Optimal capital accumulation and embodied technological progress under uncertainty*

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Abstract

This paper proposes a stochastic model of investment with embodied technological progress, in which firms invest not only to expand the capacity as in Pindyck (1988) but also to replace old machines. The scrapping decision or the age of the oldest machine is then endogenous and evolves stochastically. Uncertainty increases the optimal age of the machines in use, and due to uncertainty, not only capacity expansion but replacement as well, are postponed. By introducing heterogenous capital units, the model gets rid from the perfect "procyclicity" of investment usually implied in the literature of irreversible investment under uncertainty. The so-called cleansing effect of recessions appears since replacement can occur even in bad realizations of the stochastic process.

1. Introduction

At the plant level, Doms and Dunne (1998) observe that investment occurs infrequently and in burst. Using a 17 years sample, they show that the five years biggest investment episodes account for more than 50% of total US investment The fact that investment is lumpy and infrequent is well documented for France by Jamet (2000): on a 13 years sample the three largest investments account for 75% of total investment in the French economy. Firms also stay long periods inactive since each year, almost 20% of the firms do not invest at all. Such observations contrast with the result of the neoclassical model of investment with convex adjustment costs. Pindyck (1988) develops a model of capacity expansion, showing how uncertainty and irreversibility can affect the decision to invest. This model is able to reproduce the infrequency and lumpiness of investment at firm-level. Indeed, Pindyck (1988) obtains that firms only invest when the stochastic variable reaches

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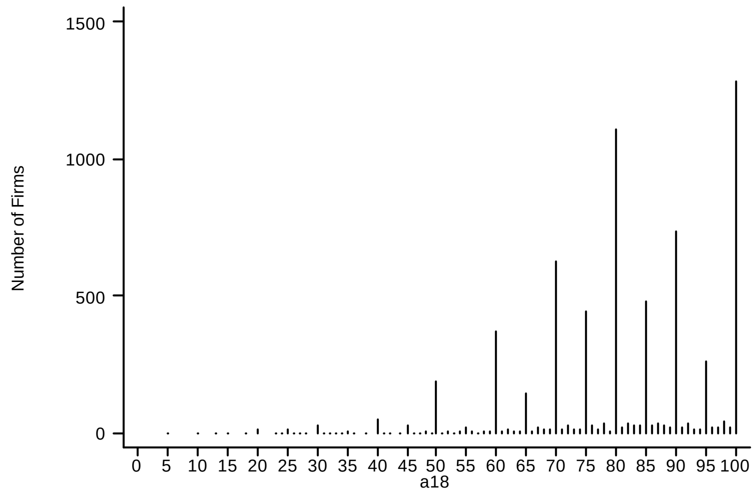
its historically most favorable values and investment only occurs to expand capacity. With homogenous units, all the installed capital (which means all the old machines) must be used before the firm starts investing. This contradicts the observation. For instance, Figure 1 presents the distribution of capacity utilization of firms having an annual growth of capital higher than 20 percent in the Spanish manufacturing industry[1] for the period 1991 to 2001. Even if the mode of the distribution is at 100 percent, there is still a large fraction of expanding firms using less than full capacity. Furthermore, the average growth rate of capital for firms with a capacity utilization less than 85% is not much lower than that for firms using full capacity (113.18% against 120.26%)[2].

Figure 1- Distribution of Capacity Utilization
of firms presenting a peak of investment higher than 20%
in the Spanish Manufacturing Industry - 1991 to 2001.



Source: Fundacion SEPI - Encuestas sobre Estrategias Empresariales

There is also evidence that technological progress is largely investment specific. We observe that the probability that a peak of investment occurs is increasing with time (see among others Caballero, Engle and Haltiwanger, (1995), Cooper, Haltiwanger and Power (1999)). Moreover, as time passes, the relative price of capital goods is declining and the ratio equipment-GDP is raising. Therefore, investment decisions and technological progress seem to be utterly interrelated. It is then relevant to consider models with embodied technology which are able to generate endogenous scrapping (see for instance Cooley et. alli (1997); Boucekkine, Germain and Licandro (1997), Pommeret and Boucekkine (2003)) that is, to explain replacement. Nevertheless, these models remain in

[1] This represents 3.654 firms out of 8.072.

[2] Computations based on the Encuestas Sobre las Estragegias Empresiales a panel for the spanish manufacturing firms, collected by Fundacion SEPI.

a deterministic environment while it has been recognized that to a large extent, investment is irreversible and that the stochastic nature of the environment matters therefore a lot to explain investment undertaken by firms.

The question is what drives investment. Surely, firms invest for two reasons: to expand capacity and to replace old machines. In this paper, we propose to explain capital accumulation by taking into account these two motivations in a stochastic framework. The model considers embodied technological progress and irreversible investment under uncertainty. It is shown to be consistent with the following empirical observations:

- Investment is lumpy and infrequent at the firm level
- Firms can invest even if they have not reached full capacity.
- Technological progress is largely investment specific

We extend the paper of Pindyck (1988), by introducing embodied technological progress[3]. The model presented here exhibits interesting characteristics: firms invest not only to expand capacity, as in Pindyck (1988), but also to replace old machines. We compare cases with disembodied and embodied technological progress under uncertainty, also focusing on the way uncertainty and the technological progress affect the investment process under embodied technological progress. To produce, firms use irreversible capital, perfectly flexible labor, and energy whose price is stochastic. Capital and energy are complementary. Under disembodied technological progress, all the machines become more energy saving at each period, contrasting with the embodied case for which only the new machines are more efficient in terms of energy requirements. Results for firms under disembodied technological progress are of course of the same nature as those of Pindyck (1988) in which there is no technological progress. All the units should be used before the firm starts investing which seems counterfactual (see figure 1). On the contrary, under embodied technology, units are no longer homogenous, and this induces firms to replace old energy-inefficient units by newer and less energy consuming units even if they do not expand capacity.

Results are then different from those obtained in a deterministic environment (see Pommeret and Boucekkine, 2003) : under uncertainty, the endogenous optimal age of the oldest machine evolves stochastically ; moreover, the optimal effective stock of capital (the one which is effectively used as opposed to the total stock of capital) is no more constant as it is in the deterministic counterpart of the model. Therefore, by allowing a stochastic environment, this paper contributes to the literature on embodied technological progress.

Introducing embodied technology in a standard model of irreversible investment under uncertainty allows to get rid of the perfect "procyclical" behavior observed when capital units are assumed to be homogenous. Under embodiment, we take into account the fact

[3]Depreciation was included in some papers, (see for instance Dixit and Pindyck (1994) chapter 6) , but none of them have, as far as we are informed, accounted for embodied technological progress.

that the firm can react to shocks in two ways, through a variation in the rate of new units acquisition or through a variation in the rate of destruction. Investment may therefore be undertaken even in unfavorable periods when firms need to replace old machines by new ones. This may generate a cleansing effect of recession as pointed out by Bresnahan and Raff (1991) and studied by Caballero and Hammour (1994, 1996) or Gooslbee (1998)[4]. Therefore, by allowing replacement investment, this paper also contributes to the literature on irreversible investment under uncertainty.

Next section presents a model of investment under uncertainty with disembodied technology. Section 3 develops the model with embodied technological progress. Section 4 presents a dynamic example to illustrate the theoretical results. Section 5 concludes.

2. Disembodied technological progress under uncertainty: a benchmark

As a benchmark, we consider a standard monopolistic competition economy under uncertainty in which the technical progress is disembodied. In fact, this section briefly presents a model that is very similar to the one proposed by Pindyck (1988). Here, the energy price is uncertain and follows a geometric Brownian motion[5].

$$dPe(t) = \mu Pe(t)dt + \sigma Pe(t)dz(t)$$

where $Pe(t)$ is the energy price at time t. μ is the deterministic energy price trend which is disturbed by exogenous random shocks. $dz(t)$ is the increment of a standard Wiener process ($E(dz) = 0$ and $V(dz) = dt$). σ is the size of uncertainty, that is, it gives the strength with which this price reacts to the shocks. The problem the firm has to solve is

$$\max E_0 \left[\int_0^\infty \left[P(t)Q(t) - Pe(t)\ E(t) - \varpi(t)L(t) - k(t)I(t) \right] e^{-rt} dt \right] \tag{2.1}$$

subject to constraints taking uncertainty into account

$$\begin{aligned}
P(t) &= bQ(t)^{-\theta} \qquad \text{with } \theta < 1 && (2.2)\\
Q(t) &= AK_{eff}(t)^{\beta} L(t)^{1-\beta} && (2.3)\\
dPe(t) &= \mu Pe(t)dt + \sigma Pe(t)dz(t) && (2.4)\\
E(t) &= K_{eff}(t)e^{-\gamma t}dz \text{ with } \gamma < r && (2.5)\\
\varpi(t) &= \overline{\varpi} && (2.6)\\
I(t) &= dK(t) \geq 0 && (2.7)\\
K(t) &\geq K_{eff} && (2.8)
\end{aligned}$$

[4] He finds that after the second oil shock, the probability of retirement of a Boeing 707 has more than doubled in the aircraft industry.

[5] See Epaulard and Pommeret (2002) for a justification.

$P(t)$ is the market price of the good produced by the firm, $Q(t)$ is the production, the demand price elasticity is $(-1/\theta)$. We assume that any capital unit that has been installed may temporarily not be used for free. We note $K_{eff}(t)$ the capital stock which is **effectively used** in the production while $K(t)$ refers to the capital the firm has installed which encompasses used units and unused units. $L(t)$ is labour $E(t)$ stands for the energy use and $I(t)$ is investment ; $\varpi(t)$ is the wage rate, $Pe(t)$ is energy price and $k(t)$ is the purchase cost of capital ; r is the firm's discount rate, and $\gamma > 0$ represents the rate of energy-saving technical progress. We assume[6] $\mu < r$ and $\gamma < r$.

The Cobb-Douglas production function exhibits constant returns to scale but there exists operating costs whose size depends on the energy requirement of the capital[7] to any capital use $K(t)$ corresponds a given energy requirement $K(t)\, e^{-\gamma t}$. Technical progress is assumed to make machines becoming less energy-consuming over time. In the disembodied case, the total stock of capital goods become more and more energy saving over time whatever their age. This is a rather unrealistic assumption which will be relaxed in the next section. We assume that labour may be adjusted immediately and without any cost and this standard problem reduces to the following conditions for optimal inputs use:

$$L^*(t) = \left[\frac{A^{1-\theta}b(1-\beta)(1-\theta)}{\varpi}\right]^{\frac{1}{1-(1-\beta)(1-\theta)}} K_{eff}(t)^{\frac{\beta(1-\theta)}{1-(1-\beta)(1-\theta)}} \tag{2.9}$$

The cash flow $cf(t)$ provided by one capital unit that is used is then

$$cf(t) = \alpha B K_{eff}(t)^{\alpha-1} - Pe(t)e^{-\gamma t} \tag{2.10}$$

where $\alpha = \frac{\beta(1-\theta)}{1-(1-\beta)(1-\theta)}$ and $B = [1-(1-\beta)(1-\theta)]\left[\left(A^{1-\theta}b\right)\left[\frac{(1-\beta)(1-\theta)}{\varpi}\right]^{(1-\beta)(1-\theta)}\right]^{\frac{1}{1-(1-\beta)(1-\theta)}}$

Note that for bad realizations of the uncertain variable, it may become negative. Since we assume that there is no cost to keep the machine unused, it is then optimal for the firm to stop using it as soon as the marginal cash-flow becomes negative. Finally, the marginal cash-flow is thus

$\alpha B K_{eff}(t)^{\alpha-1} - Pe(t)e^{-\gamma t}$ if the unit is used
0 if the unit is not used

[6] If $\mu > r$, the firm would have an incentive to infinitely get into debt to buy an infinite amount of energy.

$\gamma < r$ is a standard assumption in the exogenous growth literature since it allows to have a bounded objective function.

[7] Such a complementarity is assumed in order to be consistent with the results of several studies showing that capital and energy are complements (see for instance Hudson and Jorgenson,1974 or Berndt and Wood, 1975).

We can define the level $Pe(t)$ for which the firm stops using units: $Pe^{*D}(t) = \alpha BK(t)^{\alpha-1}e^{\gamma t}$. Since this problem is only a benchmark, we will solve it very quickly. To get greater detail, the reader can refer to Pindyck (1988), which consider a similar problem except that there is no technological progress.

2.1. Determination of the value of a marginal unit of capital

- The value $v(t)$, of a unit of capital at time t has to satisfy the following Bellman equation:
$$rv(t) = \left[\alpha BK_{eff}(t)^{\alpha-1} - Pe(t)e^{-\gamma t}\right] + E_t(dv)/dt$$
For a given effective capital stock (which means that firm does not invest at time t), this differential equation leads to the following solution:
$$v(t) = \frac{\alpha B}{r}K_{eff}(t)^{\alpha-1} - \frac{Pe(t)}{r-\mu+\gamma}e^{-\gamma t} + a_1(K_{eff}(t),t)Pe(t)^{\beta_1}$$
where $\beta_1 = \frac{1}{2} - \frac{(\mu-\gamma)}{\sigma^2} + \sqrt{\left(\frac{(\mu-\gamma)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1$. Note that $\partial\beta_1/\partial\sigma^2 < 0$.
$a_1(K_{eff}(t),t)Pe(t)^{\beta_1}$ gives the value of the option to stop using the unit; it is of course an increasing function of the energy price.

- The value w(t) of a unit of capital that is not currently used is not currently providing any cash-flow. It has to satisfy the following Bellman equation:
$$rw(t) = E_t(dw)$$
and the solution of this differential equation is:
$$w(t) = a_2(K_{eff}(t),t)Pe(t)^{\beta_2}$$
$\beta_2 = \frac{1}{2} - \frac{(\mu-\gamma)}{\sigma^2} - \sqrt{\left(\frac{(\mu-\gamma)}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} < 0$. Note that $\partial\beta_2/\partial\sigma^2 > 0$.
$a_2(K(t),t)Pe(t)^{\beta_2}$ gives the value of the option to reuse the unit; it is a decreasing function of the energy price.

- The value $q(t)$ of a unit that has not already been acquired has to satisfy
$$rq(t) = E_t(dq)$$
that is
$$q(t) = a_3(K(t),t)Pe(t)^{\beta_2}$$

where $\beta_2 < 0$ is the same as previously. $a_3(K\ (t), t)Pe(t)^{\beta_2}$ gives the value of the option the firm has to give up if investing at time t. Such an option value comes from the fact that if acquiring a unit of capital at time t, the firms makes it more difficult (in the sense that the firm will require a better realization of the stochastic variable) to invest next period since the marginal productivity of the used capital will be smaller. Note that it is a function of the installed stock of capital (and not of the effectively used one) since investment only occurs once all the hold units are used.

2.2. Optimality conditions

The utilization rule allows to derive the value of a unit currently used which in turn provides the desired stock of capital through the investment rule .

2.2.1. Utilization rule

A unit of capital will only be used if the cash flow coming from its use is positive. Moreover, the firm uses an old machine until it becomes indifferent between using it or keeping it unused: for the level $Pe^*(t)$ of the energy price the value of the oldest machine used must be same whether it is used or not. Since the model is stochastic, the transition between these two values of the unit has also to be smooth for the firm to be at the optimum. These two conditions are the value matching and smooth pasting conditions that are standard in the irreversible investment under uncertainty literature:

$$\begin{aligned} v(t) &= w(t) \qquad \text{for } Pe(t) = Pe^{*D}(t) \\ \frac{\partial v(t)}{\partial Pe(t)} &= \frac{\partial w(t)}{\partial Pe(t)} \quad \text{for } Pe(t) = Pe^{*D}(t) \end{aligned}$$

$\Leftrightarrow$

$$\frac{\alpha B}{r} K_{eff}(t)^{\alpha-1} - \frac{Pe^*(t)}{r-\mu+\gamma} e^{-\gamma t} + a_1(K_{eff}(t))Pe^{*D}(t)^{\beta_1} = a_2(K_{eff}(t), t)Pe^{*D}(t)^{\beta_2} \tag{2.11}$$

$$-\frac{1}{r-\mu+\gamma} e^{-\gamma t} + \beta_1 a_1(K_{eff}(t))Pe^{*D}(t)^{\beta_1 - 1} = \beta_2 a_2(K_{eff}(t), t)Pe^{*D}(t)^{\beta_2 - 1} \tag{2.12}$$

Taking into account he fact that $\alpha B K_{eff}(t)^{(\alpha-1)} = Pe^{*D}(t)e^{-\gamma t}$, this leads to the expression of the marginal value of a unit used at time t:

$$\begin{aligned} v(t) &= \frac{\alpha B}{r} K_{eff}(t)^{\alpha-1} - \frac{Pe(t)}{r-\mu+\gamma} e^{-\gamma t} + \\ &\quad + \underbrace{\frac{\beta_2}{\beta_1 - \beta_2} \left[\alpha B K_{eff}(t)^{\alpha-1}\right]^{1-\beta_1} e^{-\gamma \beta_1 t} \left(\frac{1}{r} - \frac{(\beta_2 - 1)}{\beta_2 (r-\mu)}\right)}_{a_1(K_{eff}(t),t)} Pe(t)^{\beta_1} \end{aligned} \tag{2.13}$$

2.2.2. Investment rule

The firm invests until it becomes indifferent between acquiring one more unit and doing nothing, that is, until the value of one more unit exactly compensates for the constant cost k to acquire it and for the value of the option to invest in the future the firm has to give up (it corresponds to the value matching condition). In order for the firm to be at the optimum of this stochastic program, such an equality has also to be true when the derivatives of these values and of these costs with respect to the energy price are considered (it corresponds to the smooth pasting condition). These conditions provide the level of the energy price for which, given the installed stock of capital, it is optimal for the firm to invest in one more unit. Symmetrically, they can be interpreted as providing the desired stock of capital for a given observed realization of the energy price. Moreover, since all units are the same, the firm will only invest when using the full capacity; this implies that $K_{eff}(t) = K(t)$ in these optimality conditions:

$$v(t) = q(t) + k \quad (2.14)$$

$$\frac{\partial v(,t)}{\partial Pe(t)} = \frac{\partial q(t)}{\partial Pe(t)} \quad (2.15)$$

$\Leftrightarrow$

$$\frac{\alpha B}{r} K(t)^{\alpha-1} - \frac{Pe(t)}{r-\mu+\gamma} e^{-\gamma t} + a_1(K(t))Pe(t)^{\beta_1} = a_3(K(t))Pe(t)^{\beta_2} + k \quad (2.16)$$

$$-\frac{1}{r-\mu+\gamma} e^{-\gamma t} + \beta_1 a_1(K(t))Pe(t)^{\beta_1-1} = \beta_2 a_3(K(t))Pe(t)^{\beta_2-1} \quad (2.17)$$

2.3. The desired stock of capital

Solving the system (the desired level of capital $K^d(t)$ for an observed level of the stochastic variable $Pe(t)$ leads to the expression of the desired stock of capital:

$$\left[\alpha B K^d(t)^{(\alpha-1)}\right]^{(1-\beta_1)} e^{-\beta_1 \gamma t} \left[\frac{\beta_2 - 1}{\beta_2 (r-\mu+\gamma)} - \frac{1}{r}\right] Pe(t)^{\beta_1} + \frac{\alpha B K^d(t)^{(\alpha-1)}}{r} = k + \frac{Pe(t)e^{-\gamma t}}{r-\mu+\gamma}\left(\frac{\beta_2-1}{\beta_2}\right) \quad (2.18)$$

Results are very to similar to Pindyck (1988) except that the desired stock of capital moves during time not only through the realization of the stochastic variable but also through the technological improvement. It can easily be checked that the later (i.e. the more advanced the technology for) a given realization of the energy price, the greater the desired stock of capital.

The firm's stock of capital, $K(t)$,will only equalize the desired one, $K^d(t)$, when it is investing. Note again that because all units are the same, the firm first reuse all its old units before investing in new ones, as in Pindyck (1988). For instance, a firm will first make sure that it uses all its old computers before investing in new ones. This seems quite unrealistic but it is a standard feature of the models which deal with irreversible investment under uncertainty. It comes from the fact that all these models abstract from embodied technological progress. Indeed, this unrealistic feature disappears when technological progress embodiment is taken into account, as it is the case in the next section.

2.4. Investment

Investment happens each time the desired stock of capital tends to exceed the installed stock of capital. Therefore, it crucially depends on the installed stock of capital. Its dynamics consists into periods with no investment (when the realizations of the energy price are too high) and period with marginal investment. Note that the introduction of technological progress leads to a value of the energy price which trigger next investment decreasing with time. For sake of comparison with next section, note that in the special case for which we observe that the energy price evolves at the rate of technological progress, the desired stock of capital is constant which implies no investment.

3. Embodied technological progress under uncertainty

Now, we consider the same program, except that the technological progress is embodied. The problem the firm has to solve becomes then:

$$\max E_0 \left[\int_0^\infty \left[P(t)Q(t) - Pe(t)\ E(t) - \varpi(t)L(t) - k(t)I(t) \right] e^{-rt} dt \right] \tag{3.1}$$

subject to constraints taking uncertainty into account

$$\begin{aligned}
P(t) &= bQ(t)^{-\theta} \qquad \text{with } \theta < 1 && (3.2)\\
Q(t) &= AK_{eff}(t)^{\beta} L(t)^{1-\beta} && (3.3)\\
K_{eff}(t) &= \int_{\tau o}^{t} I(z)dz && (3.4)\\
dPe(t) &= \mu Pe(t)dt + \sigma Pe(t)dz(t) && (3.5)\\
E(t) &= \int_{\tau o}^{t} I(z)e^{-\gamma z}dz \text{ with } \gamma < r && (3.6)\\
\varpi(t) &= \overline{\varpi} && (3.7)\\
I(t) &= dK(t) \geq 0 && (3.8)\\
K(t) &\geq K_{eff} && (3.9)
\end{aligned}$$

$K_{eff}(t)$ is still the capital stock which is effectively used but now, units are not equal in terms of energy requirement and it is oldest units which may not be used. We note τo the acquisition date of the oldest machine currently used. Again, we assume that any capital unit that has been installed may temporarily not be used for free and that labor may be adjusted immediately and without any cost. The optimal use of labor is then:

$$L^*(t) = \left[\frac{A^{1-\theta}b(1-\beta)(1-\theta)}{\overline{\overline{\omega}}}\right]^{\frac{1}{1-(1-\beta)(1-\theta)}} K_{eff}(t)^{\frac{\beta(1-\theta)}{1-(1-\beta)(1-\theta)}}$$

It can be deduced that $cf(t,\tau)$, the cash-flow generated between time t and $(t+dt)$ by one unit of capital acquired at time τ, depends on whether this unit is used or not:

$$cf(t,\tau) = \max\left[0, \left(\alpha B K_{eff}(t)^{\alpha-1} - Pe(t)e^{-\gamma\tau}\right)dt\right] \tag{3.10}$$

where B is the same as in the previous section.

3.1. Determination of the value of a marginal unit of capital

It is then possible to derive the value of a unit of capital depending on the date of acquisition of this unit and on whether this unit is currently used or not. Recall that in the case of disembodied technological progress the acquisition date of the unit was of no relevance.

- The value $V(Pe(t),\tau,t)$, of a unit of capital at time t acquired at time τ and currently used has to satisfy the following Bellman equation:

  $$rV(t,\tau) = \left(\alpha B K_{eff}(t)^{\alpha-1} - Pe(t)e^{-\gamma\tau}\right)dt + E_t(dV)$$

  For a given effective capital stock (that is, if it is optimal for the firm first not to reuse old units that were previously unused and second not to invest at time t), this differential equation leads to the following solution:

  $$V(t,\tau) = \frac{\alpha B}{r}K_{eff}(t)^{\alpha-1} - \frac{Pe(t)}{r-\mu}e^{-\gamma\tau} + b_1(K_{eff}(t),\tau)Pe(t)^{\beta_1}$$

  where $\beta_1 > 1$ is the same as previously. $b_1(K_{eff}\ (t),\tau)Pe(t)^{\beta_1}$ gives the value of the option to stop using the unit.

- The value $W(t,\tau)$ at time t of a unit of capital acquired at time τ and not currently used is not currently providing any cash-flow. It has to satisfy the following Bellman equation:

  $$rW(t,\tau) = E_t(dW)$$

  and the solution of this differential equation is:

  $$W(t,\tau) = b_2(K_{eff}(t),\tau)Pe(t)^{\beta_2}$$

  where $\beta_2 < 0$ is the same as previously. $b_2(K_{eff}\ (t),\tau)Pe(t)^{\beta_2}$ gives the value of the option to reuse the unit.

- Value $O(t)$ in time t of a unit that has not already been acquired has to satisfy

$$rO(t) = E_t(dO)$$

that is

$$O(t) = b_3(K_{eff}(t), \tau)Pe(t)^{\beta_2}$$

where $\beta_2 < 0$ is the same as previously. $b_3(K_{eff}\ (t), \tau)Pe(t)^{\beta_2}$ gives the value of the option the firm has to give up if investing at time t. Note that contrary to what we had in the case of a disembodied technological progress, the value of the option to invest ($b_3(K_{eff}\ (t), \tau)Pe(t)^{\beta_2}$) is a function of the effectively used stock of capital and not of the installed one, since it may be optimal for the firm to invest even if it is not using all the installed units of capital.

3.2. Optimality conditions

The decision scheme is not the same as in the case of disembodied technological progress. Recall that in the previous section, the firm had first to decide whether to invest or not depending on the relative values of the desired capital stock (given the observed value of the uncertain variable) and of the already installed stock. In the case in which it was not optimal to invest, the firm must then decide to use all the unit capital it has installed or only a part of it. Since any unit of capital had the same characteristics because technological progress benefits to all units, the firm would first reuse old units before investing into new ones. In a way, the decisions of using installed units and of investing in new ones were taken independently since there was no incentive for the firm to replace old units by new ones.
It is no more the case when technological progress is embodied because capital units differ according to their installation date. The intuition is the following: since a new capital unit may be a lot more energy saving than an old one, it may be interesting for the firm to stop using one old unit and to invest into a new one even if there is an acquisition cost for the new one while there is none if the firm keeps using the old one. Therefore, the firm will simultaneously have to decide to invest or not and to determine the age of the oldest machine to use. Indeed, these two decisions are now closely linked.

3.2.1. Utilization rule

As already stated (see equation (3.10)), a unit of capital will only be used if the cash flow coming from its use is positive. For an observed energy price level, the acquisition date of the oldest machine it is optimal to use, τo^*, has thus to satisfy

$$\alpha B \left[\int_{\tau o^*}^{t} I(z)dz \right]^{\alpha - 1} = Pe(t)e^{-\gamma \tau o^*} \tag{3.11}$$

This condition states that the marginal productivity, which is the same for any used machine, has to be equal to the marginal cost of using the oldest machine. Moreover, the firm uses an old machine acquired at time τ until the realization of the energy is $Pe^{*E}(t)$ such that it becomes indifferent between using it or keeping it unused: the value of the oldest machine used must be same whether it is used or not. Since the model is stochastic, the transition between these two values of the unit has also to be smooth for the firm to be at the optimum. These two conditions are the usual value matching and smooth pasting conditions.

$$V(t,\tau) = W(t,\tau) \qquad \text{for } Pe(t) = Pe^{*E}(t) \tag{3.12}$$

$$\frac{\partial V(t,\tau)}{\partial Pe(t)} = \frac{\partial W(t,\tau)}{\partial Pe(t)} \qquad \text{for } Pe(t) = Pe^{*E}(t) \tag{3.13}$$

$\Leftrightarrow$

$$\frac{\alpha B}{r} K_{eff}(t)^{\alpha-1} - \frac{Pe^{*E}(t)e^{-\gamma\tau}}{r-\mu} + b_1(K_{eff}(t),\tau)Pe^{*E}(t)^{\beta_1} = b_2(K_{eff}(t),\tau)Pe^{*E}(t)^{\beta_2} \tag{3.14}$$

$$-\frac{e^{-\gamma\tau}}{r-\mu} + \beta_1 b_1(K_{eff}(t),\tau)Pe^{*E}(t)^{\beta_1-1} = \beta_2 b_2(K_{eff}(t),\tau)Pe^{*E}(t)^{\beta_2-1} \tag{3.15}$$

Taking into account the fact that $\alpha B K_{eff}(t)^{(\alpha-1)} = Pe^{*E}(t)e^{-\gamma\tau}$, this leads to the expression of the marginal value of a unit acquired at time τ and which is currently used:

$$\begin{aligned} V(t,\tau) = & \frac{\alpha B}{r} K_{eff}(t)^{\alpha-1} - \frac{Pe(t)e^{-\gamma\tau}}{r-\mu} + \\ & + \underbrace{\frac{\beta_2}{\beta_1-\beta_2}\left[\alpha B K_{eff}(t)^{\alpha-1}\right]^{1-\beta_1} e^{-\gamma\beta_1\tau}\left(\frac{1}{r} - \frac{(\beta_2-1)}{\beta_2(r-\mu)}\right)}_{b_1(K_{eff}(t),\tau)} Pe(t)^{\beta_1} \end{aligned} \tag{3.16}$$

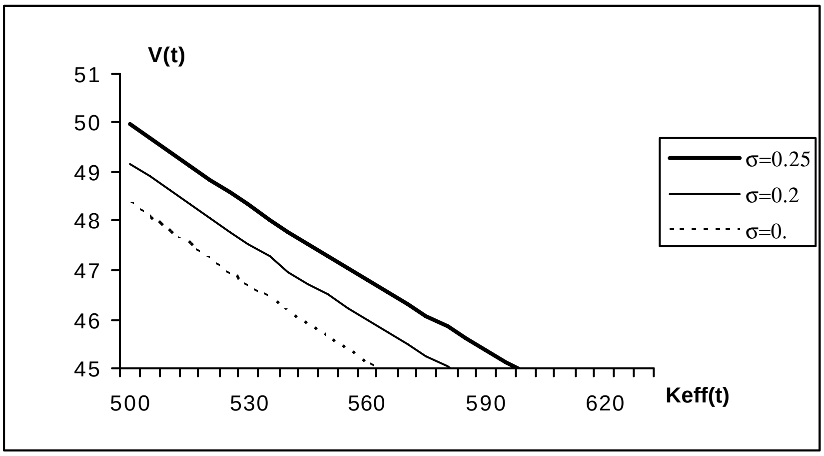
The value of the option to stop using the unit ($b_1(K_{eff}(t),\tau)Pe(t)^{\beta_2}$) negatively depends on its acquisition date τ: the sooner the unit has been installed, the less value it has to have the opportunity to stop using it.

We illustrate this value function using a numerical example. We assume $\beta = 0.3$, $\theta = 0.2$ (which correspond to a mark-up of 25%), $\mu = 0.02$ and $B = 100$. For the technological parameter, we choose $\gamma = 2\%$. Other parameters are those used in Pindyck (1988): $r = 0.05$; $k = 10$; $\sigma = 0.2$.

The value function is of course a decreasing function of the energy price. We can also observe on figure 2 in appendix that for a given energy price, the higher the uncertainty, the higher the value of the marginal unit, which is standard in the literature of investment under uncertainty. This is due to the fact that the part of this value which depends on uncertainty corresponds to the value of the option to stop using the unit which rises with uncertainty.

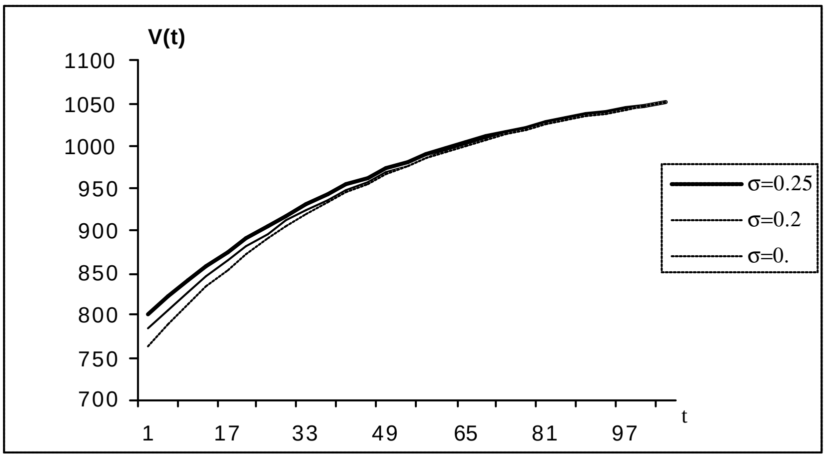
Figure 3 shows that the value of the option to stop using a unit of generation τ is increasing with $Keff(t)$, since the higher the stock of the capital, the more it is valuable to have the opportunity to stop using old units. Again, and for the same reason as previously, uncertainty increases the value of this option.

Figure 3: Value at time t of the option to stop using a unit of generation τ as a function of $Keff(t)$ $(Pe(t) = 0.5)$



The interesting result that can be seen on figure 4 is that, for a high enough τ, the value of a marginal used unit is the same, no matter how large is the uncertainty parameter. Indeed, for sufficiently recent units, the technological progress will be high enough to reduce drastically the energy requirements, and having the opportunity not to use them is of very low value, whatever the size of uncertainty.

Figure 4: Value of a marginal used unit as a function of its acquisition date τ $(Keff(t) = 1$ and $Pe(t) = 10)$



3.2.2. Investment rule

The firm invests for an energy price realization such that, for a given effective stock of capital, it is indifferent between acquiring one more unit and doing nothing, that is, until the value of a newly used unit exactly compensates for the constant cost k to acquire it and for the value of the option to invest in the future the firm has to give up (it corresponds to the value matching condition, eq 3.17). In order for the firm to be at the optimum of this stochastic program, the standard smooth pasting condition has to be satisfied as well. For given effective capital stock and technology levels, these optimality conditions provide the expression of the energy price level for which it is optimal for the firm to invest in one new unit. This expression may also be converted into that of the optimal effective stock of capital as a function of the observed energy price level and of the current level of technology.

$$V(t,\tau = t) = O(t) + k \tag{3.17}$$

$$\frac{\partial V(t,t)}{\partial Pe(t)} = \frac{\partial O(t)}{\partial Pe(t)} \tag{3.18}$$

$\Leftrightarrow$

$$\frac{\alpha B}{r} K_{eff}(t)^{\alpha-1} - \frac{Pe(t)}{r-\mu} e^{-\gamma t} + b_1(K_{eff}(t,\tau))Pe(t)^{\beta_1} = b_3(K_{eff}(t),\tau)Pe(t)^{\beta_2} + k \tag{3.19}$$

$$-\frac{1}{r-\mu} e^{-\gamma t} + \beta_1 b_1(K_{eff}(t),\tau)Pe(t)^{\beta_1 - 1} = \beta_2 b_3(K_{eff}(t),\tau)Pe(t)^{\beta_2 - 1} \tag{3.20}$$

Note that contrary to what has been obtained in the disembodied technological progress case (see equations 2.16 and 2.17), it is not the total amount of installed capital $K(t)$ but the capital which is effectively used that appears in equations (3.19) and (3.20). To decide how much to invest, firms do not care about how much capital they have but about how much capital they use. Due to the embodied technology, it may be interesting for the firm to acquire new units that are more energy saving even if all the old units are not used.

The resolution under disembodied technological progress leaves us with two conditions: one giving a requirement for the use of the capital (all the capital will be used if the total stock is less than $K_{eff}(t)$ while if $K(t)$ is greater than $K_{eff}(t)$, the stock $K(t) - K_{eff}(t)$ will stay unused) and the other giving a requirement for the investment in capital units (if $K(t)$ is less than $K^d(t)$, the firm invests until its stock reaches the desired one whereas if $K(t)$ is greater than $K^d(t)$, there is no investment). Since all the capital units are the same when the technological progress is disembodied, the desired capital does not coincides with the effective one for any realization of the uncertain variable. In fact, the expression for the desired stock of capital is only valid for the effective stock when the uncertain variable reaches its historically most favorable levels (corrected to take account of the technological progress).

Under embodied technological progress, the installed stock of capital (which may as well be in excess) is no more determinant for investment. What is more interesting is the effectively used stock of capital, and since the firm can always decide the age of the oldest capital unit in use to adjust the used stock to its optimal level, the desired level of used capital always coincides with the effective level and the expression for the desired level of effectively used capital is valid whatever the realization of the uncertain variable. This will allow to get the expression of the optimal acquisition date of the oldest machine as a function of the energy price.

The system (3.19)-(3.20) also provides the expression of the value of the option to invest:

$$O(t) = \left[\frac{\beta_1}{\beta_1-\beta_2}\left[\alpha BK_{eff}(t)^{\alpha-1}\right]^{1-\beta_1} e^{-\gamma\beta_1 t} Pe^{**E}(t)^{\beta_1-\beta_2}\left(\frac{1}{r}-\frac{(\beta_2-1)}{\beta_2(r-\mu)}\right)\right] Pe(t)^{\beta_1} \quad (3.21)$$

$$-\frac{e^{-\gamma t} Pe^{**E}(t)^{1-\beta_2} Pe(t)^{\beta_2}}{\beta_2\,(r-\mu)} \quad (3.22)$$

with $Pe^{**E}(t)$ satisfying:

$$\left[\alpha BK_{eff}(t)^{(\alpha-1)}\right]^{(1-\beta_1)}\left[\frac{\beta_2-1}{\beta_2\,(r-\mu)}-\frac{1}{r}\right] Pe^{**E}(t)^{\beta_1} e^{-\beta_1\gamma t} \quad (3.23)$$

$$+\frac{\alpha BK_{eff}(t)^{(\alpha-1)}}{r} = \frac{Pe^{**E}(t)e^{-\gamma t}}{r-\mu}\left(\frac{\beta_2-1}{\beta_2}\right)+k$$

Figure 5 shows that for a given t and K_{eff}, the value of the option decreases with the energy price, since the higher this price, the less valuable it is to have the opportunity to invest. Moreover, the higher the uncertainty, the higher this value. Note that even if uncertainty tends to zero, there will be still one option to wait for newer units because of the existence of technological progress. It can be easily checked that if $\sigma^2 \to 0$ and $\gamma = 0$, then $O(t) = 0$.

Figure 5: Option to postpone investment as a function of the energy price ($t = 10$ and $Keff(t) = 1$)

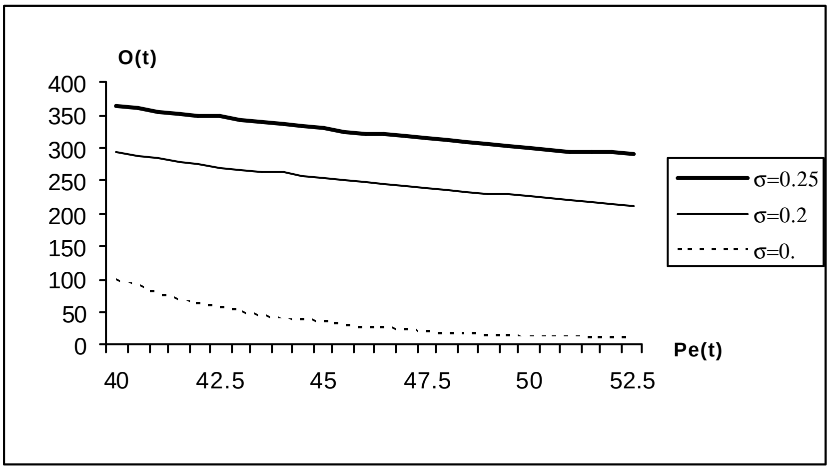
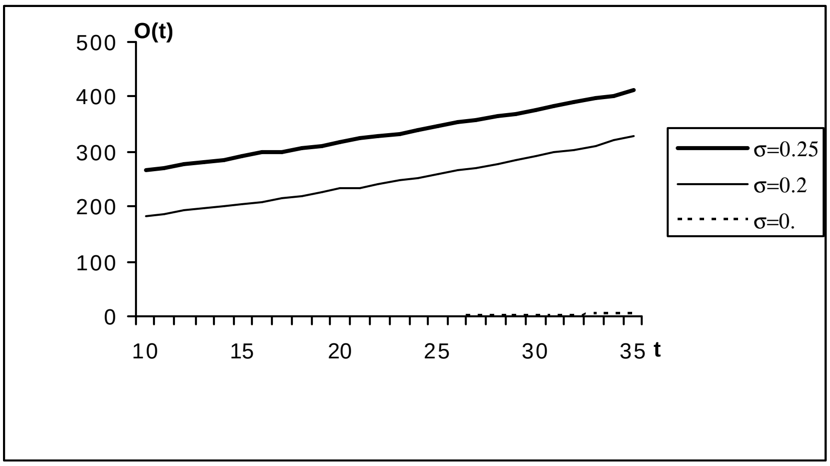


Figure 6 in appendix simply reflects decreasing returns: the more one uses the capital, the less worthy it is to hold the option to invest in future. Figure 7 illustrates one of the result of our model: as time passes, since units become more and more efficient in terms of energy requirements due to embodied technology progress, it becomes more and more worthy to have the opportunity to invest in the future.

Figure 7: Option to postpone investment as a function of time ($Keff(t) = 1$ and $Pe(t) = 80$)



3.3. Optimal effective capital stock

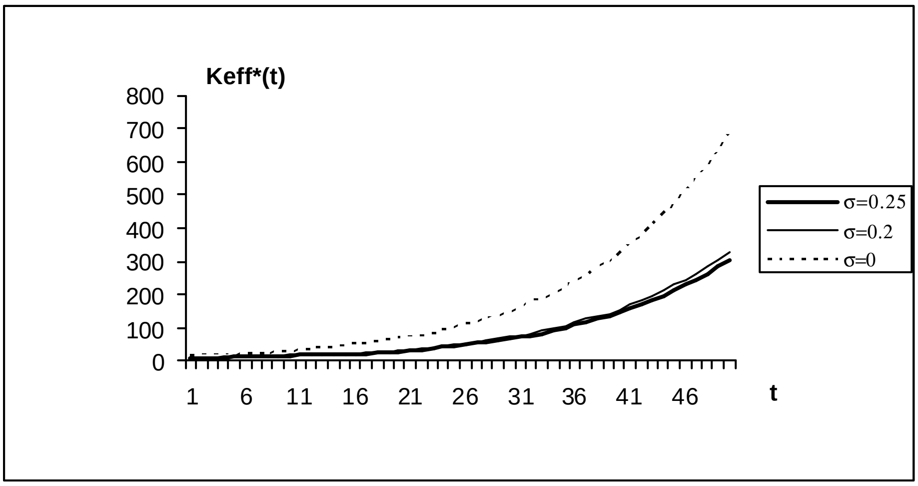
Given the observed level of the energy price and the current state of the energy-saving technology, it is optimal for the firm to have an effective capital stock equal to $K_{eff}{}^{*}(t)$ which is given by the following implicit equation :

$$\begin{aligned} &\left[\alpha B K_{eff}^{*}(t)^{(\alpha-1)}\right]^{(1-\beta_1)}\left[\frac{\beta_2-1}{\beta_2\,(r-\mu)}-\frac{1}{r}\right] Pe(t)^{\beta_1}e^{-\beta_1\gamma t} \\ &+\frac{\alpha B K_{eff}^{*}(t)^{(\alpha-1)}}{r} = \frac{Pe(t)e^{-\gamma t}}{r-\mu}\left(\frac{\beta_2-1}{\beta_2}\right)+k \end{aligned} \tag{3.24}$$

Note that thanks to potential decreases in the optimal age of the oldest machine used, it is possible for the optimal effective capital stock to be decreasing even if the total installed stock of capital is irreversible (as we have already seen, under embodiment, the installed stock of capital does not really matter as far as the firm's decisions are concerned). The optimal effective stock of capital has an expression close to[8] that derived under disembodiment for the desired stock of capital, but remember that in this latter case, $K_d(t)$ does not always match the effective stock of capital (it only does for good enough realizations of the stochastic variable).

Figure 8 in appendix shows that uncertainty reduces the optimal effective stock of capital. Of course the higher is the price of energy, the less firms use the machines. For a given energy price,the effective stock of capital increases with time (see figure 9), because more recent machines consume less energy. The higher the uncertainty, the smaller this effect.

Figure 9 : Optimal effective stock of capital as a function of time ($Pe(t) = 10$)



[8] Here of course, the discounting of the energy requirement ignores technical progress.

3.4. Optimal age of the oldest machine used

Since in this model there exists no cost to temporally not use a machine[9], there is no incentive for the firm to definitively scrap any machine. Thus we only derive an optimal age for the oldest machine used but not really an optimal scrapping age. This is a significative departure from what is obtained in a deterministic environment (see Boucekkine and Pommeret, 2001). Using equations (3.11) and (3.24) provides an implicit expression for the optimal acquisition date of the oldest machine as a function of the observed price:

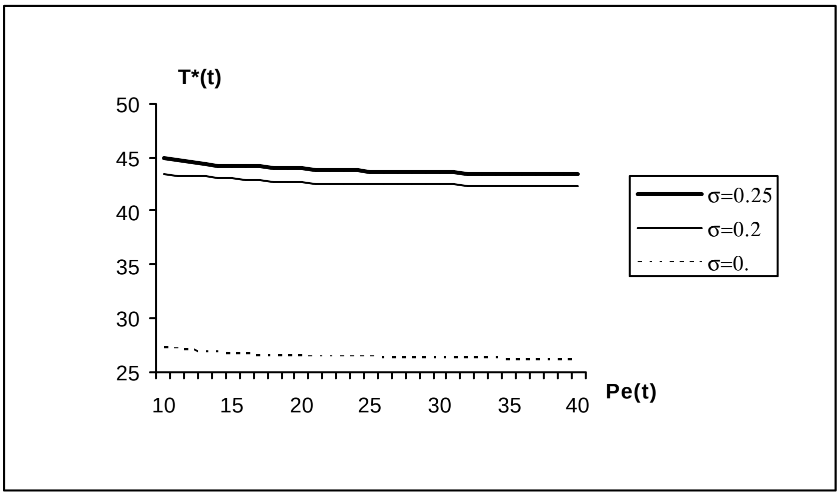
$$e^{-\gamma\tau o^*(t)} = \frac{rk}{Pe(t)} + \frac{(\beta_2 - 1)\,r}{\beta_2\,(r-\mu)}e^{-\gamma t} - \left(\frac{(\beta_2 - 1)\,r}{\beta_2\,(r-\mu)} - 1\right) e^{-\gamma\beta_1 t} e^{\gamma\tau o^*(t)(\beta_1 - 1)}$$

We also can express the optimal age of the oldest machine used $T^*(t) = t - \tau o^*(t)$:

$$e^{-\gamma\beta_1 T^*(t)} = \frac{\beta_2(\mu - r)}{\beta_2\mu - r}\left[1 - e^{-\gamma T^*(t)}\left(\frac{(\beta_2 - 1)\,r}{\beta_2\,(r-\mu)} + \frac{rk}{Pe(t)}e^{\gamma t}\right)\right]$$

Given the observed level of the energy price and the current state of the energy-saving technology, the firm desires to use only capital units that have been acquired at time $\tau o^*(t)$ or more recently. Firms dealing with uncertainty are more reluctant to renew the machines (see figure 10) ; replacement is in some sense postponed. It is also possible to see that a a higher energy price reduces the age of the oldest machine ; one could claim that the model can reproduce the "cleansing effect" : firms would tend to use newer machines in periods of higher energy prices and eventually acquire new units.

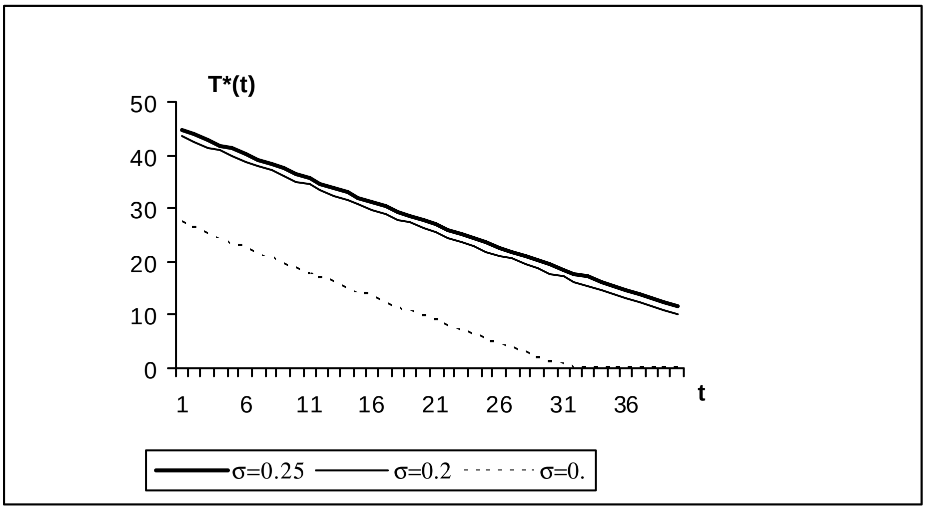
Figure 10: Optimal age of the oldest machine as a function of the energy price ($t = 1$)



[9]It is also the case in Pindyck (1988). Introducing a cost to keep the machine unused would generate an option to scrap the machine ; this would complicate the model a lot without significantly alter the results.

For a given energy price, as time passes, new technology becomes available and we have seen that the optimal effective stock of capital increases. Nevertheless, figure 11 shows that rising $K_{eff}{}^*(t)$ is not achieved through the use of older machine. Indeed, the optimal age of the oldest machine used is a decreasing function of time. This means that not only capacity is expanded through investment but that there is also replacement of old machine by new ones which takes place.

Figure 11 : Optimal age of the oldest machine
as a function of time ($Pe(t) = 10$)



3.5. Optimal investment

Equation (3.4) implies

$$dK_{eff}(t) = d\left[\int_{\tau o^*}^{t} I(z)dz\right]$$

$$\Rightarrow I^*(t) = I(\tau o^*)\frac{d\tau o^*}{dt} + \frac{dK_{eff}^*}{dt}$$

Current investment is therefore the sum of the "destruction term" $I(\tau o^*)(d\tau o^*/dt)$ that can be positive or negative, and of the variation over time of the desired effective capital stock ; that is, past history of investment matters for contemporaneous investment. Nevertheless it cannot be stated that investment exhibits echoes since variations in both the optimal date of acquisition of the oldest machine and the optimal desired capital highly depend on the realization of the uncertain variable.

Let us consider the special case for which we observe that during a time period dt the energy price evolves exactly to compensate for the gains in technology : $dPe(t)/Pe(t) =$

γdt. The optimal stock of effective capital becomes then constant as it is also the case in a deterministic framework (see Boucekkine and Pommeret, 2001): $dK^*_{eff}(t) = 0$. This does not mean that no investment is undertaken, since the optimal acquisition date of the oldest machine is not constant:

$$d\tau o^*(t) = \frac{1}{\gamma}\frac{dPe(t)}{Pe(t)} - \frac{\alpha B}{\gamma Pe(t)} e^{\gamma \tau o^*} K^*_{eff}(t)^{(\alpha-1)} \frac{dK^*_{eff}(t)}{K^*_{eff}(t)} = dt$$

Therefore, in this special case, the optimal acquisition date of the oldest machine increases exactly with time and old machines are replaced by new ones:

$$I^*(t) = I(\tau o^*)$$

4. Dynamics of the effective stock of capital, age of the oldest machine and investment

In this dynamic example, we use the same parameters as previously. Simulations are driven over 100 periods. In order to get the dynamics of $Pe(t)$, a geometric brownian motion is simulated using parameters $\mu = 0.02$, $\sigma^2 = 0.04$ and $Pe(0) = 10$ as a starting value. Figure 12 gives the sample path for $Pe(t)$. The firm observes the energy price and derives how much effective capital it is going to use. It should then decide whether it should only use more or less hold units or it should invest in new units, which would increase its total stock of capital.

Replacement

The so-called lumpiness of investment can be reproduced by all models considered here (in fact, investment irreversibility is sufficient to generate such a characteristics). However it is only under the assumption of embodied technological progress that replacement is possible. In the case of homogenous units, the firm should reach full capacity before investing. In such cases, the firm will present a very strong "procyclical" behavior, since the energy price should reach its historically lowest level (corrected to take account of the rate of technological progress if relevant) to induce the firm to use all its units and invest. In figure14, it can be seen that, under no technological progress or under disembodied technological progress, firms barely invest : they increase their total stock of capital only twice in this example, at the beginning of the program and at the very end of the period considered, when there is a significative decrease in the energy price. In the embodied case, investment is driven part by the willingness to increase the effective stock of capital and part by the possibility to acquire a more efficient machine in terms of energy requirements, that is, replacement. Indeed, in our example, investment occurs more often when technological progress is embodied rather than disembodied and periods of investment correspond to peaks in the energy price: clearly, replacement occurs which explain investment.

Uncertainty

Comparing the two cases of embodied technology shows (see figure 14) that the echoes effects is no more identifiable when firms operate in a stochastic environment. Moreover, the total stock of capital become smaller (see figure 13) and firms are more reluctant to renew the machines, leading to a higher optimal age for the oldest machine in use[10] (see figure 15). For these two reasons, firms under uncertainty will tend to invest less in new capital (see figure 14).

Technological progress

Not surprisingly, a higher rate of technological progress induces more capital accumulation (see figure 16 in appendix) ; investment peaks are higher and more frequent (see figure 17 in appendix). Replacement occurs more intensively since the age of the oldest machine used is smaller (see figure 18 in appendix).

To sum up, our model manages to reproduce the following stylized facts:

- Investment occurs in spurts, and the so-called lumpiness of the investment appears
- In the embodied case, firms can invest even if they are not using all the units they have, or in other words, it is the effective stock of capital (as opposed to the total one) that determines investment, which seems to be a much more realistic result. In this case, firms can invest for very unfavorable realizations of the uncertain variable. The higher the rate of technological progress, the more active the replacement. To some extent, this model support the cleansing effect of recessions argument.
- Uncertainty reduces both the total stock of capital and the proportion of new machines in this stock. Both capacity expansion and replacement are postponed.

[10] Since the firm may not own old enough machines, its maximal age of the oldest machine may be smaller than the optimal one. This would not affect the effective capital stock which can still be optimal, but it would of course affect investment.

Figure 12: Energy Price as a Geometric Brownian Motion, $\mu = 0.02$ and $\sigma^2 = 0.04$

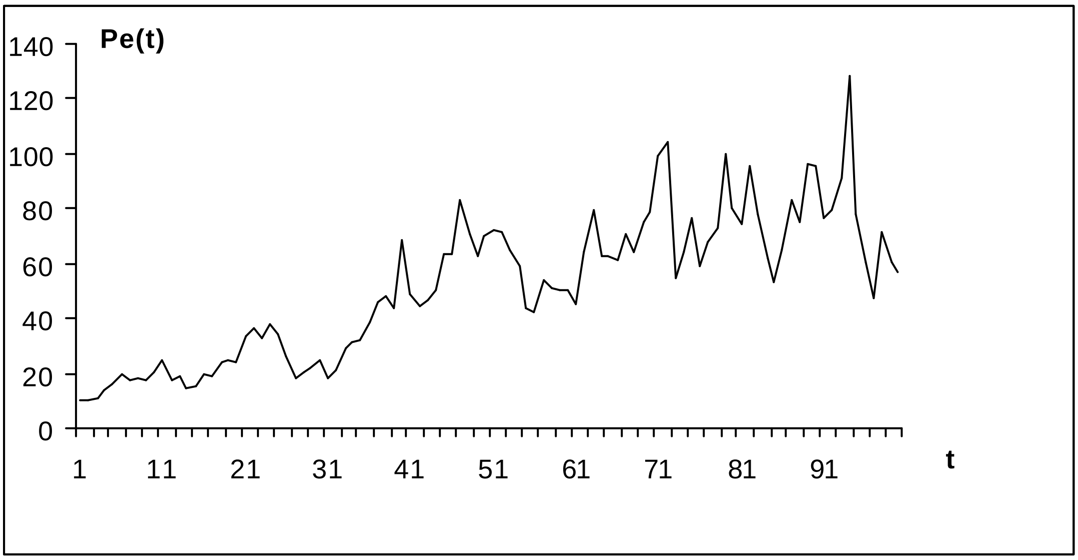


Figure 13- : Total Stock of Capital

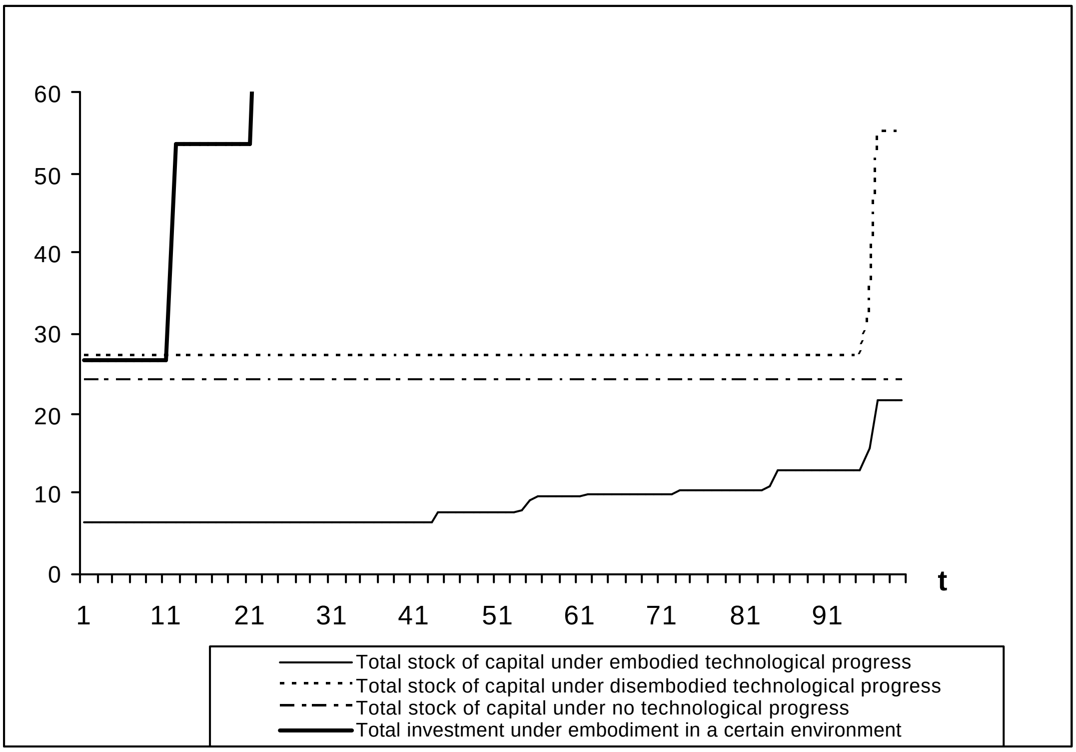


Figure 14: Investment

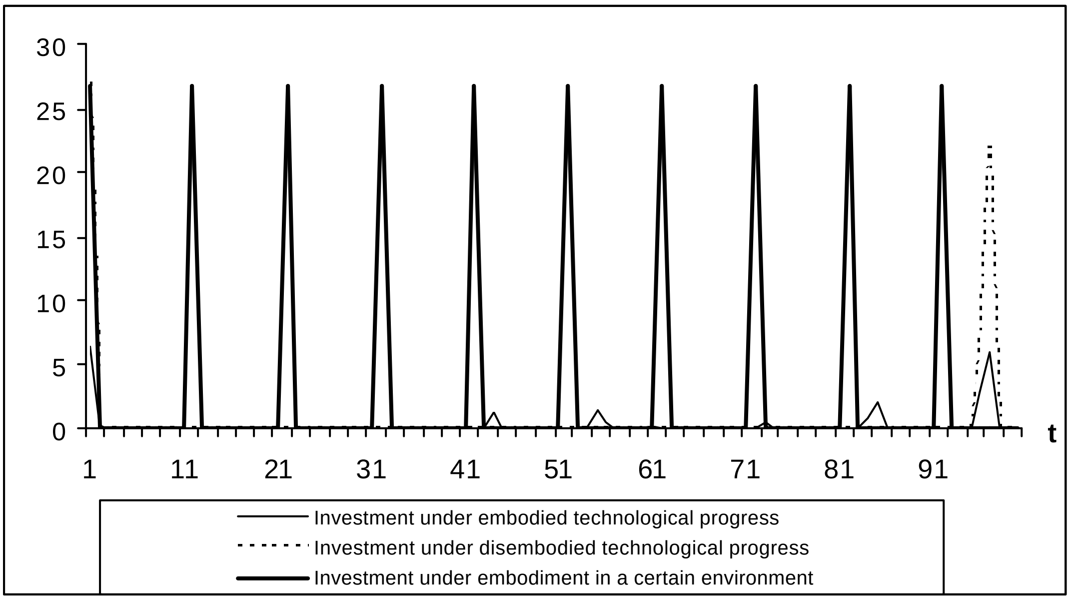
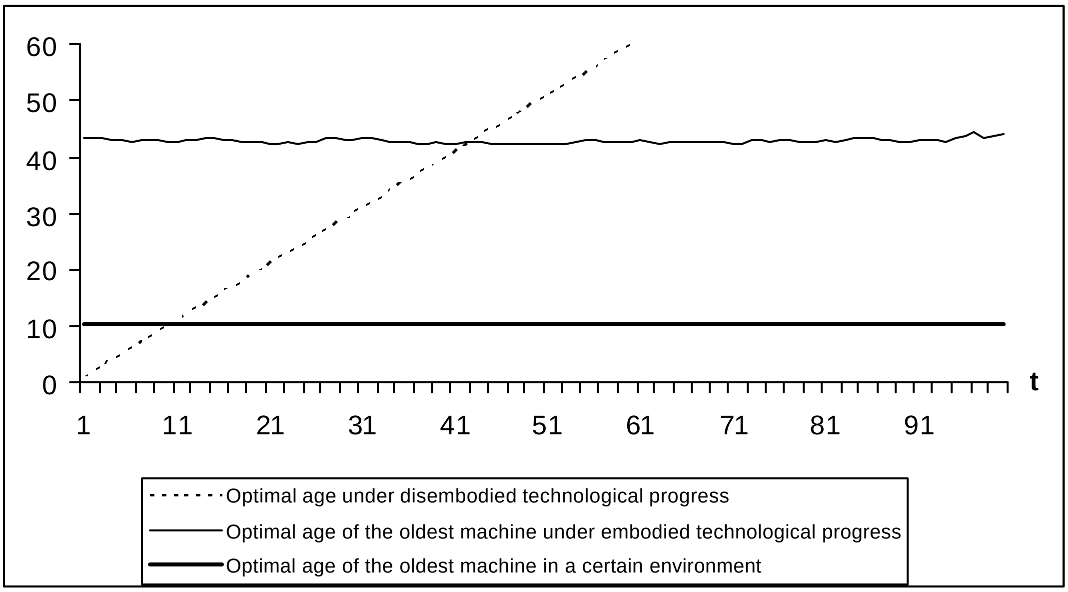


Figure 15: Age of the Oldest Machine



5. Conclusion

The literature on investment under uncertainty has mainly focused on the expansion of capacity as driving investment ; however, replacement is an important part of the story of capital accumulation, as suggested by the large literature on vintage capital. This paper has proposed a model of irreversible investment under uncertainty with embodied technological progress, in which firms invest not only to expand the capacity but also to replace old machines. The scrapping decision or the age of the oldest machine is then endogenous, but it is no longer constant as in the literature of vintages, and evolves stochastically. As shown by a dynamic example, uncertainty increases the optimal age of the machines in use, and due to uncertainty, not only capacity expansion but replacement as well, are postponed. By introducing heterogenous capital units, the model gets rid from the perfect "procyclicity" of investment usually implied in the literature of irreversible investment under uncertainty. The discussion on the firms behavior with respect to capital accumulation and on the effect of shocks on the economy is indeed significantly enriched. The so-called cleansing effect of recessions appears since replacement can occur in bad realizations of the stochastic process.

One clear extension of this paper is to introduce heterogenous firms to study the dynamics of the aggregate capital stock, and to eventually test it empirically. Note moreover that the discussion of energy utilization as well as some recent crisis in this sector have strengthened debates on how society should deal with macroeconomic impacts of energy price shocks ; an extension of the model proposed in this paper could compute the social benefits of energy policies, for instance by predicting the impact of an energy tax on the opportunity of replacement and more broadly on the economy.

6. References

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7. Appendix

Figure 2: Value at time t of a marginal unit acquired at time τ and currently used as a function of the energy price ($Keff(t) = 1$ and $\tau o^*(t) = 1$)

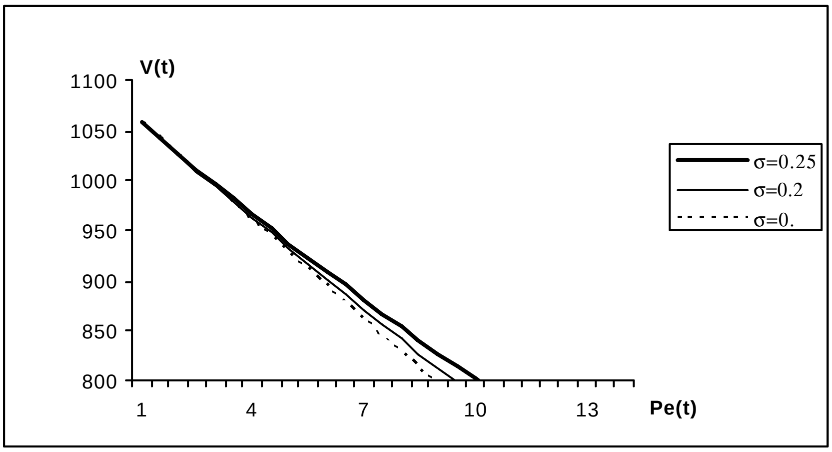

Figure 6: Option to postpone investment as a function of the effective stock of capital ($t = 10$ and $Pe(t) = 50$)

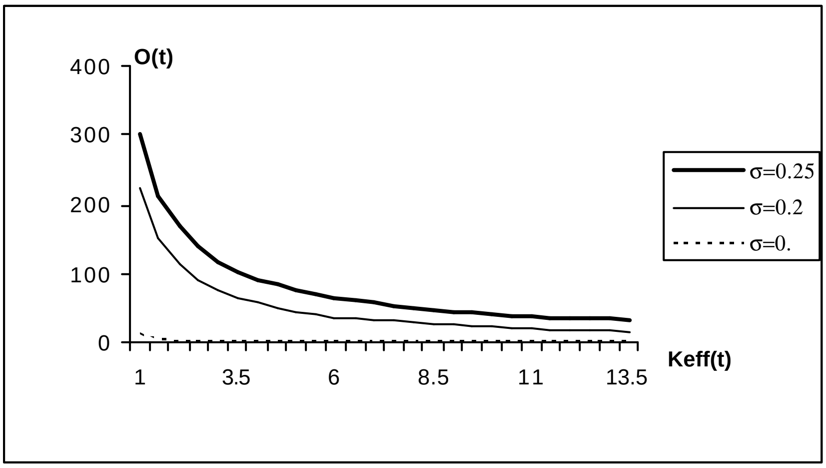

Figure 8: Optimal effective stock of capital as a function of the energy price ($t = 1$)

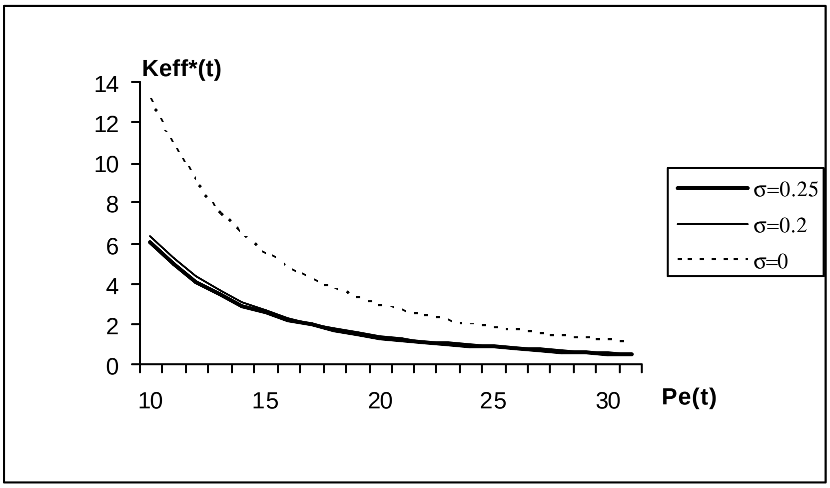

Figure 16- : Total Stock of Capital

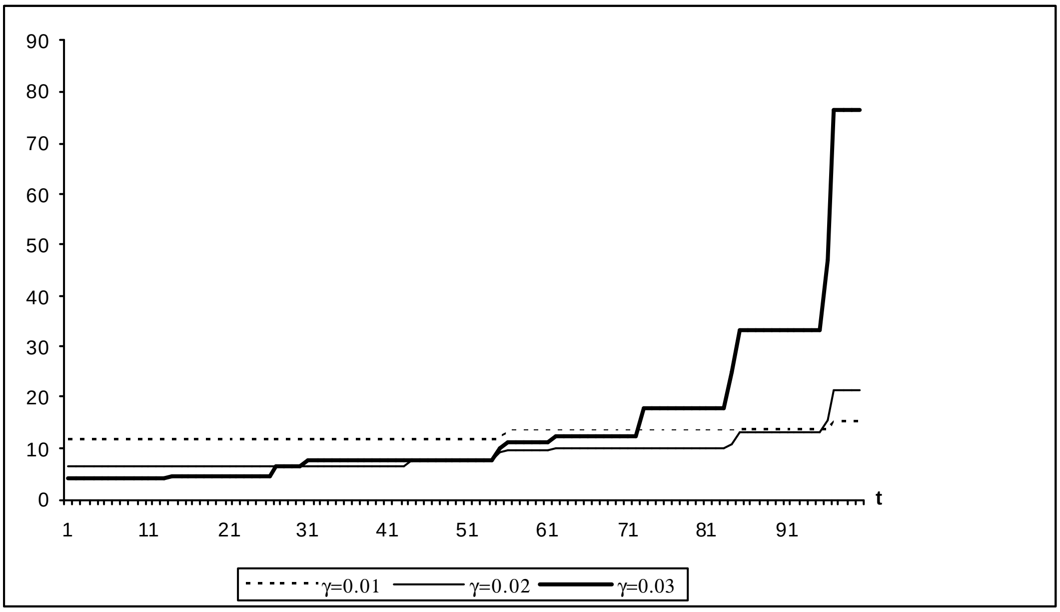

Figure 17: Investment

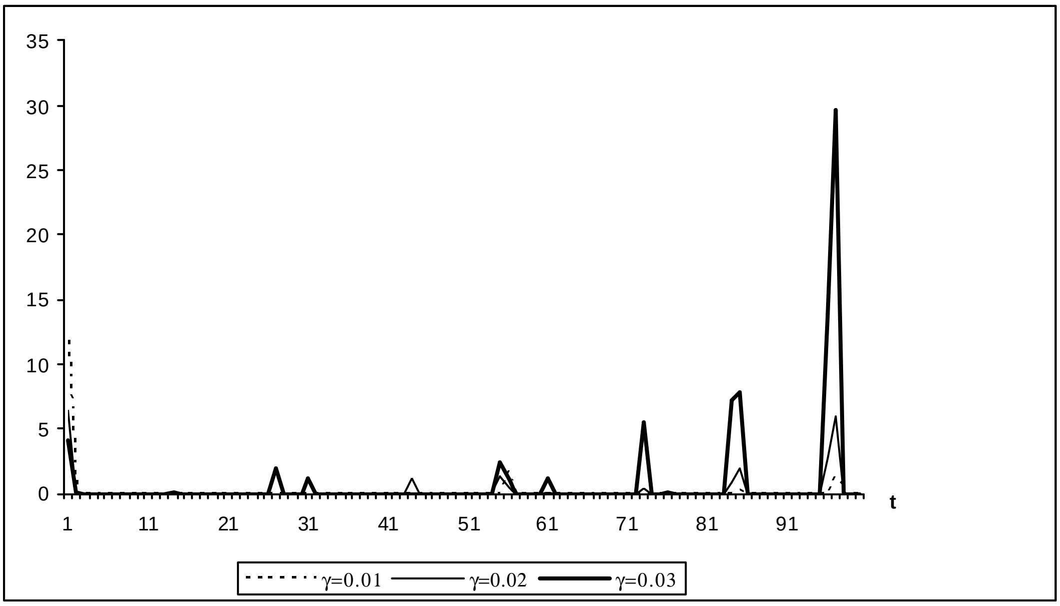

Figure 18: Age of the Oldest Machine