Games," Mimeo, University of Essex.

[2] Brunnermeier, M.K., and J. Morgan (2006): "Clock Games: Theory and Experiments," working paper.

[3] Dutta, P.K., S. Lach, and A. Rustichini (1995): "Better Late than Early: Vertical Differentiation in the Adoption of a New Technology," *Journal of Economics and Management Strategy*, 4, 563–589.

[4] Fudenberg, D., and J. Tirole (1985): "Preemption and Rent Equalization in the Adoption of New Technology," *Review of Economic Studies*, 52, 383–401.

[5] Hendricks K. (1992): "Reputation in the Adoption of a New Technology," *International Journal of Industrail Organization*, 10, 663–677.

[6] Hoppe, H.C. (2002): "The Timing of New Technology Adoption: Theoretical Models and Empirical Evidence," *The Manchester School*, 70(1), 56–76.

[7] Hoppe, H.C., and U. Lehmann-Grube (2005): "Innovation Timing Games: A General Framework with Applications," *Journal of Economic Theory*, 121, 30–50.

[8] Pitchik (1981): "Equilibria of a Two-Person Non-Zerosum Noisy Game of Timing," *International Journal of Game Theory*, 10, 207–221.

[9] Reinganum, J.F. (1981): "On the Diffusion of New Technology: A Game Theoretic Approach," *Review of Economic Studies*, 48, 395–405.

[10] Simon, L.K., and M.B. Stinchcombe (1989): "Extensive Form Games in Continuous Time: Pure Strategies," *Econometrica*, 57, 1171–1214.

[11] Walter, W. (1998): *Ordinary Differential Equations*, New York, Springer-Verlag.